Bull. Aust. Math. Soc. **92** (2015), 397–404
doi:10.1017/S0004972715000921

A SHARP VERSION OF BONSALL'S INEQUALITY

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(Received 10 January 2015; accepted 18 June 2015; first published online 19 August 2015)

Abstract

The best possible constant in a classical inequality due to Bonsall is established by relating that inequality to Young's. Further, this extends the range of Bonsall's inequality and yields a reverse inequality. It also provides a better constant in an inequality of Hardy, Littlewood and Pólya.

2010 *Mathematics subject classification*: primary 26D16; secondary 44A35.

Keywords and phrases: Bonsall's inequality, convolution, Young's inequality, Schur's inequality.

1. Introduction

In 1951, Bonsall [2] proved an inequality for integral transforms in which the kernel is homogeneous, with degree in $[-1, 0)$.

THEOREM B. *Suppose that* $p > 1$, $q > 1$, $p^{-1} + q^{-1} \geq 1$ *and* $\lambda = 2 - p^{-1} - q^{-1}$. *Suppose that* f *and* g *are real functions on* $\mathbb{R}^+ = (0, \infty)$ *and that* K *is nonnegative and homogeneous of degree* -1 *on* $\mathbb{R}^+ \times \mathbb{R}^+$ *with*

$$\int_0^\infty x^{-1/(\lambda q')} K(x, 1)\, dx = \int_0^\infty x^{-1/(\lambda p')} K(1, x)\, dx = C.$$

Then

$$\int_0^\infty \int_0^\infty K^\lambda(u, v) f(u) g(v)\, du\, dv \leq C^\lambda \|f\|_p \|g\|_q \tag{1.1}$$

and

$$\left\| \int_0^\infty K^\lambda(u, v) f(u)\, du \right\|_{q'} \leq C^\lambda \|f\|_p. \tag{1.2}$$

Here and throughout, a displayed inequality expressed with the symbol '≤' should be taken as implying that if the greater side exists and is finite, then so does the lesser, and the inequality then applies. So implicitly, in this case, $x^{-1/(\lambda q')} K(x, 1)$, $x^{-1/(\lambda p')} K(1, x)$, f^p and g^q are Lebesgue measurable and integrable on $\mathbb{R}^+$. Later there will occur reverse inequalities that are expressed using '≥'. For these the opposite convention applies; the finiteness of the lesser side will imply that of the greater. This is unconventional, so in these cases the assumptions will be spelt out. Where norms appear in this paper, they use Lebesgue integrals over the full domain of the relevant function. Also throughout, a prime will denote the Hölder conjugate, for example $1/q + 1/q' = 1$.

Bonsall did not claim that C here is the best (least) constant; the main purpose of this paper is to establish that best possible constant. Note that (1.2) implies (1.1) by the Hölder inequality, and (1.1) implies (1.2) by its converse. It follows that the best constant will be the same in both inequalities. Note also that we have shown in [8] that inequality (1.1) is strict unless either f or g is null, so (1.2) is strict unless f is null.

Bonsall proposed his result as a generalisation of the well-known Theorem 319 of Hardy, Littlewood and Pólya [6], sometimes called Schur's inequality, wherein p and q are conjugate, so that $\lambda = 1$.

Theorem HLP319. *Suppose that $p > 1$, $p^{-1} + q^{-1} = 1$, and that functions f and g are real functions on $\mathbb{R}^+$; suppose that $K(x, y)$ is nonnegative and homogeneous of degree -1 on $\mathbb{R}^+ \times \mathbb{R}^+$. Then*

$$\int_0^\infty \int_0^\infty K(x, y) f(x) g(y)\, dx\, dy \le C\, \|f\|_p \|g\|_q \tag{1.3}$$

and

$$\left\| \int_0^\infty K(x, y) f(x)\, dx \right\|_p \le C\, \|f\|_p, \tag{1.4}$$

where

$$C = \int_0^\infty K(x, 1) x^{-1/p}\, dx = \int_0^\infty K(1, y) y^{-1/q}\, dy.$$

It is known that the constant C is the least possible, although again equality in (1.3) is only attained when either f or g is null.

Bonsall's inequality is also a generalisation of Hardy, Littlewood and Pólya [6, Theorem 340], where p and q are not necessarily conjugate, but K is restricted to $K(x, y) = 1/(x + y)$.

