Can. J. Math., Vol. XXIX, No. 2, 1977, pp. 384–391

RIGID EMBEDDING OF SIMPLE GROUPS IN THE GENERAL LINEAR GROUP

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1. Introduction. Let K be a (commutative) field and n be a positive integer. Consider the K-algebra $E = \text{Mat}\ (n, K)$ of all $n \times n$ matrices over K, and the corresponding general linear group $GL(n, K)$. We shall define the set R of *rigid mappings* of E to consist of all σ in $GL_K(E)$ which can be written in one of two possible forms: either $x^\sigma = axb$ for all $x \in E$ or $x^\sigma = ax'b$ for all $x \in E$ (where a and b are fixed elements of $GL(n, K)$ and x' denotes the transpose of x). It is readily seen that R is a subgroup of $GL_K(E)$ with the structure of a wreath product $GL(n, K)\text{Wr}C_2$ where C_2 is a group of order 2 generated by the transposition mapping. Geometrically, R may be thought of as generated by "translations" $x \mapsto ax$ and $x \mapsto xb$ (a, $b \in GL(n, K)$) and a "reflection" $x \mapsto x'$.

For each subset S of E we define Fix (S) to be the set of all σ in $GL_K(E)$ such that $S^\sigma \subseteq S$. In general, $\text{Fix}(S)$ need not be a subgroup of $GL_K(E)$ because $S^\sigma \subseteq S$ need not imply $S^\sigma = S$. However, there is a long series of results by various authors (going back as far as Frobenius) which shows that for many "natural" choices of S, $\text{Fix}(S)$ is not only a subgroup of $GL_K(E)$ but indeed a subgroup of R. For example, such a theorem holds in the following cases: (i) when S consists of all elements of E with determinant 0 (see [**3**]); (ii) when K is infinite and S consists of all elements of E of rank r (for fixed $r \geqq 1$) (see [**5**]); (iii) when $K = \mathbf{R}$ (the field of real numbers) and S is either the full orthogonal group or the symplectic group in $GL(n, \mathbf{R})$ (see [**11**] and [**10**] respectively). A survey of further related results may be found in [**9**]. However, the survey [**9**] shows that although there are many results of this type known, the proofs seem special in each case. (In some situations the full set of σ in $\text{End}_K(E)$ such that $S^\sigma \subseteq S$ has been described, but consideration of singular linear transformations σ generally gives rise to series of exceptional cases.) It may be noted that the sets S for which theorems of the above type have been proved are all algebraic sets or quasi-algebraic (such as the unitary group in $GL(n, \mathbf{C})$ considered in [**8**]), and overall the scattered results suggest that for each "sufficiently twisted" algebraic subset S of E, $\text{Fix}(S)$ will be a subgroup of R. The main result of the present paper (see the Theorem in § 2) gives further support to this vague conjecture by showing that $\text{Fix}(G)$ is a

Received May 26, 1976. This research was supported in part by the National Research Council of Canada under Grants No. A7171 and T0361. The author also wishes to acknowledge the kind hospitality of the Institut des Hautes Etudes Scientifiques, Bures-sur-Yvette, during the preparation of this paper.

subgroup of R for a large class of irreducible algebraic subgroups G. In particular, the major parts of [**10**] and [**11**] are implied by this result. We also prove that Fix(S) is always a subgroup of $GL_K(E)$ when S is an algebraic subset of E (Lemma 1), and a further result on the relative independence of Fix(S) under field extensions (Lemma 2). The former of these two results would simplify the arguments of a number of papers of previous authors.

2. Statement of the theorem. When K is a subfield of the complex field $\mathbf{C}$, $E = \text{Mat}(n, K)$ is endowed with both the Zariski (algebraic) topology and the analytic topology which is induced from the usual topology of $\mathbf{C}$. Let G be an algebraic subgroup of $GL(n, K)$ (that is, a subgroup closed in the Zariski topology). Then G is a simple algebraic subgroup if it is nonabelian and connected (in the Zariski topology) and it has no normal closed connected subgroups different from 1 and G. Similarly, a Lie subgroup of $GL(n, \mathbf{C})$ is a subgroup G which is closed and connected in the analytic topology; and G is a simple Lie subgroup if in addition it is nonabelian and its only normal Lie subgroups are 1 and G.

