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THE TOPOLOGY OF GRAPH PRODUCTS OF GROUPS

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Given a finite (connected) simplicial graph with groups assigned to the vertices, the graph product of the vertex groups is the free product modulo the relation that adjacent groups commute. The graph product of finitely presented infinite groups is both semistable at infinity and quasi-simply filtrated. Explicit bounds for the isoperimetric inequality and isodiametric inequality for graph products is given, based on isoperimetric and isodiametric inequalities for the vertex groups.

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0. Introduction

In a simplicial graph we say two vertices are *adjacent* if they are joined by a single edge. Given a finite simplical graph $\mathcal{G}$ and groups associated to its vertices, the *graph product* of the vertex groups, $G\mathcal{G}$, is the free product of the vertex groups with added relations implying that elements of adjacent vertex groups commute. It is clear that the union of the generators of the vertex groups generates the graph product, and we will call this union the *standard generators.*

Graph products have been actively studied by mathematicians and computer scientists. (See [**1**], [**5**], [**9**] and the references cited there.) If all the vertex groups are free of rank one they are referred to as "graph groups" and when they are simply free monoids at each vertex, "free partially commuting monoids".

By [**8**] it is clear that if Δ is any full subgraph of $\mathcal{G}$, then the natural map $G\Delta \hookrightarrow G\mathcal{G}$ is a monomorphism. In fact, it is shown in [**5**] and could be derived from the normal forms in [**9**] that this is an isometric embedding of $G\Delta$ into $G\mathcal{G}$. That is, the word length metric on $G\Delta$ using the standard generators for $G\Delta$ is the restriction of the word length metric on $G\Delta$ using the standard generators for $G\mathcal{G}$.

Chose a single vertex v_1 in $\mathcal{G}$ and denote its associated group by A_1. Let Z be the full subgraph generated by the $n-1$ other vertices of $\mathcal{G}$ and denote by Δ the full subgraph generated by vertices adjacent to v_1. By [**8**] we then get the following decomposition:

$$G\mathcal{G}=(A_1\times G\Delta)*_{G\Delta}GZ.$$

Using this decomposition one can prove that any property which is preserved by taking direct sums and free products with amalgamation is preserved by taking graph products.

We use this basic idea to study topological properties associated to groups. One direction is to study the topoplogy at the "end(s) of a group". In Section 2 we analyse the notions of being semistable at infinity and quasi-simplified filtrated. Here there is the intriguing twist that these properties can be "created" by taking direct sums, hence if $\mathcal{G}$ contains no isolated vertex, the graph product can "create" these properties.

In Section 3 we study the topological interpretation of the word problem in terms of van Kampen diagrams. Specifically we show that the isoperimetric inequalities (as defined in [7]) and isodiametric inequalities satisfied by the vertex groups give a simple bound for isoperimetric and isodiametric inequalities for the graph product. In the isoperimetric case this overlaps recent independent work of D. E. Cohen.

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All groups in this paper are assumed to be finitely presented.

1. Semistability and quasi-simply filtrated

This section begins with a collection of various definitions and results about the topology at infinity of groups, and does not attempt to motivate or prove any of this material. The definitions are taken directly from the papers of M. Mihalik and others in the references; consult these papers for the background, motivation and main results about semistability and quasi-simple filtration. For technical reasons we assume that all our CW-complexes have piecewise linear attaching maps.

Definitions. Given a CW-complex $\tilde{X}$, two proper rays $r, s\colon [0,\infty) \to \tilde{X}$ *converge to the same end of* $\tilde{X}$ if for any compact set $C \subset \tilde{X}$ there is an integer N such that $r([N,\infty))$ and $s([N,\infty))$ are in the same component of $\tilde{X}\backslash C$. The space $\tilde{X}$ is *semistable at infinity* if there is a proper homotopy between any two proper rays which converge to the same end.

A finitely presented group G is *semistable at infinity* if the universal cover of any finite CW-complex X with $\pi_1(X) = G$ is semistable at infinity. This property is independent of the choice of X and hence independent of the choice of finite presentation for G.

The notion of a space being quasi-simply filtrated grew out of work of Casson's condition C_2 for fundamental groups of 3-manifolds.

Definitions. If X is a finite CW-complex, then X is *quasi-simply filtrated* (abbreviated QSF) if given any connected finite subcomplex $C \subset \tilde{X}$ there is a simply connected finite complex D and a cellular map $D \xrightarrow{f} \tilde{X}$ such that the restriction of f to $f^{-1}(C)$ is a homeomorphism. Intuitively, a finite complex X is QSF if finite subcomplexes in its universal cover can be approximated by finite simply connected complexes.