Theorem HLP340. *Suppose that $p > 1$, $q > 1$, $p^{-1} + q^{-1} \ge 1$ and $\lambda = 2 - p^{-1} - q^{-1}$, and that functions f and g are real functions on $(0, \infty)$. Then*

$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{(x + y)^\lambda}\, dx\, dy \le A \|f\|_p \|g\|_q \tag{1.5}$$

and

$$\left\| \int_0^\infty \frac{f(x)}{(x + y)^\lambda}\, dx \right\|_{q'} \le A \|f\|_p, \tag{1.6}$$

where A depends only on p and q.

In contrast with the previous theorem, the best constant in these inequalities is unknown. Hardy, Littlewood and Pólya [6], in discussing the series analogue, write 'the best value has not been found in the general case, and the problem of determining it appears to be difficult'. They did not give a specific value for A. Theorem B implies $A = [\pi \operatorname{cosec} \pi/(\lambda p')]^{\lambda}$, a value earlier proved by Levin (see [7] or [5, Section 3.4]). See also [9] for an experimental approach that suggested that Levin's constant may not be the best possible.

Both Theorems HLP319 and HLP340 are generalisations of Hilbert's inequality in integral form. Hilbert's result may be viewed as the intersection of those two theorems, and Bonsall's theorem as their join.

This paper will extend the range of parameters in Bonsall's theorem and demonstrate the best possible constant for the inequalities therein. This is done by converting the result to Young's convolution inequality.

2. Young's inequality

For real functions f and g defined on $\mathbb{R}$, we write the standard convolution as

$$f * g(v) = \int_{-\infty}^{\infty} f(u)g(v-u)\,du. \tag{2.1}$$

Then Young's convolution inequality in [10], as shown by Hardy, Littlewood and Pólya [6, Theorem 280], is equivalent to the following result.

THEOREM Y. *If $1 < \alpha, \beta, \gamma < \infty$ and $\alpha^{-1} + \beta^{-1} - \gamma^{-1} = 1$, and f and g are real functions on $\mathbb{R}$, then*

$$\|f * g\|_{\gamma} \le \|f\|_{\alpha}\|g\|_{\beta}. \tag{2.2}$$

Their proof (or rather the proof of the series analogue which is the only one given) will extend without difficulty to $\alpha = 1$ and to $\beta = 1$.

In 1975, Beckner [1] proved a sharper inequality. Brascamp and Lieb [3] gave an alternative proof and showed that the reverse inequality applies if $0 < \alpha, \beta, \gamma < 1$. To describe these we require the following definition.

DEFINITION. For $p \neq 0$, let $C_p = |p|^{1/(2p)}/|p'|^{1/(2p')}$. This includes the limiting cases $C_1 = C_\infty = 1$.

The modern form of Young's inequality, as extended by Beckner, Brascamp and Lieb, is as follows.

THEOREM BBL1. *Suppose that α and β are positive and $\gamma^{-1} = \alpha^{-1} + \beta^{-1} - 1$. Suppose that f and g are real functions on $\mathbb{R}$.*

(i) *If $1 \le \alpha, \beta, \gamma \le \infty$, then*

$$\|f * g\|_{\gamma} \le C_{\alpha}C_{\beta}C_{\gamma'}\|f\|_{\alpha}\|g\|_{\beta}. \tag{2.3}$$

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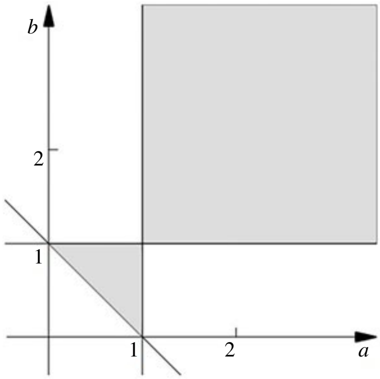

FIGURE 1. Space of parameters for Theorem BBL1.

(ii) *If* $0 < \alpha, \beta, \gamma \leq 1$ *and* f *and* g *are nonnegative in* L_α *and* L_β *respectively, then*

$$\|f * g\|_\gamma \geq C_\alpha C_\beta C_{\gamma'} \|f\|_\alpha \|g\|_\beta. \tag{2.4}$$

Equality is attained for the Gaussian functions, $f(x) = e^{-\alpha' x^2}$, $g(x) = e^{-\beta' x^2}$.