Note. For general references see [**7**] (especially p. 168) and [**2**]. An important theorem of Chevalley and Tuan [**1**] shows that an irreducible subgroup of $GL(n, \mathbf{C})$ is a simple algebraic group if and only if it is a simple Lie group. Observe that such a group need not be simple in the usual group theoretic sense, but may have a finite nontrivial centre; however modulo its centre it is simple in the usual sense. For example, the special linear group $SL(n, \mathbf{C})$ is a simple algebraic subgroup of $GL(n, \mathbf{C})$ for each $n \geqq 2$.

Now let K be any subfield of C, and let G be an algebraic subgroup of $GL(n, K)$. We shall say that G is a *classical simple group* if, when we consider G as a subgroup of $GL(n, \mathbf{C})$, the Zariski-closure of G in $GL(n, \mathbf{C})$ is a simple algebraic group. For example, the symplectic group $Sp(n, \mathbf{C})$ (for n even) and the special orthogonal group $SO(n, \mathbf{C})$ consisting of proper rotations (for $n \geqq 3$) are both simple algebraic groups. In both cases they can be defined as the set of zeros of a family of polynomials with rational coefficients, and so for each subfield K of $\mathbf{C}$ there are classical simple groups in $GL(n, K)$ whose closures in $GL(n, \mathbf{C})$ are $Sp(n, \mathbf{C})$ and $SO(n, \mathbf{C})$, respectively.

THEOREM. *Let K be a subfield of* $\mathbf{C}$, *and let G be an absolutely irreducible classical simple group in $GL(n, K)$ for some $n \geqq 2$. Then* Fix(G) *is a subgroup of the group R of rigid mappings except possibly in the following cases:*

(a) *when $n = 4$ and G is of type A_1;*
(b) *when $n = 2^m$ and G is of type B_m* $(m \geqq 2)$.

Remarks. 1) The types A_1 and B_m refer to the classification (up to isomorphism) of simple algebraic groups in terms of Lie algebras (see [**7**, p. 215]). It is not known whether the cases (a) and (b) do actually yield exceptions to the theorem.

2) The condition that G should be absolutely irreducible is clearly necessary. Otherwise G would span a proper K-subspace E_1, say, of E and we could write the latter as a direct sum $E = E_1 \oplus E_2$ of nonzero K-spaces. But in the latter case there are clearly elements of $\text{Fix}(G)$ which do not lie in R (for example, some whose restriction to E_1 is the identity).

3) With a suitable reformulation and using Lemma 2 we could extend the theorem to cover fields which contain $\mathbf{C}$ and hence to arbitrary fields of characteristic 0.

Examples. 1) Let $G = Sp(2n, K)$ be the symplectic group over any subfield K of $\mathbf{C}$. As we noted above, G is a classical simple group and it is well known that G is absolutely irreducible for all $n \geqq 1$. Since G is a simple algebraic group of type C_n, it is not in the class of possible exceptions, and so the theorem shows that $\text{Fix}(G)$ is a subgroup of R (compare with [**10**]).

2) Let $G = SO(n, K)$ be the special orthogonal group over some subfield K of $\mathbf{C}$ ($n \geqq 3$). Again G is a classical simple group which is absolutely irreducible, and of type B_m when $n = 2m + 1$ and of type D_m if $n = 2m$. Thus again the theorem shows that $\text{Fix}(G) \subseteq R$. Similarly, let $H = O(n, K)$ be the full orthogonal group, so G is the connected component of the identity in H. Suppose that $\sigma \in \text{Fix}(H)$ and $1^\sigma = a$, say. Define ρ to be the rigid mapping $x \mapsto xa^{-1}$. Then $\sigma\rho \in \text{Fix}(H)$ and $1^{\sigma\rho} = 1$. Since $\sigma\rho$ is continuous, it follows that $\sigma\rho \in \text{Fix}(G)$. Hence $\sigma \in R\rho^{-1} = R$ and we conclude that $\text{Fix}(H) \subseteq R$ (compare with [**11**]).

3. Some general lemmas. Let K be an arbitrary field, and let K^d denote the K-space of all d-tuples over K; the group $GL(d, K)$ acts on K^d from the right by matrix multiplication. Let $K[X]$ denote the ring of polynomials in d indeterminates $X = (X_1, \ldots, X_d)$ over K. For each σ in $GL(d, K)$ define $X\sigma = (X_1, \ldots, X_d)\sigma$ as a matrix product, and for each $f(X) \in K[X]$ define $f^\sigma(X) \in K[X]$ by $f^\sigma(X) = f(X\sigma^{-1})$. Then the group $GL(d, K)$ acts on the K-space $K[X]$. To each subset S of K^d we associate the annihilating ideal $\text{Ideal}(S)$ of $K[X]$ consisting of all $f(X)$ such that $f(x) = 0$ for all $x \in S$. Thus S is an algebraic set if and only if the condition $f(x) = 0$ for all $f(X) \in \text{Ideal}(S)$ implies that $x \in S$.

