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BROWNIAN MOTION ON A SYMMETRIC SPACE OF NON-COMPACT TYPE: ASYMPTOTIC BEHAVIOUR IN POLAR COORDINATES

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ABSTRACT. The results of Orihara [10] and Malliavin[2] [7] on the asymptotic behaviour in polar coordinates of Brownian motion on a symmetric space of non-compact type are obtained by means of a skew product representation on $K/M \times A^+$ of the Brownian motion on the set of regular points of X. Results of Norris, Rogers, and Williams [9] are interpreted in this context.

Introduction. This article is the second half of an exposition of results due to Orihara [10] 1970 and of Malliavin[2] [7] 1974 on the asymptotic behaviour of the Brownian motion on a symmetric space X of non-compact type in the two basic coordinate systems given by the Iwasawa and Cartan decompositions of the underlying semisimple Lie group G of isometries. In the first half [12] an introduction to such spaces was given by making use of the canonical example $G = \mathrm{SL}(n,\mathbb{R})$ and $X = \mathrm{SL}(n,\mathbb{R})/\,\mathrm{SO}(n)$, the space of symmetric positive definite matrices of determinant one, and Malliavin[2]'s result on the asymptotic behaviour of the Brownian motion relative to the Iwasawa or horocyclic coordinates was proved.

Here the case of polar coordinates is discussed. By remarking that the Brownian motion lives on the set of regular points of X—those positive definite symmetric matrices with distinct eigenvalues in the canonical example—it is possible to give a simpler and more direct proof than the one to be found in [7]. This is because the Brownian motion restricted to the regular points can be realized as a skew product on $K/M \times A^+$ (Theorem 2.7), where the role of the radius is played by A^+ and the angle by a point in the Furstenberg boundary K/M. The main result—Theorem 3.4—states the convergence results of Orihara and Malliavin[2]: a.s. the *radius* in A^+ tends to infinity in a precise direction, due to Orihara [10] 1970, and the *angle* in K/M converges, Malliavin[2] [7] 1974.

In [9] 1986 Norris, Rogers and Williams studied Brownian motion on the space of positive definite matrices with the aid of a left invariant process on the Lie group $\mathrm{GL}(n,\mathbb{R})$. Their results that have to do with the left invariant processes on the space of positive definite matrices are obtained as consequences of the result of Malliavin[2] [7]. It is to be

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noted that in fact they used right invariant motions. However, by using the map $g \longmapsto g^*$, which can be defined using the Cartan involution Θ of G, the right action becomes a left action and so their results may be stated in terms of left actions. This is more convenient here as this article follows Helgason's [4] usage of left actions.

In addition to explaining the polar coordinate convergence result of Malliavin[2] [7], the details of a remark at the end of Section 6 of [12] are given. They show how to prove Theorem 6.3 of [12] without using stochastic calculus. This theorem gives the form of Laplace-Beltrami operator on X in horocyclic coordinates. Finally, there were several technical errors in Section 7 of [12] which are corrected here. In addition I am grateful to Ewa Damek for an important remark concerning the incompleteness of Malliavin[2]'s convergence argument.

The table of contents follows:

1. **Root spaces and decompositions of the Lie algebra.** Recall that a semisimple Lie algebra $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and that there is a further orthogonal direct sum decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} \oplus \mathfrak{q} \oplus \mathfrak{a}$, where: $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{a}$, $\mathfrak{a}$ maximal abelian in $\mathfrak{p}$; $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{m}$, and $\mathfrak{m}$ is the subalgebra of $\mathfrak{k}$ that commutes with the elements of $\mathfrak{a}$. For additional details see [12] or for complete information [4]. Note that the orthogonality is relative to the positive definite form B_θ defined by the Killing form B and the Cartan involution θ. Let G denote a semisimple Lie group with finite centre and Lie algebra $\mathfrak{g}$, and let K denote the maximal compact subgroup with Lie algebra $\mathfrak{k}$.

If $G = \mathrm{SL}(n, \mathbb{R})$, $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ is the Lie algebra of $n \times n$ real matrices X of trace zero, $\mathfrak{k}$ consists of the skew-symmetric ones and $\mathfrak{p}$ of the symmetric ones, and $K = \mathrm{SO}(n)$. In terms of the Cartan involution θ, where $\theta(X) = -X^*$, $\mathfrak{k} = \{X \mid \theta(X) = X\}$ and $\mathfrak{p} = \{X \mid \theta(X) = -X\}$. It is usual to take as $\mathfrak{a}$ the diagonal matrices $H \in \mathfrak{sl}(n\mathbb{R})$. The Killing form for this example is $B(X, Y) = 2ntr(XY)$ and $B_\theta(X, Y) = -B\big(X, \theta(Y)\big) = 2ntr(XY^*)$, which is $2n$ times the inner product obtained by identifying an $n \times n$ matrix with a vector in $\mathbb{R}^{n^2}$

The maximal abelian subspace $\mathfrak{a}$ produces a commuting family of linear operators $\operatorname{ad} H$, $H \in \mathfrak{a}$, where $\operatorname{ad} H(X) = [X, H]$, that are all self-adjoint with respect to B_θ. As a result $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X\}$ is an eigenspace—called a *root space*—for all the operators $\operatorname{ad} H$ with eigenvalue α a linear functional on

$\mathfrak{a}$—*a* so-called *root*. There are a finite number of roots and the connected components of the subset of $\mathfrak{a}$ where they all have no zeros are called Weyl chambers. Given a particular choice of one Weyl chamber—then referred to as the positive Weyl chamber and denoted by $\mathfrak{a}^+$—a root α is said to be *positive* if it is positive on that chamber. A root is therefore either positive or negative. If α is a root so too is $-\alpha$ and $-\alpha = \alpha \circ \theta$.

The roots in the case of $\mathfrak{sl}(n, \mathbb{R})$ are the functionals $\alpha(H) = \lambda_i - \lambda_j$, $i \neq j$ for $i \neq j$, where H has diagonal entries λ_k. Let the cone of diagonal matrices with distinct entries in descending order be the positive Weyl chamber $\mathfrak{a}^+$.

The root spaces $\mathfrak{g}_\alpha$ determine decompositions of $\mathfrak{l}$ and $\mathfrak{q}$:

$$\text{if } \mathfrak{k}_\alpha = \mathfrak{k} \cap \left(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\right) \text{ and } \mathfrak{p}_\alpha = \mathfrak{p} \cap \left(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}\right),$$
$$\text{then } \mathfrak{l} = \Sigma_{\alpha>0}\mathfrak{k}_\alpha \text{ and } \mathfrak{q} = \Sigma_{\alpha>0}\mathfrak{p}_\alpha,$$

cf. [4] VII § 11 for the analogous case of a compact Lie algebra.

In the case of $\mathfrak{sl}(n, \mathbb{R})$, with the choice of $\mathfrak{a}$ as the set of diagonal matrices, these spaces are as follows—for $\alpha(H) = \lambda_i - \lambda_j$, $i \neq j$,

(1) $\mathfrak{g}_\alpha = \mathbb{R}E_{ij}$ where E_{ij} has zero entries except for a one at the (i,j) position;

(2) $\mathfrak{k}_\alpha = \mathbb{R}(E_{ij} - E_{ji})$; and

(3) $\mathfrak{p}_\alpha = \mathbb{R}(E_{ij} + E_{ji})$.

LEMMA 1.1. *If $X_\alpha \in \mathfrak{k}_\alpha$, then $X_\alpha = X_\alpha^+ + \theta(X_\alpha^+)$ for a unique $X_\alpha^+ \in \mathfrak{g}_\alpha$, $\alpha > 0$. Similarly, if $Y_\alpha \in \mathfrak{p}_\alpha$ then $Y_\alpha = Y_\alpha^+ - \theta(Y_\alpha^+)$, for a unique $Y_\alpha^+ \in \mathfrak{g}_\alpha$.*

PROOF. If $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$, then $X + Y \in \mathfrak{k}$ implies that $Y = \theta(X)$, since $\theta\mathfrak{g}_\alpha = g_{-\alpha}$. Similarly, if $X + Y \in \mathfrak{p}$, then $Y = -\theta(X)$, cf. [12], remark following Definition 3.2. ∎

COROLLARY 1.2. *Define $\sigma: \mathfrak{l} \mapsto q$ by setting $\sigma\left(X_\alpha^+ + \theta(X_\alpha^+)\right) = X_\alpha^+ - \theta(X_\alpha^+)$. Then σ is an isomorphism—cf. [12] Lemma 3.5 where $\tau = \sigma^{-1}$. It is also an isometry relative to the inner product B_θ.*

PROOF. The root spaces are orthogonal with respect to B_θ and θ is an isometry. Hence, if $X_\alpha = X_\alpha^+ + \theta(X_\alpha^+)$, then $\|X_\alpha\| = \sqrt{2}\|X_\alpha^+\| = \|\sigma(X_\alpha)\|$. ∎

LEMMA 1.3. *If $H \in \mathfrak{a}$, then*

$$X_\alpha \in \mathfrak{k}_\alpha \Rightarrow \operatorname{ad} H(X_\alpha) = [H, X_\alpha] = \alpha(H)\sigma(X_\alpha).$$

PROOF. $[H, X_\alpha^+] = \alpha(H)X_\alpha^+$ and $[H, \theta(X_\alpha^+)] = -\alpha(H)\theta(X_\alpha^+)$. ∎

As pointed out in [12]—cf.[4] for details—for any Lie group G the inner automorphism given by conjugation with an element g has a derivative $\operatorname{Ad}(g)$ at e which is an automorphism of the Lie algebra $\mathfrak{g}$. Also if $g = \exp X$, then $\operatorname{Ad}(g) = e^{\operatorname{ad} X}$. In the case of $G = \mathrm{SL}(n, \mathbb{R})$, $\operatorname{Ad}(g)X = gXg^{-1}$.