The space of parameters used in the theorem is shown in Figure 1 with a, b, c replacing $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$ respectively. Part (i) of the theorem assumes that $a \leq 1$, $b \leq 1$ and $c \geq 0$; the closed triangular region shown. Note that the three boundaries all have $C_\alpha C_\beta C_{\gamma'} = 1$ so the inequality there reduces to the original form due to Young. Fix b in the interior of this region and consider the variation of $\kappa = C_\alpha C_\beta C_{\gamma'}$ with a. Set $\mu = 2 \log \kappa$. Then

$$\frac{\partial^2 \mu}{\partial a^2} = \frac{1}{c} - \frac{1}{a} + \frac{1}{1-a} - \frac{1}{1-c}$$

which is greater than 0 since $c < a$. So μ is convex for $1 - b < a < 1$. Since μ is 0 at the end points, it must be negative within and hence $\kappa < 1$ at interior points in this region. For part (ii), the parameters allowed are those in the shaded semi-infinite region, $a, b \geq 1$. Again $\mu = 0$ on the boundary but this time $\partial \mu / \partial a > 0$ internally, giving $\kappa > 1$.

In group-theoretic terms, (2.1) is a convolution for functions on the group of real numbers under addition. We will use

$$f \star g(x) = \int_0^\infty f\left(\frac{x}{t}\right) g(t) \, \frac{dt}{t}, \tag{2.5}$$

the convolution of functions on the positive numbers under multiplication, with dt/t the Haar measure. Results will be simplified if we also include the Haar measure in the norms; we define a norm

$$|||f|||_p = \left[\int_0^\infty f(x)^p \, \frac{dx}{x} \right]^{1/p}$$

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so that

$$|||f|||_p = \|x^{-1/p} f(x)\|_p.$$

Of course, these convolutions are related by a logarithmic transformation. In $h(v) = \int_{-\infty}^{\infty} f(v-u)g(u)\,du$, substitute $v = \log x$, $u = \log y$, then $k(\cdot) = f(\log(\cdot))$, $\phi(\cdot) = g(\log(\cdot))$ and $\psi(\cdot) = h(\log(\cdot))$, giving

$$\psi(x) = h(\log x) = \int_0^{\infty} f\Big(\log\frac{x}{y}\Big) g(\log y)\,\frac{dy}{y} = \int_0^{\infty} k\Big(\frac{x}{y}\Big)\phi(y)\,\frac{dy}{y}.$$

Note also that

$$|||\psi|||_p = \|h\|_p$$

with similar relations for k and ϕ. Applying these to Theorem BBL1 gives the following result.

Theorem BBL2. *Suppose that α and β are positive and $\gamma^{-1} = \alpha^{-1} + \beta^{-1} - 1$. Suppose that $k : \mathbb{R}^+ \to \mathbb{R}$, $\phi : \mathbb{R}^+ \to \mathbb{R}$.*

(i) *If $1 \le \alpha, \beta, \gamma \le \infty$, then*

$$|||k \star \phi|||_\gamma \le C_\alpha C_\beta C_{\gamma'} |||k|||_\alpha |||\phi|||_\beta. \tag{2.6}$$

(ii) *If $0 < \alpha, \beta, \gamma < 1$ and k and ϕ are nonnegative in L_α and L_β respectively, then*

$$|||k \star \phi|||_\gamma \ge C_\alpha C_\beta C_{\gamma'} |||k|||_\alpha |||\phi|||_\beta. \tag{2.7}$$

Equality is attained for certain nonnull functions, except when $\alpha = 1$, or $\beta = 1$.

3. Bonsall's inequality revisited

In [4], Erdélyi noted that Theorem HLP319 may be reworded in terms of the convolution inequality

$$|||k \star f|||_p \le |||k|||_1 |||f|||_p. \tag{3.1}$$

Bonsall's inequality may be converted in a similar manner. Then application of Theorem BBL2 shows that the range of parameters p and q may be extended and provides a reverse inequality.