LEMMA 1. *Let S be an algebraic subset of K^d and put $I = \text{Ideal}(S)$. Let $\sigma \in GL(d, K)$. Then the following are equivalent:*

(i) $S^\sigma \subseteq S$;
(ii) $I^{\sigma^{-1}} \subseteq I$;
(iii) $I^{\sigma^{-1}} = I$;
(iv) $S^\sigma = S$.

In particular, the implication from (i) *to* (iv) *shows that* $\text{Fix}(S) = \{\sigma \in GL(d, K) | S^\sigma \subseteq S\}$ *is a subgroup of* $GL(d, K)$.

Proof. First note that $S^\sigma \subseteq S$ if and only if $f^{\sigma^{-1}}(x) = f(x^\sigma) = 0$ for all

$f(X) \in I$ and all $x \in S$; hence $S^\sigma \subseteq S$ if and only if $I^{\sigma^{-1}} \subseteq I$. Thus (i) implies (ii), and (iii) and (iv) are equivalent. Since (iv) clearly implies (i), the proof will be complete when we have shown that (ii) implies (iii).

Suppose that (ii) holds. For each $j \geqq 1$ let I_j be the K-subspace of I consisting of all polynomials of degree at most j. The linearity of σ^{-1} shows that $I_j^{\sigma^{-1}} \subseteq I_j$ for each j. Since σ^{-1} is injective, and I_j is finite-dimensional over K, we conclude that the restriction of σ^{-1} to I_j is bijective. Since I is the union of the I_j, σ^{-1} is therefore surjective on I and (iii) is proved. This completes the proof of the lemma.

Our next lemma deals with the question of what happens under field extensions.

LEMMA 2. *Let k be a subfield of K and let S be an algebraic subset of k^d. Put $I = \text{Ideal}(S) \subseteq k[X]$ and define $J = IK$. Let T be the algebraic subset of K^d consisting of the common zeros of the polynomials in J. Suppose that $\sigma \in GL(d, k) \subseteq GL(d, K)$. Then $S^\sigma \subseteq S$ if and only if $T^\sigma \subseteq T$.*

Proof. First suppose that $S^\sigma \subseteq S$. Then, by Lemma 1, $I^{\sigma^{-1}} = I \subseteq J$. Hence for all $x \in T$ and all $f(X) \in I$ we have $f(x^\sigma) = f^{\sigma^{-1}}(x) = 0$. Since I forms an ideal basis for J this proves that $x^\sigma \in T$ for all $x \in T$. Hence $T^\sigma \subseteq T$.

Conversely, suppose that $T^\sigma \subseteq T$. We begin by considering the special case where K is algebraically closed. In this case Hilbert's Nullstellensatz shows that Ideal(T) consists of all $f(X) \in K[X]$ such that for some integer $m \geqq 1$, $f(X)^m \in J$ (see [**6**, p. 5]). In particular, if $f(X) \in I$, then $f(X) \in \text{Ideal}(T)$, and so $f^{\sigma^{-1}}(X) \in \text{Ideal}(T)$ by Lemma 1. Hence for some integer $m \geqq 1$, $\{f^{\sigma^{-1}}(X)\}^m \in J \cap k[X] = I$. Since $I = \text{Ideal}(S)$, this implies that $f^{\sigma^{-1}}(X) \in I$. Thus we conclude that $I^{\sigma^{-1}} \subseteq I$, and so that $S^\sigma \subseteq S$ by Lemma 1. This completes the proof of the lemma in the special case where K is algebraically closed. To complete the proof in the general case we apply this special case to the two situations k and $\bar{K}$, and K and $\bar{K}$, where $\bar{K}$ denotes an algebraic closure of K. Then the general form of the lemma follows.