Let M be the connected Lie subgroup of K with Lie algebra $\mathfrak{m}$—the subgroup of K that commutes with the elements of $A = \exp\mathfrak{a}$—and let M' be the subgroup of K that normalizes $A = \exp\mathfrak{a}$, i.e. $kAk^{-1} = A$ implies $k \in M'$. Since $\operatorname{Ad}(k)$ is an automorphism of the Lie algebra, $\alpha \circ \mathrm{Ad}(k)$ is a root for all $k \in M'$. As a result, the quotient group M'/M—-the so-called *Weyl group*—preserves the Weyl chambers. In fact it permutes them in a simply transitive way, cf. [4] VII Theorem 2.12— there this result is obtained by passing to the compact dual form of the Lie algebra and making use of the fact that the roots can be obtained via complexification of the Lie algebra cf.[4] VI § 3.

In the case of $\mathrm{SL}(n, \mathbb{R})$, $m = \{0\}$ cf. [12], M is the discrete subgroup of diagonal matrices in $\mathrm{SO}(n)$ i.e. $k \in M$ if and only if it is diagonal with entries ± 1, an even number of them negative and M' consists of permutation like matrices. A matrix $k \in M'$ is obtained from a permutation matrix by replacing the non-zero entries by ± 1, subject to the requirement that the determinant be one.

LEMMA 1.4. *The Weyl group for* $\mathrm{SL}(n, \mathbb{R})$ *is the permutation group* $\mathfrak{S}_n$ *on n letters.*

PROOF. Let $k \in M'$ and $a_0 \in A^+ = \exp \mathfrak{a}^+$. The diagonal entries $\mu_i = e^{\lambda_i}$ of a_0 are in decreasing order. If $a_1 = ka_0k^{-1}$, there is a unique permutation $\pi = \pi_k \in \mathfrak{S}_n$ such that the decreasing order of the diagonal entries of a_1 is $\mu_{\pi(1)} > \mu_{\pi(2)} > \cdots > \mu_{\pi(n)}$. The map $k \longmapsto \pi_k$ is a homomorphism with kernel M.

Conversely, given a chamber $\mathfrak{a}^\dagger \neq a^+$, there is a unique permutation π of the diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $\mathfrak{a}^\dagger = \{\lambda_{\pi(1)} > \lambda_{\pi(2)} \cdots > \lambda_{\pi(n)}\}$. Let k be the corresponding permutation matrix k' if it is in $\mathrm{SO}(n)$, and the matrix obtained from k' by replacing one of the non zero entries by -1 otherwise. Then $k \in M'$ and $ka_0k^{-1} \in \mathfrak{a}^\dagger$ for any $a_0 \in \mathfrak{a}^+$. Hence, $\pi = \pi_k$. ∎

2. **The skew product representation of the Brownian motion on the set of regular points.** The left-invariant metric on the homogeneous space G/K is determined by its values on $T_0(G/K) \simeq \mathfrak{p}$. This isomorphism is given by associating with $X \in \mathfrak{p}$ the tangent vector which is the derivative at $t = 0$ of the curve $t \longmapsto \exp tX.o = \mathrm{Exp}\, tX$.

The metric on G/K is given by letting it agree with the Killing form B restricted to $\mathfrak{p}$ at the base point $o = \{K\}$ and then transporting it by G to the other points. This makes sense as B is invariant under $\mathrm{Ad}(k)$, $k \in K$.

The group K is the isotropy group of the base point o and to examine rotationally invariant, i.e. K-invariant properties of the Brownian motion, it is necessary to introduce polar coordinates on G/K so that the *angular* variable involves the action of K.

The Cartan decomposition $G = KAK$ of the group G implies that $G/K = KA.o$. In the case of $G = \mathrm{SL}(n, \mathbb{R})$ this amounts to the fact that that every $n \times n$ matrix g of determinant one can be written as the product $g = k_1ak_2$, where the matrices k_1 and k_2 are orthogonal of determinant one and a is diagonal cf. [2] and [12]. Note that a is unique if its diagonal entries are in decreasing order cf. [4] IX Theorem 1.1.

In order to introduce polar coordinates into the symmetric space $X = G/K = KA.o$ it is necessary to restrict attention to the so-called *regular* points.

DEFINITION 2.1. *A point* $a \in A$ *will be said to be regular if it is of the form* $\exp H$, $H \in \mathfrak{a}' = \{H \in \mathfrak{a} \mid \alpha(H) \neq 0, \forall \alpha \in \Sigma\}$. *Let* A' *denote the set of regular points of* A. *Let* $G' = KA'K$ *denote the set of regular points of* G. *If* $X = G/K$, *let* $X' = G'.o$

Note that $\mathfrak{a}'$ is the union of all the Weyl chambers. In the case of $\mathrm{SL}(n,\mathbb{R})$, A' is the set of diagonal matrices of determinant one all of whose diagonal entires are distinct.

For the points in X', the K-orbits are all diffeomorphic to K/M as stated in [4] IX Corollary 1.2—the general result on polar coordinates. In the particular case of $G = \mathrm{SL}(n,\mathbb{R})$ this result is relatively easy to verify. It is stated as the following proposition.

PROPOSITION 2.2 (POLAR COORDINATES). *Let $x \in X'$. Then there is a unique $a \in A^+ = \exp \mathfrak{a}^+$ such that $x = ka.o$.*

Further,
(1) for any $a \in A'$, $Ka.o \simeq K/M$;
(2) if $a_1 \neq a_2 \in A^+$ then $Ka_1.o \cap Ka_2.o = \emptyset$.
Consequently, there is an injective map:

$$K/M \times A^+ \mapsto X = G/K$$

given by $(kM, a) = (\dot{k}, a) \mapsto ka.o$. In addition, X'—the image of this map—is open and dense in X.

PROOF FOR $\mathrm{SL}(n,\mathbb{R})$. Since $M'/M = W$ is transitive on the Weyl chambers, it follows that $KA'.o = KA^+.o$. The fact that A' is dense in A, implies that $X' = KA^+.o$ is dense in $X = KA.o$.

Let $a \in A^+$ and assume $ka.o = a.o$. Then $kak^{-1}.o = a.o$. If $a = \exp H$, the following Lemma 2.A implies that $\mathrm{Ad}(k)H = H$ and so $kak^{-1} = a$. Since the diagonal entries of a are all distinct, k is diagonal and hence in M—in the general case this is the difficult point to establish. Therefore, M is the subgroup of K that fixes a and so K/M can be identified with $Ka.o$ by mapping $\dot{k} = kM \mapsto ka.o$.

LEMMA 2.A. *The map* $\mathrm{Exp}\colon \mathfrak{p} \mapsto X$ *given by* $\mathrm{Exp}\, Y = \exp Y.o$ *is a bijection.*

PROOF. The general result is [4] VI Theorem 1.1. In the case of $\mathrm{SL}(n,\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, this result is I § IV Proposition 5 of [2]. It is easy to see that Exp is onto by using the fact that the correspondence $g.o \leftrightarrow gg*$ between cosets and symmetric matrices is bijective; and $s = kdk^{-1}$, d diagonal and so $s = \exp(kDk^{-1})$, $\exp D = d$. ∎

The next Lemma will imply that the orbits of distinct points in $A^+.o$ are distinct

LEMMA 2.B. $\mathrm{Ad}(k)H_0 = H_1$, $H_i \in \mathfrak{a}' \Longrightarrow k \in M'$.

PROOF. Since $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha$, the maximality of $\mathfrak{a}$ as an abelian subspace of $\mathfrak{p}$ implies that $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. Consequently, if $X = X_0 + \sum_{\alpha\in\Sigma} X_\alpha$, and $[H_1, X] = 0$ for $H_1 \in \mathfrak{a}'$, then $X \in \mathfrak{p}$ implies that $X \in \mathfrak{a}$ since $[H_1, X] = \sum_{\alpha\in\Sigma} \alpha(H_1)X_\alpha$.

Hence, if $X \in \mathfrak{a}$, $0 = [H_1, X] = [\mathrm{Ad}(k)H_0, X]$ and so $0 = [H_0, \mathrm{Ad}(k^{-1})X]$. Therefore, $\mathrm{Ad}(k^{-1})X \in \mathfrak{a}$ i.e. $k \in M'$. ∎

If $Ka_1.o \cap Ka_2.o \neq \emptyset$, then there exists $k \in K$ such that $ka_1.o = a_2.o$. Consequently, by Lemma 2.A, $ka_1k^{-1} = a_2$. By Lemma 2.B this implies that $k \in M'$. However, since the diagonal entries of a_1 and a_2 are both given in descending order, by Lemma 1.4—or directly—the matrix k is in M. Therefore, $a_1 = a_2$. This completes the proof modulo a discussion of smoothness—and hence the openness of the image—for which we refer the reader to [4]. ∎

In order to study the asymptotic behaviour of the Brownian motion on $X = G/K$ started from $o = \{K\}$, it turns out that it is sufficient to study Brownian motion on X'. This is because, as explained in the Appendix, the set $X' \setminus X$ of non-regular points is a set of capacity zero and therefore with probability one Brownian motion never visits this singular set. In order to get a handle on the Brownian motion on X', it is important to describe the Riemannian metric of the orbits $Ka.o, a \in A^+$. As shown in [5] p. 267, X' can then be seen to satisfy the hypotheses of the general Theorem II 3.7 of [5] and so a radial part of the Laplace-Beltrami operator is determined. In fact, it satisfies the hypotheses of the general result of Pauwels and Rogers [11] concerning skew products which extends Helgason's polar coordinate formula [5] II 5.24 for the Laplacian.