Theorem B2. *Suppose that p, q are in the extended reals and set $\lambda = 2 - p^{-1} - q^{-1}$. Suppose that $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is nonnegative and homogeneous of degree -1, that $f : \mathbb{R}^+ \to \mathbb{R}$, $g : \mathbb{R}^+ \to \mathbb{R}$ and that*

$$\int_0^{\infty} x^{-1/(\lambda q')} K(x,1)\,dx = \int_0^{\infty} x^{-1/(\lambda p')} K(1,x)\,dx = C.$$

(i) *If $1 \le p \le \infty$, $1 \le q \le \infty$ and $p^{-1} + q^{-1} \ge 1$ (so $0 \le \lambda \le 1$) then*

$$\int_0^{\infty}\int_0^{\infty} K^\lambda(x,y) f(x) g(y)\,dx\,dy \le C_{1/\lambda} C_p C_q C^\lambda \|f\|_p \|g\|_q \tag{3.2}$$

and

$$\Big\| \int_0^{\infty} K^\lambda(u,v) f(u)\,du \Big\|_{q'} \le C_{1/\lambda} C_p C_q C^\lambda \|f\|_p. \tag{3.3}$$

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(ii) *If* $0 < p < 1$, $-\infty < q < 0$ *and* $p^{-1} + q^{-1} < 1$ *(so* $1 < \lambda < \infty$*),* f *and* g *are nonnegative in* L_p *and* L_q *respectively and* $C < \infty$, *then*

$$\int_0^\infty \int_0^\infty K^\lambda(x,y) f(x) g(y)\, dx\, dy \geq C_{1/\lambda} C_p C_q C^\lambda \|f\|_p \|g\|_q \tag{3.4}$$

and

$$\left\| \int_0^\infty K^\lambda(u,v) f(u)\, du \right\|_{q'} \geq C_{1/\lambda} C_p C_q C^\lambda \|f\|_p. \tag{3.5}$$

In both parts, equality is attained for certain nonnull functions, unless p *or* λ *is* 1.

Proof. We identify α, β and γ of Theorem BBL2 with $1/\lambda$, p and q' given here, so that the definition of λ ensures that $\gamma^{-1} = \alpha^{-1} + \beta^{-1} - 1$. Define $k(z) = z^{\lambda-(1/p')} K^\lambda(1,z)$ and $\phi(z) = z^{1/p} f(z)$. This gives

$$|||k|||_\alpha = \left\{ \int_0^\infty x^{-1/(\lambda p')} K(1,x)\, dx \right\}^\lambda = C^\lambda,$$

$$k \bigstar \phi(y) = y^{\lambda-(1/p')} \int_0^\infty x^{-\lambda} K^\lambda(1, y/x) f(x)\, dx = y^{1/q'} \int_0^\infty K^\lambda(x,y) f(x)\, dx,$$

$$|||k \bigstar \phi|||_\gamma = \left\| \int_0^\infty K^\lambda(x,y) f(x)\, dx \right\|_{q'} \tag{3.6}$$

and

$$|||\phi|||_\beta = \|f\|_p. \tag{3.7}$$

Assuming the conditions of part (i), $1 \leq p \leq \infty$ gives $1 \leq \beta \leq \infty$, $1 \leq q \leq \infty$ gives $1 \leq \gamma \leq \infty$ and $0 \leq \lambda \leq 1$ gives $1 \leq \alpha \leq \infty$. All the conditions of Theorem BBL2 part (i) are satisfied. Thus we have

$$|||k \bigstar \phi|||_\gamma \leq C_\alpha C_\beta C_{\gamma'} |||k|||_\alpha |||\phi|||_\beta$$

which, via (3.6) and (3.7), is equivalent to (3.3). And (3.2) follows from the Hölder inequality.

If instead we take the conditions of part (ii), then $0 < p < 1$ implies $0 < \beta < 1$, $-\infty < q < 0$ implies $0 < \gamma < 1$ and $1 < \lambda < \infty$ gives $0 \leq \alpha < 1$. Here the conditions of Theorem BBL2 part (ii) are satisfied. Thus we have the reverse inequality to (3.3) which is (3.5). Again (3.4) will follow but this time the reverse Hölder inequality, the second part of [6, Theorem 189], is required.

So far we have shown that all K and f that are allowed in Theorem B2 map to functions allowed in Theorem BBL2. The converse is also required. Given $k : \mathbb{R}^+ \to \mathbb{R}$, we construct $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ using $K(1,z) = z^{1/(p'\lambda)-1} |k|^{1/\lambda}(z)$ for all $z > 0$ and $K(x,y) = x^{-1} K(1, y/x)$ for all $x, y > 0$. Then K is positive and homogeneous of degree -1 and (3.6) holds. Also given $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, define $f(z) = z^{-1/p} \phi(z)$ so (3.7) holds.