Now let $n_1, \ldots, n_r$ be positive integers and consider the (outer) tensor product $GL(n_1, K) \otimes \ldots \otimes GL(n_r, K)$. The latter group can be embedded in a natural way as a subgroup of $GL(n, K)$ where $n = n_1 \ldots n_r$, and this embedding is uniquely determined to within conjugacy in $GL(n, K)$. Moreover, if G_i is an algebraic (or Lie) subgroup of $GL(n_i, K)$ for $i = 1, \ldots, r$, then the same is true for the image of $G_1 \otimes \ldots \otimes G_r$ under this embedding. In the special case where $E = \text{Mat}(n, K)$ we shall identify $GL(n, K) \otimes GL(n, K)$ with a subgroup of $GL_K(E)$ by defining $x^{a \otimes b} = a'xb$ for all $a, b \in GL(n, K)$ and $x \in E$.

LEMMA 3. *Let G_1 and G_2 be simple Lie subgroups of $GL(n_1, \mathbf{C})$ and $GL(n_2, \mathbf{C})$, respectively. Then the only normal Lie subgroups of $G_1 \otimes G_2$ are $1, 1 \otimes G_2, G_1 \otimes 1$ and $G_1 \otimes G_2$.*

Remark. A similar proof would show that an analogous result holds for simple algebraic groups. The corresponding theorem for discrete groups is also well known.

Proof. It is readily seen that each of the four subgroups listed is a normal Lie subgroup of $G_1 \otimes G_2$. We also note that the simplicity of G_i shows that the connected component of the identity in the centre $Z(G_i)$ equals 1 for $i = 1, 2$. This latter shows that, if N is a normal subgroup of $G_1 \otimes G_2$ different from 1, $1 \otimes G_2$ and $G_1 \otimes 1$, then N must contain an element $a \otimes b$ with $a \notin Z(G_1)$ and $b \notin Z(G_2)$. Consider the closed normal subgroup H of G_1 defined by $H \otimes 1 = N \cap (G_1 \otimes 1)$. For each $c \in G_1$ we have

$$c^{-1}a^{-1}ca \otimes 1 = (c \otimes 1)^{-1}(a \otimes b)^{-1}(c \otimes 1)(a \otimes b) \in N$$

and so $c^{-1}a^{-1}ca \in H$. Since $a \notin Z(G_1)$ and $c \mapsto c^{-1}a^{-1}ca$ is a continuous mapping of G_1 into H, H contains a nontrivial connected subset. Since H is normal in G_1, the connected component of the identity in H is therefore a nontrivial normal Lie subgroup of G_1. The simplicity of G_1 shows that $H = G_1$, and hence we have $N \cap (G_1 \otimes 1) = G_1 \otimes 1$. Thus N contains $G_1 \otimes 1$, and a similar argument shows that it also contains $1 \otimes G_2$. This proves that $N = G_1 \otimes G_2$ as required.

4. Proof of the theorem. First note that it follows from Lemma 2 that it is enough to prove the theorem in the case when $K = \mathbf{C}$. Then by the note in § 2 we can restate the theorem in terms of simple Lie subgroups of $GL(n, \mathbf{C})$; and since the groups are simple we are in fact dealing with subgroups of $SL(n, \mathbf{C})$.

The key step in the proof is based on a deep classification by Dynkin of the (connected) semisimple Lie subgroups of $SL(n, \mathbf{C})$ and their inclusions (see [**6**]). We should observe that Dynkin himself suggests in [**6**, p. 249] that his results can be used to prove theorems of this kind, but he does not seem to have ever given details of such an application. To clarify the proof of our theorem we shall summarize the three results from [**6**] which we shall use.

(A) (Theorem 2.1 of [**6**]). Each irreducible Lie subgroup H of $SL(n, \mathbf{C})$ is conjugate to a subgroup of the form $H_1 \otimes \ldots \otimes H_r$ for some $r \geqq 1$ where each H_i is an irreducible simple Lie subgroup of $SL(n_i, \mathbf{C})$ for certain integers $n_1, \ldots, n_r$ with $n_1 \ldots n_r = n$.

(B) (Theorem 2.2 of [**6**]). If H has a decomposition $H_1 \otimes \ldots \otimes H_r$ as described in (A), and H^* is an irreducible Lie subgroup of H, then there exist irreducible Lie subgroups H_i^* of H_i ($i = 1, \ldots, r$) such that $H^* = H_1^* \otimes \ldots \otimes H_r^*$.

(C) (Theorem 2.3 of [**6**]). Table 5 of [**6**] gives a complete list of all inclusion types $H^* \subset H \subset SL(n, \mathbf{C})$ such that H and H^* are distinct irreducible Lie subgroups and H is a simple Lie group which is not conjugate in $GL(n, \mathbf{C})$ to any of $SL(n, \mathbf{C})$, $SO(n, \mathbf{C})$ or $Sp(n, \mathbf{C})$.