Explanations now follow for the above assertions. First of all, if $L_g: G/K \mapsto G/K$ is the left action of g on G/K, the tangent space

$$T_{ka.o}(G/K) = dL_{ka}\big(T_o(G/K)\big) = dL_{ka}(\mathfrak{q}) \oplus dL_{ka}(\mathfrak{a}),$$

since the tangent space $\big(T_o(G/K)\big)$ is identified with $\mathfrak{p} = \mathfrak{q} \oplus \mathfrak{a}$. It is therefore clear that the subspace $dL_{ka}(\mathfrak{q})$ of $T_{ka.o}(G/K)$ has some relation to the orbit Ka.o. In fact it is $T_{ka.o}(Ka.o)$ as is shown by the following proposition. Note that if the orbit $Ka.o$ is identified with K/M, then the tangent vector S in Proposition 2.3 is the value at $kM = \dot{k}$ of the left invariant vector corresponding to $U \in \mathfrak{l}$.

PROPOSITION 2.3. *Assume that $a \in A^+$. Let $U \in \mathfrak{l}$ and define the tangent vector $S \in T_{ka.o}(Ka.o)$ by the formula*

$$Sf(ka.o) = \frac{d}{dt} f(k\{\exp tU\}a.o)|_{t=0}.$$

Let $a = \exp H$ and define $\sinh(\operatorname{ad} H): \mathfrak{l} \mapsto \mathfrak{q}$, by setting $\sinh(\operatorname{ad} H)X_\alpha = \sinh \alpha(H)\sigma(X_\alpha)$, $X_\alpha \in \mathfrak{k}_\alpha$. Then $\sinh(\operatorname{ad} H)$ is invertible and

$$Sf(ka.o) = \frac{d}{dt} f(ka \exp tV.o)|_{t=0} = dL_{ka}(V)f,$$

where V is the unique $V \in \mathfrak{q}$ such that $V = \sinh(-\operatorname{ad} H)U$.

PROOF. First note that $k \exp tUa.o = ka \exp t \operatorname{Ad}(a^{-1})U.o$. To compute $\operatorname{Ad}(a^{-1})U$, it is necessary to use the structure of $\mathfrak{l}$ that is given by the roots $\alpha > 0$.

Lemma 1.3 shows how to compute $\operatorname{Ad}(a^{-1})U$. Let U_α be the component of $U \in \mathfrak{k}_\alpha$. Then, if $a = \exp H$, it follows that $\operatorname{Ad}(a^{-1})U = e^{-\operatorname{ad} H}U = \Sigma_{\alpha>0} e^{-\operatorname{ad} H}U_\alpha = \sinh(-\operatorname{ad} H)U + \cosh(\operatorname{ad} H)U$, as $e^{-\operatorname{ad} H}U_\alpha = -\sinh \alpha(H)\sigma(U_\alpha) + \cosh \alpha(H)U_\alpha$. Since $\cosh(\operatorname{ad} H)U \in \mathfrak{l} \subset \mathfrak{k}$, the tangent vector to the curve $t \mapsto \exp t \operatorname{Ad}(a^{-1})U.o$, at $t = 0$ is $V = -\sinh(\operatorname{ad} H)U$.

Because $H \in \mathfrak{a}'$, $\alpha(H) \neq 0\ \forall \alpha \in \Sigma^+$, and so the operator $\sinh(\operatorname{ad} H)$ is invertible. ∎

REMARK. Orihara on p. 77 of [10] computed tangent vectors by identifying the points of $E = \exp \mathfrak{p}$ with elements g^*g of the group, where implicitly he defined g^*

to be $\Theta(g^{-1})$. This introduces a 2 into the computation and so his later calculations differ from those that follow which have to do with behaviour of the *radial* component in A^+. This way of looking at things affects the way the action of AdA is computed.

As a result, with the closed subgroup of isometries being K and the transversal submanifold $A^+.o \subset X = G/K$, the orbits $Ka.o$ intersect $A^+.o$ in the point a.o and the tangent space $T_{ka}.o(X) = T_{ka}.o(Ka.o) \oplus T_{ka}.o(A^+.o)$ and so the hypotheses of [5] II Theorem 3.7 are satisfied. The orbits are all diffeomorphic to K/M, the induced Riemannian metric on an orbit $Ka.o$ corresponds to a left K- invariant metric g^a on K/M, and it will be shown that the corresponding Riemannian measures are all proportional. Proposition 2.3 enables us to compute the density function $\delta(a.o) = \delta(a), a \in A^+$ which scales the Riemannian measure of the orbit $Ka.o$.

The tangent space $T_{\dot e}(K/M)$ of the homogeneous space K/M at $\dot e = \{M\}$ is identified with $\mathfrak{l}$: $U \in \mathfrak{l}$ defines the tangent vector $U_{\dot e}f = \frac{d}{dt}f(\exp tU.\dot e)|_{t=o}$. Now $\mathrm{Ad}(M)\mathfrak{l} = \mathfrak{l}$ as $[\mathfrak{m},\mathfrak{l}] \subset \mathfrak{l}$ since $X \in \mathfrak{g}_\alpha, U \in \mathfrak{m} \Rightarrow 0 = \big[H,[U,X]\big] + \big[U,[X,H]\big] + \big[X,[H,U]\big] = \big[H,[U,X]\big] + \big[U,[X,H]\big] = \big[H,[U,X]\big] - \alpha(H)[U,X]$, i.e. $[\mathfrak{m},\mathfrak{g}_\alpha] \subset \mathfrak{g}_\alpha$. Further, the Killing form B restricted to $\mathfrak{l}$ is invariant under Ad(M). As a result, there is a unique K-left invariant metric g on K/M which has as orthonormal basis at $\dot e$ a basis $U_1, U_2, \ldots, U_\ell$ of $\mathfrak{l}$ orthonormal with respect to the form $-B$. Let this basis be subordinate to the direct sum decomposition $\mathfrak{l} = \Sigma_{\alpha>0}\mathfrak{k}_\alpha$. Denote by $d\nu$ the corresponding Riemannian measure.

The compact group K has a unique left-invariant metric which is determined by the form $-B$ on $\mathfrak{k}$. If $M_1, M_2, \ldots, M_m$ is an orthonormal basis of $\mathfrak{m}$, then

$$M_1, M_2, \ldots, M_m, U_1, U_2, \ldots, U_\ell$$

is an orthonormal basis of $\mathfrak{k}$. Let $d\mu$ denote the Riemannian measure on K.

LEMMA 2.4. *Let $M_1, M_2, \ldots, M_m$ be a basis of $\mathfrak{m}$ and $U_1', U_2', \ldots, U_\ell'$ be a basis of $\mathfrak{l}$. Denote by γ the left-invariant metric on K for which $M_1, M_2, \ldots, M_m, U_1', U_2', \ldots, U_\ell'$ is an orthonormal basis of $\mathfrak{k}$, and by g the left-invariant metric on K/M for which $U_1', U_2', \ldots, U_\ell'$ is an orthonormal basis of $T_{\dot e} = \mathfrak{l}$. Then the natural projection $K \mapsto K/M$ is a submersion and the image of $d\mu'$—the Riemannian measure on K—is $d\nu'$—the Riemannian measure on K/M.*

PROPOSITION 2.5 ([5] II LEMMA 5.25 P. 310). *With respect to g^a, $a = \exp H$, an orthonormal basis of $T_{\dot e}(K/M)$ is given by the vectors*

$$u_1(a)U_1, u_2(a)U_2, \ldots, u_\ell(a)U_\ell, \text{ where } u_i(a) = -1/\sinh\alpha_i(H) \text{ if } U_i \in \mathfrak{k}_{\alpha_i}.$$

Consequently, $d\nu^a = \delta(a)d\nu$, where $\delta(a) = \Pi_{\alpha>0}\big(\sinh\alpha(H)\big)^{m_\alpha}$ and $m_\alpha = \dim\mathfrak{g}_\alpha$.

PROOF. The first statement follows immediately from Proposition 2.3 since the orthonormal basis $U_1, U_2, \ldots, U_\ell$ of $\mathfrak{l}$ and the isometry $\sigma : \mathfrak{l} \mapsto \mathfrak{q}$ determines an orthonormal basis $V_1, V_2, \ldots, V_\ell$ of $\mathfrak{q} \subset \mathfrak{p} = T_o(G/K)$ which in turn is transported by dL_{ka} to $T_{ka.o}(Ka.o) = dL_{ka}(\mathfrak{q})$.

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To compute $\delta(a)$ it suffices, in view of Lemma 2.4, to show that $d\mu^a = \delta(a)d\mu$, where $d\mu^a$ is the Riemannian measure on K for which $M_1, M_2, \ldots, M_m$, $u_1(a)U_1$, $u_2(a)U_2, \ldots$, $u_\ell(a)U_\ell$ is an orthonormal basis of $\mathfrak{k}$.

View these tangent vectors as left-invariant vector fields. The exterior product of the dual basis of one-forms determines the Riemannian measure. Let $\mu_i = M_i^*$, $\omega_j = U_j^*$ and $\omega_j(a) = \big(u_j(a)U_j\big)^*$. Then,

$$d\mu^a = \pm\mu_1 \wedge \mu_2 \wedge \cdots \wedge \mu_m \wedge \omega_1(a) \wedge \omega_2(a) \wedge \cdots \wedge \omega_\ell(a) = \prod_{\alpha>0} \big(\sinh \alpha(H)\big)^{m_\alpha} d\mu. \ \blacksquare$$

REMARK 2.6. In the case of $SL(n, \mathbb{R})$ as pointed out earlier, the real dimension of all the root spaces is one and so $\delta\big(\exp(H)\big) = \prod_{i<j} \sinh(\lambda_i - \lambda_j)$, when H is the diagonal matrix with entries λ_i.

These results show that on X', the Riemannian metric satisfies the hypotheses of not only Helgason's theorem on the existence of a radial part of the Laplace-Beltrami operator ([5] II Theorem 3.7), but also the more general result of Pauwels and Rogers ([11] Theorem 4) which guarantees that the Laplace-Beltrami operator has the form of a so-called skew product. In this context, it means that the following result holds

THEOREM 2.7 (CF. [5] II THEOREM 5.24). *Let L denote the Laplace-Beltrami operator on $X' = KA^+.o$. Viewing X' as $K/M \times A^+$, then*

$$Lf(\dot{k}, a) = \big\{L^a_{K/M} f(\cdot, a)\big\}(\dot{k}) + \big\{\big[L_{A^+} + grad\, log\,\delta\,\big]f(\dot{k}, \cdot)\big\}(a),$$

where $L^a_{K/M}$ is the Laplace-Beltrami operator of the fibre $K/M \times \{a\}$ equipped with the metric g^a, and L_{A^+} is the Laplace-Beltrami operator of A^+ The operator L_{A^+} + grad log δ on A^+ is the radial part of L.

