Plücker's first equation connecting the singularities of Curves.

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[The following notes are exactly as I left them in the hands of the Committee of the Society eleven years ago. They are printed now in the hope that, chiefly because of their brevity, they may be found useful to members, who may not have leisure or opportunity to read up the subject in the recognised text-books. —Jan., 1894. C. G. K.]

Let $U = f(x, y, z) = 0$ be the equation in trilinear co-ordinates of a curve of the n^{th} degree.

The number of terms is obviously

$$1 + 2 + 3 + \ldots\ldots + (n+1) = \tfrac{1}{2}(n+1)(n+2).$$

Hence $\frac{1}{2}(n+1)(n+2) - 1 = \frac{1}{2}n(n+3)$ points determine a general curve of the n^{th} degree.

According to Plücker, a curve may be considered as kinematically described by a point which moves along a line which continually rotates about that point.

This conception gives very simple notions as to the nature of singularities—double points, cusps, points of inflexion, double tangents, and so on.

Of these, only the cusp and point of inflexion are true singularities, the former being produced by a stationary point, the latter by a stationary line.

Still, a singular point may be more generally defined as a point which has descriptive properties not, in general, possessed by other points.

For example, a double point has two tangents, a triple point three, etc., a double tangent touches the curve in two distinct points, or, more strictly, meets the curve in two pairs of coincident points.

In the equation $\qquad U = f(xyz) = 0$

substitute for x the expression $\lambda x + \mu x_1$, for y, $\lambda y + \mu y_1$, and for z, $\lambda z + \mu z_1$. Then by Taylor's theorem U becomes

$$\lambda^n U + \lambda^{n-1}\mu\nabla U + \lambda^{n-2}\mu^2\frac{1}{\underline{|2}}\nabla^2 U + \ldots + \frac{\mu^n}{\underline{|n}}\nabla^n U$$

where $$\nabla = x_1\frac{\partial}{\partial x} + y_1\frac{\partial}{\partial y} + z_1\frac{\partial}{\partial z}.$$

This gives n values of λ/μ if equated to zero. That is, any arbitrary line cuts the curve in n points real or imaginary.

If the point xyz is on the curve as well as the point $(\lambda x + \mu x'$, etc.) then $U = 0$, and there are $(n-1)$ other points of intersection.

If, further, $\nabla U = 0$, then there are two coincident points $(\lambda/\mu = 0)$, and

$$\nabla U = x_1\frac{\partial U}{\partial x} + y_1\frac{\partial U}{\partial y} + z_1\frac{\partial U}{\partial z} = 0$$

is the equation of the tangent at the point (xyz), $x_1\ y_1\ z_1$ being any point on the tangent.

If the point $x_1y_1z_1$ is fixed, the equation $\nabla U = 0$ of degree $(n-1)$ in xyz represents the First Polar, a curve of the $(n-1)^{th}$ degree cutting the curve

$$U = 0$$

in $n(n-1)$ points, which include all the points to which tangents can be drawn from the points $x_1y_1z_1$.

The greatest number of tangents which can be drawn from any point to the curve, is called the *class* of the curve.

In the general curve of the n^{th} degree the class is $n(n-1)$; but if singularities exist the class is not so great.

But the equation $\nabla U = 0$ may be true whatever $x_1y_1z_1$ may be *i.e.*, if

$$\frac{\partial U}{\partial x} = 0, \quad \frac{\partial U}{\partial y} = 0, \quad \frac{\partial U}{\partial z} = 0 \text{ at } x, y, z.$$

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That is, any line through xyz cuts the curve in two coincident points, or in a double point.

Hence $\nabla U=0$ must pass through all the double points.

Or the First Polar passes through the double points of a curve.

Hence, if there are δ double points, since each double point counts for two, the other intersections cannot be greater than

$$n(n-1)-2\delta,$$

and this is of course a superior limit to the number of tangents than can be drawn from a given point to the curve.

Consider more closely the conditions

$$\frac{\partial U}{\partial x}=0, \quad \frac{\partial U}{\partial y}=0, \quad \frac{\partial U}{\partial z}=0.$$

Here there is, generally speaking, if $\nabla^2 U$ does not identically vanish, a node. The particular character of the node is got by considering the properties of $\nabla^2 U$.

If $\nabla^2 U=0$ as well as $U=0$, $\nabla U=0$, there are three coincident points.

For a double point the equation $\nabla^2 U=0$ gives two lines—two values of $x_1 y_1 z_1$.

In certain cases, however, these tangent lines coincide, namely, when

$\nabla^2 U$ is a complete square,

i.e., when

$$\frac{\partial^2 u}{\partial x^2}\,\frac{\partial^2 u}{\partial y^2}-\left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 = Z = 0$$

$$\frac{\partial^2 u}{\partial y^2}\,\frac{\partial^2 u}{\partial z^2}-\left(\frac{\partial^2 u}{\partial y\,\partial z}\right)^2 = X = 0$$

$$\frac{\partial^2 u}{\partial z^2}\,\frac{\partial^2 u}{\partial x^2}-\left(\frac{\partial^2 u}{\partial z\,\partial x}\right)^2 = Y = 0.$$

But in this case the tangent to any First Polar

$$\nabla U=0$$

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is given by the condition

$$\left(x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right) \nabla U = 0$$

independently of the values $x_1 y_1 z_1$ in ∇U

or $$\begin{vmatrix} \frac{\partial^2 U}{\partial x^2} & \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial x \partial z} \\ \frac{\partial^2 U}{\partial x \partial y} & \frac{\partial^2 U}{\partial y^2} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = 0 = H$$

must be satisfied. But evidently $X = 0$, $Y = 0$, $Z = 0$ satisfy this.