Remarks. The term *inclusion type* is defined as follows. Two pairs of subgroups $A^* \subset A$ and $B^* \subset B$ of $SL(n, \mathbf{C})$ are of the same inclusion type if both A is conjugate to B and A^* is conjugate to B^* (in $GL(n, \mathbf{C})$). Table 5 of [**6**] lists 36 families or isolated cases of exceptional pairs of subgroups.

Now suppose that G satisfies the hypotheses of the theorem (and $K = \mathbf{C}$). Define G_0 as the subgroup of $SL(n, \mathbf{C})$ consisting of all elements whose transpose lies in G. Note that G_0 is isomorphic to G under the mapping $x \mapsto (x')^{-1}$, and so G_0 is also an irreducible simple Lie subgroup of $SL(n, \mathbf{C})$. Put $H^* = G_0 \otimes G$ where H^* is embedded in $SL_{\mathbf{C}}(E) \simeq SL(n^2, \mathbf{C})$ as described in § 3. Since G is absolutely irreducible, G spans the $\mathbf{C}$-space E by Burnside's Theorem (see [**4**, p. 36]), and so H^* is irreducible. Since G is a Lie group, H^* is a Lie subgroup of $SL_{\mathbf{C}}(E)$, and evidently $H^* \subseteq \mathrm{Fix}(G)$.

On the other hand, by Lemma 1, $\mathrm{Fix}(G)$ is a subgroup of $GL_{\mathbf{C}}(E)$, and it is clearly closed in the analytic topology. Therefore, its connected component of the identity, say H, is a Lie subgroup of $SL_{\mathbf{C}}(E)$, and indeed the largest Lie subgroup which is contained in $\mathrm{Fix}(G)$. In particular, $H^* \subseteq H$ and H is irreducible because H^* is. We shall now show that $H^* = H$ except in a few specified cases.

First suppose that H is not a simple Lie group. Since $H \supseteq H^* = G_0 \otimes G$, it follows from (A), (B) and Lemma 3 that there exist irreducible Lie subgroups H_1 and H_2 of $SL(n, \mathbf{C})$ containing G_0 and G, respectively, such that $H = H_1 \otimes H_2$. However, $H \subseteq \mathrm{Fix}(G)$, and so $G^{a \otimes b} = a'Gb \subseteq G$ for all $a \in H_1$ and $b \in H_2$. Therefore, $H_1 = G_0$ and $H_2 = G$, and $H = H^*$ as claimed.

Secondly, consider the situation where H is a simple Lie group. We begin by showing that H does not contain any of the symplectic groups or special orthogonal groups of degree n^2 in $GL_{\mathbf{C}}(E)$. In fact, to each of these latter groups there is associated a nondegenerate quadratic form $q: E \rightarrow \mathbf{C}$ such that $\sigma \in GL_{\mathbf{C}}(E)$ lies in the group if and only if $\sigma \in SL_{\mathbf{C}}(E)$ and $q \circ \sigma: E \rightarrow \mathbf{C}$ is a nonzero scalar multiple of q; in suitable coordinates q is the standard alternating form or the standard diagonal form in the two cases (the alternating form only arises when n is even). Now suppose that H contained one of these groups. Since G spans E over $\mathbf{C}$, there exists $a \in G$ such that $q(a) \neq 0$. Then for each $b \in E$ with $q(b) \neq 0$ there is a nonzero $\lambda \in \mathbf{C}$ such that $q(b) = q(\lambda a)$, and so by Witt's theorem there exists $\sigma \in GL_{\mathbf{C}}(E)$ such that $q \circ \sigma = q$ and $a^\sigma = b$. Multiplying by a suitable scalar we obtain $\rho \in SL_{\mathbf{C}}(E)$ such that $\mu a^\rho = b$ for some nonzero $\mu \in \mathbf{C}$ and $q \circ \rho$ is a scalar multiple of q. Thus, by hypothesis, $\rho \in H \subseteq \mathrm{Fix}(G)$ and so $b = \mu a^\rho \in \mathbf{C}^*.G$ where $\mathbf{C}^*$ is the group of nonzero scalars; in particular, the determinant det b is nonzero. Thus we have shown that $\det b = 0$ implies that $q(b) = 0$. If we now apply this to various cases where b is chosen as a matrix with at most two nonzero entries, and use the nondegeneracy of q, then it is clear that: (i) $n = 2$; (ii) q has the form