In other words, L is a skew product (cf. [13]) of two elliptic operators, one—the radial part—being an operator on A^+ which acts in the directions along A^+ and the other operator $L^a_{K/M}$ acting along the fibre $K/M \times \{a\}$. Notice that the vector field grad log δ on A^+—given by the Riemannian metric on A^+—is the image under the natural map of the corresponding vector field on $K/M \times A^+$. Also, in the Theorem II 5.24 of Helgason [5], the *spherical* part of the operator is written as an operator on K. This is the *horizontal* operator on K corresponding to the appropriate Laplace-Beltrami operator on K/M.

REMARK 2.8. In view of the isomorphism exp: $\mathfrak{a} \longmapsto A$, the radial part can also be viewed as an elliptic operator on the cone $\mathfrak{a}^+$. Since $\mathfrak{a} \simeq \mathbb{R}^r$ and the restriction of the Killing form B to $\mathfrak{a}$ is an inner product this operator is an elliptic operator $L_{\mathfrak{a}^+,B}$ with constant coefficients on the cone $\mathfrak{a}^+$. It is

$$L_{\mathfrak{a}^+,B}\varphi(H) = \Delta_B\varphi(H) + \nabla_B \log\delta\big(\exp(H)\big) \cdot \nabla_B\varphi(H),$$

where the subscript B denotes the dependence on this inner product. Changing B by a positive scaling factor c to $B' = cB$ scales this operator by c^{-1} i.e. $L_{\mathfrak{a}^+,cB} = c^{-1}L_{\mathfrak{a}^+,B}$.

Let $H_1, H_2, \dots, H_r$ be an orthonormal basis of $\mathfrak{a}^+$ relative to B. Then, if $H = \sum_{i=1}^r x_i H_i$, $\varphi(H) = \varphi(x)$, and $\alpha(H) = \alpha \cdot x = \sum_{i=1}^r a_i x_i$—where $a_i = \alpha(H_i)$—it follows that

$$(*) \qquad L_{\mathfrak{a}^+,B}\varphi(H) = \sum_{i=1}^r \frac{\partial^2 \varphi}{\partial x_i^2}(x) + \sum_{i=1}^r \sum_{\alpha>0} m_\alpha \coth(\alpha \cdot x)\, a_i \frac{\partial \varphi}{\partial x_i}(x).$$

REMARK 2.9. The result in Theorem 2.7 may be used to compute the skew products studied in examples (vi) and (vii) of [11]. It is shown below that in these examples the symmetric spaces $GL(n, \mathbb{R})/O(n)$ and $\mathrm{GL}(n, \mathbb{C})/U(n)$ are naturally isomorphic to $\{\mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)\} \times \mathbb{R}^+$ and $\{\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)\} \times \mathbb{R}^+$—products of a symmetric space of non-compact type with one of euclidean type. Let $\mathrm{GL}_0(n, \mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, be the subgroup of $\mathrm{GL}(n, \mathbb{F})$ for which the determinant is positive. Since every hermitian symmetric matrix has the form gg^*, the matrix g can be assumed to lie in $\mathrm{GL}_0(n, \mathbb{C})$. The above symmetric spaces are therefore isomorphic to $\mathrm{GL}_0(n, \mathbb{R})/\mathrm{SO}(n)$ and $\mathrm{GL}_0(n, \mathbb{C})/SU(n)$. Now $\mathrm{GL}_0(n, \mathbb{F}) \simeq \mathrm{SL}(n, \mathbb{F}) \times \mathbb{R}^+$—$g \leftrightarrow \left(\det(g)^{-1/n} g, \det(g)^{1/n}\right)$—from which the isomorphism follows. The metric corresponds to a product metric and so the invariant diffusion on, for example, $\mathrm{GL}(n, \mathbb{C})/U(n) \simeq \left\{\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)\right\} \times \mathbb{R}^+$ is the direct sum of the invariant diffusion on the symmetric space $\mathrm{SL}(n, \mathbb{R})/SU(n)$ with Brownian motion on the symmetric space $\mathbb{R}^+$ which has, up to a constant, the generator $t^2 u''(t) + t u'(t)$. Adding this operator to the radial part of the generator for the diffusion on $\mathrm{SL}(n, \mathbb{C})/SU(n)$ gives the skew product representation for the diffusion on $\mathrm{GL}(n, \mathbb{C})/U(n)$.

To determine the radial part in the case of $\mathrm{GL}(n, \mathbb{R})/O(n)$, recall that for the Killing form B, $B(H, H) = 2ntr(H^2) = 2n \sum_1^n \lambda_i^2$, when H has the λ_i as its diagonal entries. In [11], the metric that is used amounts to scaling B by $1/2n$ with the consequent scaling of the radial part. The Lie algebra of the abelian group $A \times \mathbb{R}^+$ is $\mathfrak{a} \oplus \mathbb{R}(1, 1, \dots, 1) \simeq \mathbb{R}^n$. So by using $2n$ times the standard inner product as inner product on $\mathbb{R}^n$, and Remark 2.8, it follows that the radial part of the generator is $1/2n$ times

$$(1/2)\Delta_{\mathbb{R}^n} + (1/2)\nabla(\delta \circ \exp)\cdot = (1/2)\sum_{k=1}^n \frac{\partial^2}{\partial \lambda_k^2} + (1/2)\sum_{k=1}^n \left\{\sum_{j \neq k} \coth(\lambda_k - \lambda j)\right\} \frac{\partial}{\partial \lambda_k},$$

which is the generator of the diffusion given by the SDE on p. 256 of [11].

The radial part in the case of $\mathrm{GL}(n, \mathbb{C})/U(n)$ is very similar. A difference in δ is produced by the fact that the real dimension of all the root spaces is 2. To see this it suffices to rerun the computations for $\mathrm{SL}(n, \mathbb{R})$ and to notice that everything is almost the same: in particular $\mathfrak{a}$ and the roots are the same and the root space $\mathfrak{g}_\alpha = \mathbb{C}E_{ij}$ if $\alpha(H) = \lambda_i - \lambda_j$. This is what increases the real dimension of the root spaces to 2. As a result, $\delta\left(\exp(H)\right) = \prod_{i<j} \sinh^2(\lambda_i - \lambda_j)$ and so when computing the gradient of $\log \delta$ an extra 2 appears as remarked on p. 257 of [11]. The Killing form is again $2n$ times the trace (cf. [4] p. 187) and so the radial part of the generator is $1/2n$ times

$$(1/2)\sum_{k=1}^n \frac{\partial^2}{\partial \lambda_k^2} + \sum_{k=1}^n \left\{\sum_{j \neq k} \coth(\lambda_k - \lambda j)\right\} \frac{\partial}{\partial \lambda_k}.$$

3. **Asymptotic behaviour of Brownian motion: polar coordinates.** Start Brownian motion from any point $x_0 \in X = G/K$ and stop it when it first hits the geodesic sphere S centered at o of radius $2d(o, x_0) + 1$. In view of the capacity of the singular set—see the Appendix—a.s. the Brownian motion lies in $S \cap X'$. Consequently, if the asymptotic behaviour of the Brownian motion on X' is the same for all starting points, it follows from the Strong Markov Property that the Brownian motion on X will also have the same asymptotic property.

DEFINITION 3.1. Let H_α represent the root α i.e. $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{a}$. The vector H_ρ is defined to be $(1/2)\sum_{\alpha>0} m_\alpha H_\alpha$.

The drift vector D in the radial part of the Laplace-Beltrami operator $L_{G/K}$ relative to the horocyclic coordinates given by the Iwasawa decomposition is $-H_\rho$ when this radial part is viewed as an operator on $\mathfrak{a}$ cf. [12] Corollary 5.5.

THEOREM 3.2, ORIHARA [10]. *Let $x_0 \in X'$. Denote the path of the Brownian motion started from $x_0 = k_0 a_0^+ \cdot o$ by $X_t'(\omega) = k_t(\omega)a_t^+\omega) \cdot o$. If $a_t^+(\omega) = \exp H_t^+(\omega)$, then a.s. $H_t^+(\omega) = tH_\rho + o(t)$.*

PROOF. Orihara [10] proved this result by considering the stochastic differential equation satisfied by the radial component in skew coordinates determined by a fundamental system $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of positive roots α_i i.e. a basis (cf. [3] and the Appendix) of positive roots. Let E' be the product of r copies of $(0, +\infty)$. Define the linear transformation $T\colon \mathfrak{a} \longmapsto \mathbb{R}^r$ by setting $T(H) = \sum_{i=1}^r \alpha_i(H)\mathbf{e}_i = \sum_{i=1}^r y_i\mathbf{e}_i = y$, where $\mathbf{e}_i$ is the i^{th} standard basis vector of $\mathbb{R}^r$. Clearly, $T(\mathfrak{a}^+) = E'$. If $\varphi(H) = \varphi(x) = \psi(y)$, then

$$
\begin{aligned}
L_{\mathfrak{a}^+,B}\varphi(H) &= \sum_{i=1}^r \frac{\partial^2\varphi}{\partial x_i^2}(x) + \sum_{i=1}^r \sum_{\alpha>0} m_\alpha \coth\alpha(H)\, a_i \frac{\partial\varphi}{\partial x_i}(x) \\
&= \sum_{\ell,k=1}^r a^{\ell k}\frac{\partial^2\psi}{\partial y_\ell \partial y_k}(y) + \sum_{k=1}^r \sum_{\alpha>0} c^k(\alpha) m_\alpha \coth\alpha(H)\frac{\partial\psi}{\partial y_k}(y),
\end{aligned}
$$

where $a^{\ell k} = (\alpha_\ell, \alpha_k) = \sum_{i=1}^r \alpha_\ell(H_i)\alpha_k(H_i)$ and $c^k(\alpha) = (\alpha, \alpha_k) = \sum_{i=1}^r \alpha(H_i)\alpha_k(H_i)$. With this slight correction of the formula for A in Orihara's Proposition 2—his coefficient $a^{\ell k}$ should be divided by 4—the rest of his proof applies.