A finitely presented group G is *quasi-simply filtrated* if any finite CW-complex X with $\pi_1(X) = G$ is QSF. This also is independent of choice of X and hence independent of choice of finite presentation.

The central theorems we will use hold for both semistable at infinity and QSF groups hence we combine their statements, even though their proofs are quite different.

In the case of semistability, the following theorem is proved in [**11**] and the proof for the QSF case is in [**4**].

Theorem 1.1. *If* $1 \to H \to G \to L \to 1$ *is a short exact sequence of finitely presented infinite groups, then* G *is* 1*-ended and semistable at infinity* (*QSF*).

We will only make use of Theorem 1.1 in the case where G is the direct sum of two finitely presented infinite groups.

The proofs of the following theorem are in [**12**] (semistability) and [**3**] (QSF).

Theorem 1.2. *If* G *and* H *are finitely presented semistable at infinity groups* (*QSF groups*) *and* C *is any finitely generated subgroup, then* $G *_C H$ *is semistable at infinity* (*QSF*).

Proposition 1.3 below follows immediately from the previous two theorems using the decomposition of graph products into free products with amalgamation.

Proposition 1.3. *The graph product of semistable at infinity* (*QSF*) *groups is semistable at infinity* (*QSF*). □

This result can be significantly strengthened using Theorem 1.1. Theorem 1.4 has essentially the same proof in either the semistable at infinity case or the QSF case, hence we give only the argument for the semistable case. The reader wanting to read the proof for quasi-simply filtration should substitute the phrase "quasi-simply filtrated" everywhere "semistable at infinity" occurs.

Theorem 1.4. *For a connected, non-trivial graph, the graph product of finitely presented infinite groups is semistable at infinity* (*QSF*).

Proof. The proof is by induction on the number of vertices of the underlying graph $\mathscr{G}$. Because the graph is connected, we may begin with a graph composed of a single edge, and the result follows from Theorem 1.1.

Assume the theorem is true for graph products of finitely presented infinite groups based on connected graphs with $n-1$ vertices. Then using the notation from the introduction, the graph product on n vertices can be expressed as

$$G\mathscr{G} = (A_1 \times G\Delta) *_{G\Delta} GZ$$

where the vertex v_1 may be chosen so that the subgraph Z is connected.

Since A_1 and $G\Delta$ are finitely presented infinite groups, $A_1 \times G\Delta$ is semistable at infinity by Theorem 1.1. By the induction hypothesis GZ is semistable at infinity so by Theorem 1.2 $G\mathscr{G}$ is semistable at infinity. □

Corollary 1.5. *If* $\mathscr{G}$ *is a finite simplicial graph with no isolated vertex, then the graph* *product of finitely presented infinite groups based on the graph $\mathcal{G}$ is semistable at infinity (QSF).*

Proof. Since there is no isolated vertex, each connected component of $\mathcal{G}$ satisfies the hypotheses of Theorem 1.4, hence the graph product based on each connected component is semistable at infinity (QSF). Since the graph product based on $\mathcal{G}$ is the free product of the graph products of each connected component of $\mathcal{G}$, the corollary follows from Theorem 1.2. $\square$

2. Isoperimetric and isodiametric inequalities

Assume there is a fixed finite presentation for a given group. If a word in the generators of a group represents the trivial element, then it can be expressed as the product of conjugates of the relations. This can be interpreted as a 1-connected planar 2-complex with labeled edges, such that the boundary path reads off the original word, and the path about any cell gives a relation. Such a diagram is called a *van Kampen diagram.* (See **[10]** for more details.) Isoperimetric and isodiametric inequalities provide bounds on the complexity of the word problem in terms of van Kampen diagrams.

If ω is a null-homotopic word, let $\Delta(\omega)$ be minimal number 2-cells in a van Kampen diagram for ω. A *Dehn function* (or an *isoperimetric function*) for a given presentation is a map $D(n)$ from the natural numbers to the natural numbers such that for any set of null-homotopic words $\{\omega_i\}$, whose lengths sum to less than or equal to n, $\sum \Delta(\omega_i) \leqq D(n)$. Thus if ω is a null-homotopic word of length n it can be expressed as the product of less than $D(n)$ conjugates of the relators.

This definition of Dehn functions is slightly non-standard (although it is the one used in **[7]**) in that we work with sets of words instead of a single null-homotopic word. We do this to avoid using Brick's notion of the "subnegative closure"; using the more standard definition as in **[2]** would require the insertion of "subnegative closure" into the statements of the results in this section.

