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LOCAL BIFURCATIONS OF CRITICAL PERIODS IN THE REDUCED KUKLES SYSTEM

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ABSTRACT. In this paper, we study the local bifurcations of critical periods in the neighborhood of a nondegenerate centre of the reduced Kukles system. We find at the same time the isochronous systems. We show that at most three local critical periods bifurcate from the Christopher-Lloyd centres of finite order, at most two from the linear isochrone and at most one critical period from the nonlinear isochrone. Moreover, in all cases, there exist perturbations which lead to the maximum number of critical periods. We determine the isochrones, using the method of Darboux: the linearizing transformation of an isochrone is derived from the expression of the first integral. Our approach is a combination of computational algebraic techniques (Gröbner bases, theory of the resultant, Sturm's algorithm), the theory of ideals of noetherian rings and the transversality theory of algebraic curves.

1. **Introduction.** The study of isochronous systems different from the linear isochrone goes back to Huygens who studied the cycloidal pendulum. This pendulum has isochronous oscillations in contrast with the monotonicity of the period of the usual pendulum. As soon as one looks at quadratic systems one finds centres for which the period is not monotonous but has some critical points [CD]. A global study of the number of critical points of the period is a very difficult question. A simpler question is the local problem of the number of critical periods which can appear by perturbation of a system in the neighborhood of a centre. This question is attacked by calculating the Taylor series of the period function in the neighborhood of the centre and by determining the order of its first non-constant term. This calculation is purely algorithmic. When it is performed on a polynomial family of vector fields the coefficients of the period function are polynomials in the coefficients of the system. The vanishing of the period coefficients then leads to questions like determining the Gröbner basis of an ideal of polynomials. Ultimately this local analysis allows us to determine the isochronous centres in the family, which are the ones for which all coefficients of the Taylor series vanish, except the first. More precisely this method yields necessary conditions for isochronicity. Their sufficiency is given by ad hoc methods.

The method described above has been used to study the local critical periods and the isochronous centres among quadratic systems [CJ], and among cubic systems symmetric with respect to a centre [RT]. In the quadratic (resp. cubic symmetric) case the necessary and sufficient conditions for isochronicity had been determined previously by Loud [L] (resp. Pleshkan [P]).

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Similarities were found in the quadratic and cubic symmetric systems with an isochronous centre at the origin. Indeed, in each case the system has a rational first integral and the two complex separatrices of the origin belong to different irreducible algebraic curves. Moreover the system is reversible in the sense of Żołądek [Z]. Also, regarding bifurcations of local periods, in each case we can attain a greater number of local critical periods starting from a weak centre than from an isochronous centre.

Starting from these remarks it is interesting to investigate other families of vector fields and to try to deduce general laws. One question which is of particular importance for us is to understand the mechanisms by which a centre is isochronous. This mechanism is completely hidden when one applies the algebraic method described above. There is no hope of characterizing geometrically, from the phase portraits alone, systems having isochronous centres since the phase portrait does not determine the velocity along the trajectories. Our hope is nevertheless to find necessary geometric conditions for isochronicity. When they are not satisfied we find obstructions to isochronicity.

There are very few families for which the centre conditions are known. The reduced Kukles system is one of these and this explains why we choose to study it.

Hence we consider, in this paper, the bifurcations of critical periods of periodic solutions in the neighborhood of a nondegenerate centre of the reduced Kukles system:

$$(K_0)\qquad \begin{aligned} \dot{x} &= -y \\ \dot{y} &= x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2. \end{aligned}$$

Necessary and sufficient conditions for the centre have been given by Christopher and Lloyd [CL]. Using the Gröbner Basis packages [B], [DST] on Maple V, Rousseau, Schlomiuk and Thibaudeau [RST] determined the basis of the ideal generated by the five first Lyapunov constants and were able to verify the Christopher-Lloyd conditions. We address the problem of the maximum number of critical periods bifurcating from the origin in (K_0) and solve it completely. There are four strata of centres for the reduced Kukles system, one consisting of quadratic systems and the remaining three consisting of truly cubic systems. For each stratum of cubic systems we calculate the coefficients of the period function. Only one of the strata contains nonlinear isochrones: these form a 1-parameter family. We show that at most three critical periods can bifurcate from the centres of finite order or from the linear isochrone and at most one from the nonlinear isochrones. We reduce the proof of the existence of perturbations leading to the maximum number of critical points to the proof that some algebraic curves have transversal intersections. Moreover, using Darboux's method [D], [Sc], we give necessary and sufficient conditions for the origin of the reduced Kukles system to be isochronous. We can also derive a linearizing transformation. As in the quadratic case and in the cubic system symmetric with respect to a centre, we note that the system with an isochronous centre has a rational first integral and that it is reversible in the sense of Żołądek [Z]. The complex separatrices of the origin again belong to two different algebraic curves.