It uses some basic results about root systems—cf. [3] and the Appendix—and the stochastic differential equations

$$dX(t) = X(0) + \tilde{a}dB(t) + b\big(X(t)\big)dt,$$

where $(\tilde{a})^2 = (a^{\ell k})$, $\big(B(t)\big)_{t\geq 0}$ is a standard r-dimensional Brownian motion and $b(y) = \big(b_1(y), b_2(y), \ldots, b_r(y)\big)$, with $b_k(y) = \sum_{\alpha>0} c^k(\alpha)m_\alpha \coth\alpha(H)$, and

$$dY(t) = Y(0) + \tilde{a}dB(t) + \tilde{b}\big(Y(t)\big)dt,$$

where $\tilde{b}_k(y) = (\alpha_k, \alpha_k)[m_{\alpha_k}\coth\alpha_k(H) + m_{2\alpha_k}\coth\alpha_{2\alpha_k}(H)]$, with $m_{2\alpha_k} = 0$ if $2\alpha_k$ is not a root. ■

REMARK. This theorem has an interpretation in terms of harmonic measures. Consider the exit law on the geodesic sphere S_R of radius R for the Brownian motion started from its centre o at the time of its first exit from the ball of radius R. This law is the harmonic measure associated with o and it is left K-invariant. It is therefore uniformly distributed on the K-orbits of the points of $S_R \cap (A^+.o)$. Theorem 3.2 implies that it tends to concentrate on the orbit corresponding to the direction of H_ρ as R tends to infinity.

As in the case of the horocyclic skew product, the behaviour of the radial part determines the behaviour of the other component. This is because of the nature of the skew product and the fact stated in [12] Corollary 6.4—and proved in [13]—that for skew products, the regular conditional probabilities obtained by conditioning on the behaviour of the radial component can be identified with the laws of a stochastic differential equation on the *angular* component.

Using the polar coordinates given by Proposition 2.3, a point $x_0 \in X'$ can be identified with a point $(\dot{\ell}_0, a_0) \in K/M \times A^+$ and a continuous path on X' with a pair of paths i.e. a point in $\mathbf{W}(K/M) \times \mathbf{W}(A^+)$, where for any topological space E, $\mathbf{W}(E)$ denotes the space of continuous paths $w: \mathbb{R}^+ \longmapsto E$. Let $\mathbf{P}_0$ denote the law of the Brownian motion on X' started from x_0. This is a probability on $\mathbf{W}(X') \simeq \mathbf{W}(K/M) \times \mathbf{W}(A^+)$. The natural projection $pr: K/M \times A^+ \longmapsto A^+$ intertwines the Laplacian on $X' \simeq K/M \times A^+$ and its radial part $L_{A^+} + \operatorname{grad}\log\delta$. Let $\mathbf{Q}_0$ denote the law of the radial process—with generator the radial part—started from $a_0 = pr(\dot{\ell}_0, a_0)$. Then by using the skew product set-up of [13], the regular conditional probability $\mathbf{p}(\omega_2, \cdot)$ of $\mathbf{P}_0$ given $\omega_2 \in \mathbf{W}(A^+)$ is a.s. the projection onto $\mathbf{W}(K/M)$ of the law of the following Stratonovich *SDE* on K:

$$(3.\omega_2) \qquad \begin{aligned} dk(t) &= \sum_{i=1}^{\ell} u_i\big(\omega_2(t)\big) U_i\big(k(t)\big) \circ dW(t) \\ k(0) &= \ell_0. \end{aligned}$$

Here, the functions $u_i(a)$ are as defined in Proposition 2.5 and $W(t)_{t\geq 0}$ is an ℓ-dimensional standard Brownian motion. This equation arises because, by fixing a path $\omega_2 \in \mathbf{W}(A^+)$, the resulting diffusion on K/M is given by a time dependent operator—at time t it is the Laplace-Beltrami operator for the metric $g^{\omega_2(t)}$. These operators are most easily studied by lifting them to K, cf. [12] Theorem A2.1, as $K \longmapsto K/M$ is a principal bundle (cf. [8] Nomizu p. 45) and the metrics are M-right invariant.

Alternatively, one may observe, following Pauwels and Rogers [11] p. 243, that there is a natural way to construct from the skew product on $K/M \times A^+$ a skew product on $K \times A^+$ such that the map $(\pi, id): K \times A^+ \longmapsto K/M \times A^+$, where $\pi(k) = kM = \dot{k}$, intertwines them. The basic ingredients for this construction were used in Lemma 2.4.

The $\mathbf{Q}_0$-a.s asymptotic behaviour of the paths ω_2 given in Theorem 3.2 and the possibility to do the computation conditionally implies

COROLLARY 3.3. *Let $x_0 \in X'$. Then, a.s.* $\lim_{t\to\infty} \dot{k}_t(\omega)$ *exists and is uniformly distributed on K/M—i.e. its law is the unique left K-invariant probability on K/M.*

PROOF. The law on K/M of the limit is necessarily left K-invariant. It is unique as K has a unique left invariant probability. Hence, it suffices to show that any smooth function f on K that is M-right invariant converges a.s along the paths of the SDE $(3.\omega_2)$ if ω_2 is a path that has the asymptotic behaviour given in Theorem 3.2.

Now

$$\begin{aligned} f\big(k(t)\big) &= f\big(k(0)\big) + \int_0^t \sum_{i=1}^{\ell} \big\{u_i\big(\omega_2(s)\big)U_i f\big\}\big(k(s)\big) \circ dW^i(s) \\ (**) \qquad &= f\big(k(0)\big) + \int_0^t \sum_{i=1}^{\ell} \big\{u_i\big(\omega_2(s)\big)U_i f\big\}\big(k(s)\big)\, dW^i(s) \\ &\quad + (1/2)\int_0^t \sum_{i=1}^{\ell} \big\{u_i^2\big(\omega_2(s)\big)U_i^2 f\big\}\big(k(s)\big)\, ds. \end{aligned}$$

If $\omega_2(t) = \exp H(t)$, for large t, $\big|u_i\big(\omega_2(t)\big)\big| = 1/\big|\sinh \alpha_i\big(H(t)\big)\big| \le e^{-ct}$ for some $c > 0$ in view of the asymptotic behaviour of ω_2.

Consequently, the martingale $\Psi_i(t) = \int_{t_k}^t \big\{u_i\big(\omega_2(s)\big)U_i f\big\}\big(k(s)\big)\, dW^i(s)$ has $\langle \Psi_i \rangle (t) \le \frac{C^2}{c} e^{-2ct_k} = Ce^{-2ct_k}, C = \| U_i f \|$.

It follows from Lemma 7.3* below, applied to $\big(e^{ct_k}\Psi_i(t)\big)_{t \ge t_k}$ that

$$P\Big[\sup_{t \ge t_k} |\Psi_i(t)| \ge \beta_k e^{-ct_k}\Big] \le Ce^{-\beta_k},$$

and so

$$P[\sup_{t \ge t_k} |\sum_{i=1}^{q} \Psi_i(t)| \ge q\beta_k e^{-ct_k}] \le qCe^{-\beta_k}.$$

Now as in Malliavin[2] [7], choose $\beta_k \longrightarrow +\infty$ and $t_k \uparrow +\infty$ such that

$$\sum_{k=1}^{\infty} e^{-\beta_k} < \infty \text{ and } \beta_k e^{-ct_k} \longrightarrow 0.$$

It follows from the Borel-Cantelli Lemma that a.s. $\exists\ k_0(\omega) = k_0$, such that $\big|\int_{t_k}^t \sum_{i=1}^q \big\{u_i\big(\omega_2(s)\big)U_i f\big\}\big(k(s)\big)\, dW^i(s)(\omega)\big| < \beta_k e^{-ct_k}$, as long as $k \ge k_0$.

It follows from $(**)$, that

$$f\big(k(t)\big) - f\big(k(t_k)\big) = \Psi_i(t) + \int_{t_k}^t \sum_{i=1}^{\ell} \big\{u_i^2\big(\omega_2(s)U_i^2 f\big)\big\}\big(k(s)\big)\, ds.$$

The absolute value of the integral in this expression is less than Ce^{-2ct_k}. Consequently, a.s. $\big|f\big(k(t)\big) - f\big(k(t_k)\big)\big| < (\beta_k + C)e^{-2ct_k}$ for sufficiently large k and so the function f converges a.s. along the paths. ∎

The following theorem summarizes the results of this section.

THEOREM 3.4. *Let $x_0 \in X'$ and let $X'_t(\omega) = \big(\dot{k}_t(\omega), a_t^+(\omega)\big)$ denote the Brownian motion on X'. Then, a.s.,*

(1) $\lim_{t \to +\infty} \frac{1}{t} H_t^+(\omega) = H_\rho$, *where* $a_t^+(\omega) = \exp H_t^+(\omega)$; *and*

(2) $\lim_{t \to \infty} \dot{k}_t(\omega)$ *exists in* K/M, $\dot{k} = kM$.

As a result, if $(X_t)_{t \ge 0}$ is the Brownian motion on $X = G/K$ started at $x_1 \in X$, then a.s.