Hence at a cusp every first polar passes through the curve and has the same tangent with it—that is, meets the curve in three coincident points.

Hence if there are κ cusps, each cusp counts for three points, and the other points of intersection, or, what is the same thing, the *class* of the curve, is

$$n(n-1) - 2\delta - 3\kappa.$$

The equation $H = 0$ is called the Hessian. Its order is $3(n-2)$. It intersects the curve in $3n(n-2)$ points; and if there are no nodes or cusps these are points of inflexion of the original curve and the Hessian meets the curve in three coincident points.

Hence, generally, if ι = number of inflexions

$$\iota = 3n(n-2).$$

But if there are nodes and cusps, this number ι is reduced. The analytical condition for the Hessian is

$$U = 0, \quad \nabla U = 0, \quad \nabla^2 U = 0.$$

But for a double point U and ∇U vanish independently of $x_1 y_1 z_1$; and the polar conic $\nabla^2 U = 0$ reduces to the two tangents at the double point; hence with these conditions $x_1y_1z_1$ may be a point anywhere on either of the tangents. ∇U vanishes because

$$\frac{\partial U}{\partial x}=0 \quad \frac{\partial U}{\partial y}=0 \quad \frac{\partial U}{\partial z}=0$$

or

$$\left.\begin{array}{l} x\dfrac{\partial^2 U}{\partial x^2}+y\dfrac{\partial^2 U}{\partial x\partial y}+z\dfrac{\partial^2 U}{\partial x\partial z}=(n-1)\dfrac{\partial U}{\partial x}=0 \\ \quad\ldots \text{ etc. } \ldots \quad \ldots \quad \ldots \quad =0 \\ \quad\ldots \text{ etc. } \ldots \quad \ldots \quad \ldots \quad =0 \end{array}\right\}$$

Whence eliminating xyz

$$H=0, \text{ the Hessian.}$$

Or the Hessian also passes through all the double points.

But, the general condition of there being three coincident points in which H intersects U obviously requires that H must touch U there, must have the two tangents also two tangents, must itself have a double point there.

Hence a double point counts for 6 intersections between H and U.

For rigid proof, take the node as origin. Hence U contains no term in x and y lower than the second degree. Consider the lowest dimensions of $H=0$ in x and y, thus:

$$\begin{vmatrix} \dfrac{\partial^2 U}{\partial x^2} & \dfrac{\partial^2 U}{\partial x\partial y} & \dfrac{\partial^2 U}{\partial x\partial z} \\ \dfrac{\partial^2 U}{\partial x\partial y} & \dfrac{\partial^2 U}{\partial y^2} & \dfrac{\partial^2 U}{\partial y\partial z} \\ \dfrac{\partial^2 U}{\partial x\partial z} & \dfrac{\partial^2 U}{\partial y\partial z} & \dfrac{\partial^2 U}{\partial z^2} \end{vmatrix} \quad \text{is} \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix}$$

Hence the order of the lowest terms in x and y is 2. Therefore $H=0$ has also a double point at the origin.

Again, let the tangent at the origin be the line x. Then x will be a factor in the equation $U=0$ and must be present in the expressions $\dfrac{\partial^2 U}{\partial y^2}$, $\dfrac{\partial^2 U}{\partial z^2}$, $\dfrac{\partial^2 U}{\partial y\partial z}$, one of which is present in every term of $H=0$. Hence x is also a tangent to H.

Hence if there are δ nodes

$$\iota = 3n(n-2) - 6\delta.$$

Take now the case of the cusp. Take it as origin and let $x=0$ be the tangent. Then x^2 is a factor in $U=0$. Hence the lowest dimensions in x and y of H are

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{vmatrix}$$

Hence 3 is the order of the lowest term in $H=0$, and each term contains x^2 as a factor. Hence this point is a triple point on H formed by a simple branch passing through a cusp; and the coincident tangents coincide with the cuspidal tangent. But a triple point and a double point meet in 6 coincident points, and the common pair of tangents means other two points. Hence H cuts U at a cusp in 8 points.

Hence if there are δ nodes and κ cusps

$$\iota = 3n(n-2) - 6\delta - 8\kappa.$$

The possible double points to the curve are limited in number; they cannot exceed $\frac{1}{2}(n-1)(n-2)$. For, if possible, let there be $\frac{1}{2}(n-1)(n-2)+1$.

Then, these points together with $(n-3)$ other points will determine a curve of degree $(n-2)$ because these together give

$$\tfrac{1}{2}(n-2)(n+1) = \tfrac{1}{2}\{n-2\}\{n-2+3\}.$$

Hence this curve of degree $(n-2)$ intersects the curve of degree n in

$$\begin{aligned}(n-1)(n-2)+2+n-3 \text{ points} \\ =(n-1)^2 = n(n-2)+1\end{aligned}$$

which is impossible.

The number

$$D = \tfrac{1}{2}(n-1)(n-2) - \delta - \kappa$$

is called the "deficiency" of the curve. It is of great importance in the theory of transformations.

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The curve for which $D=0$ is unicursal.

The curve for which $D=1$ is bicursal, and so on.

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JOHN STURGEON MACKAY, Esq., M.A., President, in the Chair.

Some notes on Quaternions.

By CARGILL G. KNOTT, D.Sc., F.R.S.E.

Some theorems on Radical Axes.

By DAVID MUNN, F.R.S.E.