$$q\left(\begin{bmatrix} \xi_1 & \xi_2 \\ \xi_3 & \xi_4 \end{bmatrix}\right) = \alpha\xi_1\xi_4 + \beta\xi_2\xi_3$$

for some $\alpha, \beta \in \mathbf{C}$; and so (taking all ξ_i equal) (iii) $\alpha = -\beta \neq 0$. But it is now clear that all rigid mappings of the form $x \mapsto xc$ ($c \in SL(2, \mathbf{C})$) leave q invariant and lie in $SL_{\mathbf{C}}(E)$ and so belong to Fix(G). This immediately implies that $G = SL(2, \mathbf{C})$; hence Fix(G) = Fix$(GL(2, \mathbf{C}))$ = Fix$(E\backslash GL(2, \mathbf{C}))$ by Lemma 1, and so Fix$(G) = R$ by [**3**]. This is impossible because the connected component H of the identity of Fix(G) is simple. Thus we have shown that H contains no symplectic or special orthogonal group of order n^2 in $GL_{\mathbf{C}}(E)$. It now follows from (C), Table 5 of [**6**] and the particular form $G_0 \otimes G$ of H^* (recall that $G_0 \cong G$) that $H^* = H$ unless: either G is of type A_1, H is of type D_5 and $n^2 = 2^4$; or G is of type B_m, H is of type D_{2m+1} and $n^2 = 2^{2m}$ ($m = 2, 3, \ldots$).

To complete the proof of the theorem it remains to show that $H = H^*$ implies that Fix$(G) \subseteq R$. Since H is the connected component of the identity in Fix(G), it is normal in Fix(G). Therefore it is enough to show that each $\sigma \in GL_{\mathbf{C}}(E)$ which normalizes $H^* = G_0 \otimes G$ lies in R.

Thus suppose that σ normalizes $G_0 \otimes G$. Then $\sigma^{-1}(1 \otimes G)\sigma$ is a normal Lie subgroup of $G_0 \otimes G$, so Lemma 3 shows that it must equal either $1 \otimes G$ or $G_0 \otimes 1$. Now if $\sigma^{-1}(1 \otimes G)\sigma = G_0 \otimes 1$, then $(\sigma\tau)^{-1}(1 \otimes G)(\sigma\tau) = 1 \otimes G$ where τ is the transposition mapping. Since $\sigma \in R$ if and only if $\sigma\tau \in R$, this shows that it is enough to consider the case where $\sigma^{-1}(1 \otimes G)\sigma = 1 \otimes G$; and in this case we must also have $\sigma^{-1}(G_0 \otimes 1)\sigma = G_0 \otimes 1$ by Lemma 3. Thus conjugacy under σ defines two group automorphisms α and β of G_0 and G, respectively, satisfying the condition

$$x^\alpha \otimes y^\beta = \sigma^{-1}(x \otimes y)\sigma \quad \text{for all } x \in G_0 \text{ and } y \in G.$$

In particular, taking traces we find

$$n \operatorname{tr} x^\alpha = \operatorname{tr}(x^\alpha \otimes 1) = \operatorname{tr} \sigma^{-1}(x \otimes 1)\sigma = \operatorname{tr}(x \otimes 1) = n \operatorname{tr} x$$

and so $\operatorname{tr} x^\alpha = \operatorname{tr} x$ for all $x \in G_0$. Now α and the natural embedding of G_0 into $GL(n, \mathbf{C})$ can both be considered as irreducible representations of G_0 into $GL(n, \mathbf{C})$, and from what we have just seen they both afford the same character. Therefore by the Frobenius-Schur theorem (see [**4**, p. 33]) these two representations are equivalent, and so there exists $a \in GL(n, \mathbf{C})$ such that $x^\alpha = a^{-1}xa$ for all $x \in G_0$. Similarly there exists $b \in GL(n, \mathbf{C})$ such that $y^\beta = b^{-1}yb$ for all $y \in G$. Hence

$$\sigma^{-1}(x \otimes y)\sigma = (a \otimes b)^{-1}(x \otimes y)(a \otimes b) \quad \text{for all } x \otimes y \in G_0 \otimes G.$$

Since $G_0 \otimes G$ is irreducible, Schur's lemma shows that there exists a scalar λ such that $\sigma(a \otimes b)^{-1} = \lambda$. Thus $\sigma = \lambda a \otimes b$ lies in R and the theorem is proved.

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