(1) $\lim_{t\to+\infty} \frac{1}{t} H_t^+(\omega) = H_\rho$, where $\exp H_t^+(\omega) = a_t^+(\omega)$ is the unique element $a^+ \in \overline{A^+}$ such that $X_t(\omega) = k_1 a^+ k_2$ (cf. [4] IX Theorem 1.1); and

(2) $X_t(\omega)$ enters and remains in the set X' of regular points and its resulting component in K/M, $\dot{k}_t(\omega)$ converges.

4. **The results of Norris et al [9].** In [9] the authors examined Brownian motion on $\mathrm{GL}(n, \mathbb{R})/O(n)$. Rather than study it in terms of cosets they used a left Brownian motion $(g_t)_{t\geq 0}$ on $G = \mathrm{GL}(n, \mathbb{R})$ started at the identity and transported it to the positive definite matrices via the map $g \longmapsto gg*$ to get a process $(Y_t)_{t\geq 0}, Y_t = g_t g_t^*$ which corresponds to the left invariant Brownian on $\mathrm{GL}(n, \mathbb{R})/O(n)$ started at the identity matrix when $gg*$ is identified with the coset $gO(n)$. Note that as mentioned in the introduction, they in fact used a right invariant motion and one needs to convert left to right by using the map $g \longmapsto g*$ to get their statements.

If, as indicated in Remark 2.9, one views $\mathrm{GL}(n, \mathbb{R})/O(n)$ as $\mathrm{SL}(n, \mathbb{R})/SO(n) \times \mathbb{R}^+$, then one obtains the radial form of the Laplace-Beltrami operator from that of the Brownian motion on $\mathrm{SL}(n, \mathbb{R})SO(n)$. The asymptotic behaviour of the eigenvalues is as follows: the matrix of logarithms of the normalized eigenvalues is asymptotic to tH_ρ and the determinant (the product of the eigenvalues) performs an independent Brownian motion on $\mathbb{R}^+$.

Norris et al also consider another process which they view as more natural: namely the process $(Z_t)_{t\geq 0}$, $Z_t = g_t^* g_t$—denoted by X not Z in [9]. While this is a process on the space of symmetric positive definite matrices it is no longer invariant under the group action: $g_1 \cdot (gg^*) = g_1 gg^* g_1^*$.

The uniqueness of the $\overline{A^+}$-component in the decomposition of an element g in a semi-simple group G as $g = k_1 a^+ k_2$—cf.[4] IX Theorem 1.1—implies that if, for g in $\mathrm{GL}(n, \mathbb{R})$, $gg^* = k_1 a^+ k_2$ with the distinct eigenvalues of the diagonal matrix a^+ in descending order then $g^* g = k_3 a^+ k_4$. As a result the eigenvalue processes associated with $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are the same. This observation together with the formula in Remark 2.9 implies formula (7.3) in their Theorem A. The fact that the eigenvalues do not collide follows as the capacity of the set $X \backslash X' \times \mathbb{R}^+$, $X = \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$ is zero.

Norris et al also obtain a limit theorem for the orthonormal frame process associated with $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ In their Theorem B, the orthonormal frame process associated with $(Y_t)_{t\geq 0}$ may be viewed as the lift via the covering map $K \longmapsto K/M$ of the process $(\dot{k}_t)_{t\geq 0}$ on K/M—the flag space $\mathcal{F}_n$ in this instance (cf. [12]). It converges a.s. as they state in Theorem B because the process $(\dot{k}_t)_{t\geq 0}$ does so.

The results of the present paper do not cover the behaviour of the other orthonormal frame process. It is related to a theorem proved by Baxendale in [1] on real hyperbolic space.

Note that the process $(Z_t)_{t\geq 0}$ has an analogue on a symmetric space $X = G/K$ of non-compact type. Let Θ be the group automorphism of G whose derivative at e is the Cartan involution θ. Define $g^* = \Theta(g^{-1})$. If $(g_t)_{t\geq 0}$ is a left Brownian motion on G then the process $(g_t^* \cdot o)_{t\geq 0}$ on X corresponds to the process $(Z_t)_{t\geq 0}$.

This second process may be considered as natural as it gives a random linear flow on the symmetric space $x \mapsto g_t^*.x$. Note that this flow is not the flow associated with the Brownian motion on G/K: $g.o \mapsto gg_t.o$

5. **Relation between the two radial asymptotic behaviours.** The Iwasawa decomposition and horocyclic coordinates and the Cartan decomposition and polar coordinates give rise to two skew product diffusions and therefore two *radial* processes. Malliavin[2] [7] gave a proof of Theorem 3.2 of Orihara by using the result about the behaviour of the radial process for horocyclic coordinates to deduce the behaviour of the other radial process. This amounts to determining the K-orbit of the random point $x_t = n_t a_t.o$, i.e. the A^+-component of x_t in polar coordinates, where n_t and a_t are the horocyclic coordinates of x_t in G/K (cf.[12]).

If $a = \exp H_1.o$ and $H_1 \in -\mathfrak{a}^+$, then a^+—its A^+-component—is $a^{-1} = \exp(-1)H_1$. Hence, a.s., $a_t^+(\omega) = tH_\rho + o(t)$.

Since, as shown in [12], $n_t(\omega)$ converges, the distance of $n_t(\omega)a_t(\omega).o$ to $a_t(\omega).o$ is bounded. It is therefore sufficient to prove Theorem 3.2 from the asymptotic behaviour of the Brownian motion in horocyclic coordinates (cf. [12]) to show that the distance of the A^+-component of $n_t(\omega)a_t(\omega).o$ from that of $a_t(\omega).o$ is bounded.

LEMMA 5.1 (CF. MALLIAVIN[2] [7] LEMME P. 211). *Let $a.o$ and $na.o$ be two regular points in $X = G/K$. Assume that $n = \exp Y$, $Y \in \mathfrak{n}$. Let $\pi: X' = \{K/M \times A^+\}.o \mapsto A^+$ denote the projection on A^+.*

Consider the curve $t \mapsto \exp tYa.o$. There are $\tau_0 = 0 < \tau_1 < \ldots < \tau_k < \tau_{k+1} = 1$ such that $\exp tYa.o \in X'$, $\tau_i < t < \tau_{i+1}, 0 \leq i \leq k$. Further, for $\tau_i < t < \tau_{i+1}$, $0 \leq i \leq k$, the length of the tangent vector to the curve at t is $\leq (1/\sqrt{2})\|Y\|$.

Consequently, if $n = \exp Y$,

$$d\big(\pi(na.o), \pi(a.o)\big) \leq (1/\sqrt{2})\|Y\|.$$

PROOF. First consider the curve for $\tau_i < t < \tau_{i+1}$. Let $n(t) = \exp tY$. Then $n(t)a = k_1(t)\exp H^+(t)k_2(t)$, where $H^+(t) \in \overline{\mathfrak{a}^+}$ and is unique by Proposition 2.2. In the case of $SL(n,\mathbb{R})$ this follows immediately: the diagonal entries are in decreasing order. Consequently, if $n(t)a$ is regular, $\pi\big(n(t)a.o\big) = \exp H(t)$. Now $n(t)k_1(t_0)\exp H(t_0)k_2(t_0) = n(t+t_0)a$, and so $\exp H(t+t_0) = \pi\big(n(t+t_0)a.o\big) = \pi\big(n(t)k_1(t_0)\exp H(t_0).o\big)$.

If $\gamma_1(t)$ and $\gamma_2(t)$ are two curves with $\gamma_1(0) = \gamma_2(0) = g_0$, set $\gamma_1 \sim \gamma_2$ if they determine the same tangent vector at g_0.

Since π is smooth, the fact that $Y = W + V$, $W \in \mathfrak{k}$ and $V \in \mathfrak{q}$ implies that $\exp tY \sim \exp tW \exp tV$, cf. [4] II Lemma 1.8. Therefore,

$$\begin{aligned}\pi\big(n(t)k_1(t_0)\exp H(t_0).o\big) &\sim \pi\big(\exp tVk_1(t_0)\exp H(t_0).0\big)\\ &= \pi\Big(\exp t\big\{\mathrm{Ad}\big(k_1^{-1}(t_0)\big)V\big\}\exp H(t_0).o\Big).\end{aligned}$$

Let $H(t_0) = H_0$ and $\mathrm{Ad}\big(k_1^{-1}(t_0)\big)V = X_0 = V_1 + H_1 \in \mathfrak{q} \oplus \mathfrak{a}$. Then,

$$\begin{aligned}\exp tX_0 \exp H_0 . o &= \exp H_0 \exp t\{ e^{-\,\mathrm{ad}\, H_0}(X_0)\} . o \\ &= \exp t\{\cosh(\mathrm{ad}\, H_0)V_1 - \sinh(\mathrm{ad}\, H_0)\sigma^{-1}V_1 + H_1\} . o,\end{aligned}$$

—cf. the proof of Proposition 2.3—and so

$$t \longmapsto \exp tX_0 \exp H_0 . o \sim t \longmapsto \exp t\{\cosh(\mathrm{ad}\, H_0)V_1 + H_1\} . o,$$

as $V \in \mathfrak{q}$ implies $\sinh(\mathrm{ad}\, H_0)V_1 \in \mathfrak{k}$.

Now $V \in \mathfrak{q}$ implies $\cosh(\mathrm{ad}\, H_0)V_1 \in \mathfrak{q}$ and so $dL_{\exp H_0}(H_1)$ is the tangent vector to the curve $t \longmapsto \pi\Big(\exp t\big\{\mathrm{Ad}\big(k_1^{-1}(t_0)\big)V\big\} \exp H(t_0). o\Big)$. The length of this tangent vector is $\|H_1\| \le \|X_0\| = \|V\| \le (1/\sqrt{2})\|Y\|$. The existence of the τ_i follows from the next Lemma and the final conclusion of Lemma 5.1 then follows from what has been established.