In particular, the functions k and ϕ that give equality in Theorem BBL2 will map to functions giving equality in Theorem B2. These are the nonnull functions whose existence is asserted if $p \neq 1$ and $\lambda \neq 1$. □

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Theorem B2 as presented seems odd in that the domain of q is not contiguous. It is $1 \le q \le \infty$ in part (i) and $-\infty < q < 0$ in part (ii). However, in the context of extended reals, these intervals are connected at $q = \pm\infty$. The theorem would perhaps seem more natural if expressed in terms of q' which takes all values in $(0, \infty]$.

The question remains as to why Theorem B involves a strict inequality for nonnull f whereas Theorem B2 allows equality. If $p^{-1} + q^{-1} > 1$, then $\lambda < 1$ and $C_{1/\lambda}C_pC_q < 1$. Then the constant C^λ in Theorem B is greater than the best constant, and equality cannot apply in (1.2). If, however, $p^{-1} + q^{-1} = 1$ then $\lambda = 1$ and $C_{1/\lambda}C_pC_q = 1$, but Theorem B2 no longer asserts equality.

Finally, we consider special cases of Theorem B2. In the case where p and q are conjugate, as mentioned above, $\lambda = 1$ and $C_{1/\lambda}C_pC_q = 1$, and so part (i) of Theorem B2 immediately yields Theorem HLP319. Further, part (ii) will give the following result.

Suppose that $0 < p < 1$, $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *is homogeneous of degree* -1 *with*

$$\int_0^\infty x^{-1/p} K(x, 1)\, dx = \int_0^\infty x^{-1/q'} K(1, x)\, dx = C < \infty,$$

and f *and* g *are nonnegative functions on* R^+ *in* L_p *and* $L_{p'}$ *respectively. Then*

$$\int_0^\infty\int_0^\infty K(x, y) f(x) g(y)\, dx\, dy \ge C\, \|f\|_p \|g\|_{p'} \tag{3.8}$$

and

$$\left\| \int_0^\infty K(u, v) f(u)\, du \right\|_p \ge C\, \|f\|_p. \tag{3.9}$$

This is Theorem 335 of Hardy, Littlewood and Pólya [6], except for a subtle difference in the conditions. Inequality (3.9) here requires the finiteness of the right-hand side and implies that of the left, whereas this is reversed in Theorem 335. A similar but more elaborate difference occurs with (3.8).

The second special case of Theorem B2 is obtained using $K(x, y) = 1/(x + y)$. Here part (ii) does not apply since C is no longer finite. Part (i) immediately gives Theorem HLP340, with the constant A as $C_{1/\lambda}C_pC_qC^\lambda$. This is a tighter bound than that afforded by Levin's constant C^λ. Note, however, that determination of the best constant in Theorem B does not completely settle the question of the best constant in Theorem HLP340, since restriction of the kernel there may allow a smaller constant.

References

[1] W. Beckner, 'Inequalities in Fourier analysis', *Ann. of Math.* (2) **102** (1975), 159–182.

[2] F. F. Bonsall, 'Inequalities with non-conjugate parameters', *Q. J. Math.* **2** (1951), 135–150.

[3] H. J. Brascamp and E. H. Lieb, 'Best constants in Young's inequality, its converse, and its generalization to more than three functions', *Adv. Math.* **20** (1976), 151–173.

[4] A. Erdélyi, 'An extension of a Hardy–Littlewood–Polya inequality', *Proc. Edinb. Math. Soc.* (2) **21** (1978), 11–15.

[5] S. R. Finch, *Mathematical Constants* (Cambridge University Press, Cambridge, 2003).

[6] G. H. Hardy, J. E. Littlewood and G. Pólya, *Inequalities*, 2nd edn (Cambridge University Press, Cambridge, 1952).
[7] V. Levin, ‘On the two parameter extension and analogue of Hilbert’s inequality’, *J. Lond. Math. Soc.* (2) **11** (1936), 119–124.
[8] T. C. Peachey, ‘Some inequalities related to Hilbert’s’, *J. Inequal. Pure Appl. Math.* **4**(1) (2003), Art. 19.
[9] T. C. Peachey and C. M. Enticott, ‘Determination of the best constant in an inequality of Hardy, Littlewood, and Pólya’, *Exp. Math.* **15**(1) (2006), 43–50.
[10] W. H. Young, ‘On the multiplication of successions of Fourier constants’, *Proc. Roy. Soc.* A **87**(596) (1912), 331–339.

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