Dehn functions bound the area of van Kampen diagrams in terms of their perimeter. Similarly one can bound the diameter of van Kampen diagrams in terms of their perimeter.

If ω is a null-homotopic word, let $\imath(\omega)$ be the minimal maximum distance between vertices in a van Kampen diagram for ω, where the minimum is taken over all van Kampen diagrams for ω. An *isodiametric function* for a presentation is a function $I(n)$ from the natural numbers to the natural numbers, such that for the any set of words $\{\omega_i\}$ whose lengths sum to less than or equal to n, $\sum \imath(\omega_i) \leqq I(n)$.

We will want to give rough comparisons of the Dehn and isoperimetric functions, hence we introduce a partial ordering on functions. We say that $f(x) \preceq g(x)$ if there are positive integers A, B and C such that $f(n) \leqq Ag(Bn) + Cn$ for all $n > 0$. Two functions are *equivalent*, $f \simeq g$, if $f \preceq g$ and $g \preceq f$. Any two polynomials of the same degree are equivalent under this definition, as are any two exponential functions k^x and l^x as long as $k, l > 1$. Any two finite presentations of a group yield equivalent Dehn and isodiametric functions (see Proposition 2.4 of [7]).

We say a function $f(n)$ is of *degree* d if $f(n) \preceq n^d$. Notice that because of the linear term in our equivalence relation, the degree of a function is minimally 1.

The following bounds (Propositions 2.1, 2.2 and 2.3) for Dehn functions for products of groups were given by Brick in [2]. Essentially the same arguments can be used to establish the bounds for isodiametric functions which we give below.

Proposition 2.1. *Let $D_G(n)$ and $D_H(n)$ be Dehn functions for finitely presented groups G and H. Then there is Dehn a function $D(n)$ for $G \times H$ with $D(n) \preceq \max\{D_G(n), D_H(n), n^2\}$. In particular, if G and H admit polynomial Dehn functions, then* $\deg(D(n)) = \max\{\deg(D_G(n)), \deg(D_H(n)), 2\}$.

Proposition 2.2. *Let $I_G(n)$ and $I_H(n)$ be isodiametric functions for finitely presented groups G and H. Then there is an isodiametric function $I(n)$ for $G \times H$ with $I(n) \preceq \max\{I_G(n), I_H(n)\}$. In particular, if G and H admit polynomial isodiametric functions, then* $\deg(I(n)) = \max\{\deg(I_G(n)), \deg I_H(n))\}$.

The following proposition is not explicitly stated in [2], however it can be proven in a manner analogous to the proof of Proposition 3.2 in [2]. The only significant additional step is to realize that if w is a word in the generators of G or H which evaluates to an element c of C, then c can be expressed as a word w' in the generators of C. Since C isometrically embeds in G and H, the length of w' is no longer than the length of w. The interested reader can combine this fact with the proof of Proposition 3.2 in [2] to establish the following result.

Proposition 2.3. *Assume that C isometrically embeds in G and H, and $D_G(n)$ and $D_H(n)$ are Dehn functions for G and H. Then there is a Dehn function $D(n)$ for $G *_C H$ such that $D(n) \preceq \max(D_G(n), D_H(n))$. Thus* $\deg(D(n)) = \max(\deg(D_G(n), D_H(n))$, *assuming that $D_G(n)$ and $D_H(n)$ are polynomial. The same statement is true for isodiametric functions for the groups G and H.*

Theorems 2.4 and 2.5 follow from 2.1 through 2.3, using the decomposition of a graph product in terms of a free product with amalgamation.

Theorem 2.4. *Let $D_i(n)$ be Dehn functions for finitely presented groups. A_i. Then there is a Dehn function $D(n)$ for any graph product with vertex groups A_i, where $D(n) \prec \max\{D_i(x), n^2\}$. In particular,* $\deg(D(n)) = \max\{\deg(D_i(n)), 2\}$, *assuming the $D_i(n)$ are polynomial.*

Theorem 2.4 has also been independently proven by D. E. Cohen [6], by working "bare handed" with the noption of "pruning" defined in [9].

Theorem 2.5. *Let $I_i(n)$ be isodiametric functions for finitely presented groups A_i. Then* *there is an isodiametric function $I(n)$ for a graph product with vertex groups A_i, where $I(n) \preceq \max\{I_i(x)\}$. In particular, if the $I_i(n)$ are polynomial,* $\deg(I(n)) = \max\{\deg(I_i(n))\}$.

□

Example. Since infinite cyclic groups admit linear Dehn and isodiametric functions, any graph group admits a quadratic Dehn function and a linear isodiametric function.

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