LEMMA 5.2. *Let $\gamma(t)$, $0 \le t \le 1$ be a real analytic curve on a semisimple Lie group G. Assume that $\gamma(0)$ and $\gamma(1)$ are regular. Then the curve meets the set $G \setminus G'$ of singular points at most a finite number of times.*

PROOF. Consider the group $\mathrm{Ad}(G) \subset \mathfrak{gl}(\mathfrak{g})$. In the case of $G = SL(n, \mathbb{R})$, this is the group of linear transformations $X \longmapsto gXg^{-1}$, $g \in G$. These transformations preserve the bilinear form $\mathrm{tr}(XY)$ and those that correspond to orthogonal matrices also preserve the inner product on $\mathfrak{sl}(n, \mathbb{R})$ given by $\mathrm{tr}\, XY^*$ which is $(1/2n)$ times the inner product B_θ on $\mathfrak{g}$ that is derived from the Killing form B and the Cartan automorphism θ. In general, with respect to B_θ, the subgroup $\mathrm{Ad}(K)$ consists of orthogonal matrices. Further for any $g \in G$, the transformation $s = \mathrm{Ad}(g)\,\mathrm{Ad}(g)^*$ is symmetric, where * indicates the transpose relative to B_θ. If $g = k_1 a k_2$, then the eigenvalues of s are those of $\mathrm{Ad}(a)\,\mathrm{Ad}(a)^*$.

Assume $a = \exp H, H \in \mathfrak{a}$. The orthogonal decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$ implies that, with respect to a suitable basis, $\mathrm{Ad}(a)$ is represented by a diagonal matrix. Hence, the eigenvalues of $\mathrm{Ad}(a)\,\mathrm{Ad}(a)^*$ are $\nu(\alpha) = e^{2\alpha(H)}$. Consequently, the characteristic polynomial of $s = \mathrm{Ad}(k_1 a k_2)\,\mathrm{Ad}(k_1 a k_2)^*$ is

$$p(x) = (x-1)^{m_0} \prod_{\alpha \in \Sigma} \big(x - \nu(\alpha)\big)^{m_\alpha}, \text{ where } m_\alpha = \dim \mathfrak{g}_\alpha .$$

This implies that $g = k_1 a k_2$ is regular if and only if 1 is a root of multiplicity $m_0 = \dim \mathfrak{g}_0$ of the characteristic polynomial of $\mathrm{Ad}(g)\,\mathrm{Ad}(g)^*$.

Let $p(t,x)$ be the characteristic polynomial of $\mathrm{Ad}\big(\gamma(t)\big)\,\mathrm{Ad}\big(\gamma(t)\big)^*$. Then its coefficients are real analytic functions of t as are the coefficients of $p^{(m_0)}(t,x)$—the m_0^{th}-derivative in x of this polynomial. Since $t \longmapsto p^{(m_0)}(t, 1)$ is real analytic, the result follows. ■

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6. **Appendix.**

6.1. *The capacity of the set of non-regular points.* Let $o \neq x_0 = k_0 a_0 . o$ be a non-regular point. If $a_0 = \exp H_0$ there is at least one root $\alpha_0 > 0$ with $\alpha_0(H_0) = 0$. Consider the map $\kappa: K \mapsto G/K$ given by $k \mapsto k a_0 . o$. A tangent vector $U \in \mathfrak{k}_\alpha$ is in the kernel of $d\kappa_e$ if and only if $\alpha(H_0) = 0$. To see this consider the curve $t \mapsto \exp tU a_0 . o = a_0 \exp t \operatorname{Ad}(a_0^{-1})U . o$. The proof of Proposition 2.3 shows that $\operatorname{Ad}(a_0^{-1})U = -\sinh \alpha(H_0)\sigma(U) + \cosh \alpha(H_0) U$. It follows that the derivative of the curve at t=0 is zero if and only if $\alpha(H_0) = 0$.

From this it follows that the rank of κ at $e \in K$, and hence at all points of K, is the dimension of K/M minus the sum of the dimensions of the spaces $\mathfrak{k}_\alpha$, $\alpha > 0$ for which $\alpha(H_0) = 0$. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be the positive roots that vanish at H_0 and let $\pi = \{H \mid \alpha_i(H) = 0, 1 \leq i \leq k \text{ and } \alpha(H) > 0 \text{ if } 0 < \alpha \neq \alpha_i, 1 \leq i \leq k\}$. The K-orbit of $\exp \pi$ is a submanifold of X of codimension at least 2. The set of non-regular points is a finite union of submanifolds of this type.

In the case of a symmetric space of compact type the above argument is outlined by Borel in an article in the Cartan Seminar of 1949/50, referred to by Malliavin[2] [7].

Let M be a smooth manifold on which a second order elliptic operator is defined. It is known in potential theory that a subset of a submanifold of codimension at least 2 has capacity zero relative to the corresponding diffusion or potential theory (cf. Theorem 4.2, Chapter 11 of Friedman [3]. For the reader's convenience here is a short indication as to why it is true.

In terms of the diffusion, a set A is of capacity zero if, for any starting point x_0, the diffusion started from x_0 a.s never hits A. It follows that the countable union of sets of capacity zero is also of capacity zero and so it suffices to consider the question for an operator on an open set O in $\mathbb{R}^n$. The set may be assumed to be such that there is a Green function G for the operator. The singularity of the Green function at a pole y is the same as that of the Newtonian kernel if $n \geq 3$ and is logarithmic in dimension 2. It is this fact that gives the capacity result. If a relatively compact set A lies in a hypersurface S of codimension 2 and $x_1 \notin S$, then because of the canonical singularity, there is a Green potential v which is $+\infty$ on A and finite at x_1. This implies that the probability of hitting A starting from x_1 is zero as it is the value at x_1 of the infimum of all Green potentials on O that are ≥ 1 on A. Since this probability is an excessive (i.e. superharmonic) function, it is lower semi-continuous and so identically zero.

6.2. *Some remarks on roots.* The set R of roots of $\mathfrak{a}$ is finite subset of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$. It is known that R is a so-called root system (cf. [4] p. 456, which requires the reader to do some algebraic manipulations to obtain the result from the result for a compact symmetric space). The inner product on $\mathfrak{a}$ given by the Killing form B induces one on $\mathfrak{a}^*$ which will be denoted by $\langle \ , \ \rangle$. A root α is said to be *simple* if $\frac{1}{2}\alpha$ is not a root.

In the case of $\mathfrak{sl}(n, \mathbb{R})$, taking $\mathfrak{a}$ to be the diagonal matrices of trace zero, the simple roots are given by $\alpha(H) = \pm(\lambda_{i+1} - \lambda_i)$, $1 \leq i \leq n-1$. It is always possible to find a basis for $\mathfrak{a}^*$ given by simple roots α_i—which can be taken to be > 0—and then every root $\alpha = \sum_{i=1}^{r} n_i\alpha_i$, with $n_i \in \mathbb{Z}$ and all have the same sign. Furthermore, $\langle \alpha_i, \alpha_j \rangle \leq 0$ if $i \neq j$.

For $\mathfrak{sl}(n, \mathbb{R})$, $\alpha_i(H) = \lambda_i - \lambda_{i+1}$, $1 \leq i \leq n-1$ is a basis for the $\mathfrak{a}^*$ and the root $\alpha(H) = \lambda_i - \lambda_{i+k} = \sum_{\ell=i}^{i+k-1} \alpha_i$. Also $\langle \alpha_i, \alpha_{i+k} \rangle = 0$, $k \geq 2$, and $= -1/2n, k = 1$, since the vector H_{α_i} representing α_i with respect to the Killing form is $(1/2n)$ times the diagonal matrix whose non-zero entries are 1 at position (i, i) and -1 at $(i+1, i+1)$.

Associated with each root α there is a reflection $s_\alpha\colon \mathfrak{a}^* \mapsto \mathfrak{a}^*$ which is given by $s_\alpha\beta = \beta - a_{\beta,\alpha}\alpha$, where $a_{\beta,\alpha} = 2\langle \beta, \alpha \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z}$.

PROPOSITION (CF. LEMMA X.3.11 [4]). *The vector H_ρ representing $\rho = \frac{1}{2}\sum_{\alpha>0} m_\alpha \alpha$ is in $\mathfrak{a}^+$.*

Furthermore,

$$2\alpha_i(H_\rho) = 2\langle \rho, \alpha_i \rangle = \{ m_{\alpha_i} + 2m_{2\alpha_i} \} \langle \alpha_i, \alpha_i \rangle,$$

where $m_{2\alpha_i} = 0$ if $2\alpha_i$ is not a root.

PROOF. Let R_i be the set of positive roots distinct from α_i and $2\alpha_i$. Then, as in the proof of Lemma X.3.11 of [4], $\alpha \in R_i \Rightarrow s_{\alpha_i}\alpha = \beta \in R_i$.

Also, by Lemma VII.2.4 of [4], s_{α_i} is realized by Ad(k) for some $k \in K$. This implies that $m_\alpha = m_\beta$ and so $s_{\alpha_i}\rho = \rho - \{ m_{\alpha_i}\alpha_i + m_{2\alpha_i} 2\alpha_i \}$. This fact, together with the formula $a_{\alpha,\alpha_i} = 2\langle \alpha, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle$, implies that $2\langle \rho, \alpha_i \rangle / \langle \alpha_i, \alpha_i \rangle = \{ m_{\alpha_i} + 2m_{2\alpha_i} \}$. ∎

COROLLARY. *If $\rho = \sum_{k=1}^{r} \mu_k \alpha_k$, then*

$$\|\rho\|^2 = \|H_\rho\|^2 = \frac{1}{2}\sum_{k=1}^{r} \mu_k \{ m_{\alpha_k} + 2m_{2\alpha_k} \} \|\alpha_k\|^2.$$

REMARK. In the case of $\mathfrak{sl}(n, \mathbb{R})$, the vector $4nH_\rho$ has diagonal entries starting at $(n-1)$ decreasing by 2 at each step to terminate with $-(n-1)$. Hence, $\alpha_i(H_\rho) = 1/2n$, $1 \leq i \leq n-1$.

7. **Iwasawa decomposition: correction.** In [12] Section 7 some technical errors need correction. To begin with Lemma 7.3 should read as follows

LEMMA 7.3* (CF. MCKEAN [12] P. 3). *Let $\Psi(t) = \int_0^t \Phi(s)\, dB(s)$ where $\int_0^t \Phi^2(s)\, ds = \langle \Psi \rangle (t) \leq C$ for all t. Then, if $\beta > 0$,*

$$P[\sup_{t\geq 0} |\Psi(t)| \geq \beta] \leq 2e^{C/2-\beta}.$$

PROOF. Let $\Theta(t) = \exp\{\Psi(t) - (1/2)\int_0^t \Phi^2(s)\, ds\}$. Then $\big(\Theta(t)\big)_{t\geq 0}$ is a martingale of expectation one.

Further, $\Psi(t) \geq \beta \Rightarrow \Theta(t) \geq e^{\beta - C/2}$. Hence, by Doob's inequality,

$$[\sup_{t\geq 0} \Psi(t) \geq \beta] \leq e^{C/2-\beta}.$$ ∎

The second point is that the convergence argument in Malliavin[2] [7] is incomplete—it appears to suppose that there is a uniformity in the choice of a constant given by a Borel-Cantelli argument. They showed that a large class of bounded smooth functions f converge along the process on N. To get convergence of the process it is necessary to know that this class of functions is large enough to ensure that the process is a.s bounded. This observation is due to E. Damek.

The convergence argument of Malliavin[2] was not correctly presented in [12]. A corrected version follows the proof of Corollary 3.3 above. It proves

PROPOSITION 7.4* (CF. MALLIAVIN[2][7] PP. 199–200). *Let $f \in C^\infty(N)$ be such that $|Y_i f| \leq C$ and $|Y_i^2 f| \leq C$, $1 \leq i \leq q$. Then, a.s. $\lim_{t\to\infty} f\big(n(t)\big)$ exists.*

To prove Theorem 7.1 in [12] it is necessary to verify

THEOREM 7.5*. *Almost surely the process $\big(n(t)\big)_{t\geq 0}$ converges.*

PROOF. E. Damek pointed out that the convergence argument in Malliavin[2][7], i.e. Proposition 7.4*, can be completed by showing that a.s. the set $\{n(t,\omega) | t \geq 0\}$ is bounded. Damek showed the author that this follows from Proposition 7.4* by making use of the following result of A. Hulanicki. The convergence of the function f in Lemma 7.6 along almost all paths proves that a.s. the path is relatively compact. With this established, Proposition 7.4* applied to functions of compact support gives the a.s convergence.

LEMMA 7.6 (HULANICKI [6]). *Let $\tau(x) = d(x,e)$ where d is a left invariant Riemannian metric on a Lie group G and let dy denote a (right) Haar measure. If $\varphi \in C_0^\infty(G)$ and $f(x) = (\tau \star \varphi)(x) = \int \tau(xy^{-1})\varphi(y)\,dy$ then, for any left invariant vector field X on G,*

$$Xf = \tau \star X\varphi \text{ and } X^2 f = \tau \star X^2\varphi,$$

are both bounded.

PROOF. Note that $|\tau(\exp tX)| \leq |t|\ \|X\|$, where $\|X\|$ is the length of X at e. Now

$$|f(x\exp tX) - f(x)| \leq \int |\tau\big(xy^{-1}\exp t\,\mathrm{Ad}(y)X\big) - \tau(xy^{-1})|\ |\varphi(y)|\ dy$$

which by the triangle inequality is

$$\leq \int |\tau\big(\exp t\,\mathrm{Ad}(y)X\big)|\ |\varphi(y)|\,dy$$
$$\leq |t| \int \|\,\mathrm{Ad}(y)X)\|\ |\varphi(y)|\,dy = |t|C.$$

It follows that $|Xf| \leq C$. The change of variable $y = ux$ shows that $\int \tau(u^{-1})\varphi(ux)\,dy = f(x)$ and so $Xf = \tau \star X\varphi$. ∎

8. **Iwasawa decomposition: simplification.** Viewing X as the group $S = NA$, it is clear that the metric on X corresponds to a left invariant metric q on the Lie group S. The metric is therefore completely determined by the corresponding quadratic form Q on the Lie algebra $\mathfrak{s}$ of S.

Denote by $V_1, V_2, \ldots, V_d$ be an orthonormal basis of $\mathfrak{s}$. If $V \in \mathfrak{s}$ let V also denote the unique left-invariant vector field on S whose value at e is V.

As shown for example in [13] Proposition A 1.1, the divergence of a left-invariant vector field X on S is a constant computable from the Lie algebra $\mathfrak{s}$.

PROPOSITION 8.1. *If V is a left-invariant vector field on S then its divergence with respect to the metric q is a constant equal to* $-\operatorname{tr}(\operatorname{ad} V)$.

As a consequence, the Laplace-Beltrami operator associated with q may be computed.

COROLLARY 8.2. *Let L be the Laplace-Beltrami operator associated with the left-invariant Riemannian metric q on S. If $V_1, V_2, \ldots, V_d$ is an orthonormal basis of $\mathfrak{s}$, then*

$$L = \sum_{i=1}^{d} \{ V_i^2 + c_i V_i \}, \tag{8.1}$$

where $c_i = \operatorname{div} V_i = -\operatorname{tr}(\operatorname{ad} V_i)$.

PROOF. $Lf = \operatorname{div} \operatorname{grad} f$ and $\operatorname{grad} f = \sum_{i=1}^{n} (V_i f) V_i$. The result follows from Proposition 8.1. ■

Therefore to compute the Laplace-Beltrami operator for X it suffices to identify the quadratic form Q on the Lie algebra $\mathfrak{s}$ that corresponds to the metric on X at o and to compute the appropriate traces. With the identification of $T_o(X)$ as $\mathfrak{p}$, where $X \in \mathfrak{p}$ is identified with the tangent vector at $t = 0$ to the curve $t \longmapsto \exp tX \cdot o$, the metric on X at o corresponds to the Killing form B restricted to $\mathfrak{p}$.

Now $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ and is not a subset of $\mathfrak{p}$. If $H \in \mathfrak{a}$, the tangent vector to the curve $t \longmapsto \exp tH \cdot o$ at $t = 0$ is H. If $Y \in \mathfrak{n}$, then Lemma 1.1 implies that $2Y = X + U$, with $X \in \mathfrak{q}$—recall $\mathfrak{q}$ is the direct summand of $\mathfrak{a}$ in $\mathfrak{p}$—and $U \in \mathfrak{k}$. Therefore, the tangent vector $\in T_o(X)$ corresponding to X is the tangent vector at $t = 0$ to the curve $t \longmapsto \exp 2tY \cdot o$.

In view of Corollary 1.2, $\sqrt{2}\|Y\| = \|X\|$. As a result, given a suitable orthonormal basis X_i—with respect to B—of $\mathfrak{q}$, there corresponds to it an orthogonal basis—with respect to B_θ—of $\mathfrak{n}$, all of whose vectors Y_i have length $\frac{1}{\sqrt{2}}$ wrt B_θ and all of which lie in one of the root spaces.

The quadratic form Q on $\mathfrak{s}$ is therefore defined by taking as an orthonormal basis the B_θ orthogonal vectors $2Y_1, 2Y_2, \ldots, 2Y_q$ together with an orthonormal basis H_1, $H_2, \ldots, H_r$ for $\mathfrak{a}$. In other words, $Q = \frac{1}{2} B_\theta$ on $\mathfrak{n}$ and $B_\theta (= B)$ on $\mathfrak{a}$.

To complete the computation, it remains to calculate the traces tr ad V, V in the basis.

Since the vectors Y_i lie in root spaces it is easy to see from the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta$ is a root, or $\{0\}$ otherwise, that $tr \operatorname{ad} Y_i = 0$, $1 \le i \le q$. If $H \in \mathfrak{a}$, then $\operatorname{ad} H(Y) = \alpha(H) Y$ when $Y \in \mathfrak{g}_\alpha$. From this it follows that

$$tr \operatorname{ad} H|_{\mathfrak{s}} = \sum_{\alpha > 0} m_\alpha \alpha(H) = 2\rho(H) \tag{8.2}$$

— recall that $\rho(H) = \frac{1}{2}\sum_{\alpha>0} m_\alpha \alpha(H)$ and that $m_\alpha =$ dimension $\mathfrak{g}_\alpha$.

From (8.2) it follows that the linear term in (8.1) is

$$-\sum_{j=1}^{r}(tr\,\mathrm{ad}\,H_j)H_j = -\sum_{j=1}^{r}\sum_{\alpha>0} m_\alpha \alpha(H_j)H_j = -2H_\rho,$$

where H_ρ is the vector $\in \mathfrak{a}$ that represents $\rho : B(H_\rho, H) = \rho(H)$.

This completes the proof of

THEOREM 8.3. *The Laplace-Beltrami operator L_X corresponds to the left invariant operator $L = L_S$ on the group S, where*

$$L = 4\sum_{i=1}^{q} Y_i^2 + \sum_{j=1}^{r} H_j^2 - 2H_\rho. \tag{8.3}$$

COROLLARY 8.4. *Iwasawa radial part of L_X. Let f be an N-invariant function on $X = G/K$. Then*

$$(L_A f|_\mathfrak{a}) = \sum_{j=1}^{r} H_j^2 - 2H_\rho.$$

If $f(\exp H \cdot o) = \varphi(H)$, then

$$(L_A f|_\mathfrak{a}) = \triangle_\mathfrak{a}\varphi(H) - 2H_\rho \cdot \nabla\varphi(H). \tag{8.4}$$

PROOF. It suffices to show that if $Y \in \mathfrak{n}$ then $Yf = 0$. Since $na\exp tY = n\exp t\,\mathrm{Ad}(a)Ya$ and $\mathrm{Ad}(a)Y \in \mathfrak{n}$, $Yf(na.o) = \frac{d}{dt}f(na\exp tY.o)|_{t=0}) = 0$. ■

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