One originality of the paper comes from the computer assisted proofs.

2. **Preliminaries.** Let $X(x, y, \lambda)$ be a family of plane analytic vector fields parametrized by $\lambda \in \mathbb{R}^n$ with a nondegenerate centre at the origin for every λ. Upon blowing up, the nondegenerate centre is replaced by a regular closed trajectory. The period function $P(x, \lambda)$ is then an analytic function of the coordinate x parametrizing the x-axis: $P(x, \lambda) = \sum_{k=0}^{\infty} p_k(\lambda)x^k$. Moreover, this function is even [MRT] and has the following Taylor series

$$P(x, \lambda) = 2\pi + \sum_{k=1}^{\infty} p_{2k}(\lambda)x^{2k}, \tag{2-1}$$

for $|x|$ and $|\lambda - \lambda_*|$ sufficiently small. The coefficients $p_k(\lambda)$ can be calculated by an algorithm via a symbolic manipulator such as Maple V. It can be shown that they are polynomials in the components of the bifurcation parameter λ [CJ]. By the Hilbert basis theorem there exists $N \in \mathbb{N}$ such that the ideal of all coefficients is finitely generated by the first N coefficients. Calculating the coefficients p_{2k} until we get $\langle p_2, \ldots, p_{2N}\rangle = \langle p_2, \ldots, p_{2(N+1)}\rangle$ leads to the conjecture that $p_2 = \cdots = p_{2N} = 0$ are sufficient conditions for isochronicity. One can then try to prove the isochronicity of the systems using ad hoc methods. In this case we find a linearizing transformation derived from the first integral.

DEFINITION 2.1. If $p_2 = p_4 = \cdots = p_{2k} = 0$ and $p_{2k+2} \neq 0$, then the origin is a *weak linear centre of finite order k*.

If $p_{2k} = 0$ for each $k \geq 1$, then the origin is of infinite order; it is an *isochronous* centre.

DEFINITION 2.2. *k local critical periods* bifurcate from the weak centre corresponding to the parameter λ_* if:

(1) for every $\alpha > 0$, sufficiently small, there exists a neighborhood W of λ_* such that for any $\lambda \in W$, $P(x, \lambda)$ has at most k critical points in $U = (0, \alpha)$.

(2) Moreover, any neighborhood of λ_* contains a point λ^1 such that $P(x, \lambda^1)$ has exactly k critical points in $U = (0, \alpha)$.

DEFINITION 2.3. Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of systems with a centre at the origin and period coefficients $p_{2k}(\lambda)$. The family satisfies condition (P) if for any $\lambda^* \in \Lambda$ such that $p_2(\lambda^*) = \cdots = p_{2k}(\lambda^*) = 0$, $p_{2k+2}(\lambda^*) \neq 0$ and any neighborhood $W \subset \Lambda$ of λ^* in which $p_{2k+2}(\lambda) \neq 0$ there exists $\lambda^1 \in W$ such that

$$\begin{aligned} p_{2k+2}(\lambda^1)p_{2k}(\lambda^1) &< 0 \\ p_2(\lambda^1) = \cdots = p_{2k-2}(\lambda^1) &= 0. \end{aligned} \tag{2-2}$$

The system X_{λ^*} is said to satisfy condition (P_k).

PROPOSITION 2.4 [T]. *Let $\{X_\lambda\}_{\lambda \in \Lambda}$ be a family of systems with a centre at the origin, satisfying condition (P) and X_{λ^*} satisfy condition (P_k). Then, for any sufficiently small $\epsilon > 0$, for any neighborhood V of λ^* and for any $0 \leq n \leq k$ there exists $\lambda^{**} \in V$ such that $X_{\lambda^{**}}$ has exactly n periodic orbits with critical periods passing through points $(x, 0)$, with $x \in (0, \epsilon)$.*

PROOF. Critical periods correspond to zeros of the derivative $Q(x, \lambda) = \frac{\partial P}{\partial x}(x, \lambda)$ of the period function with respect to x.

The proof goes by induction on k and n. It is obviously true for $k = 0$ or for $k > 0$ and $n = 0$.

We start with $\epsilon > 0$, sufficiently small so that $Q(x, \lambda^*) = (2k+2)p_{2k+2}x^{2k+1} + o(x^{2k+1})$ does not vanish for $0 < x < \epsilon$. For the sake of simplicity let us suppose $p_{2k+2}(\lambda^*) > 0$. By continuity of Q we can choose $x_1 \in (0, \epsilon)$ such that $Q(x, \lambda^*) > 0$ on $(0, x_1]$. Condition (P_k) allows us to choose λ^1 sufficiently close to λ^* so that $Q(x_1, \lambda^1) > 0$ and (2-2) is satisfied. Then there exists $0 < x_2 < x_1$ and such that $Q(x_2, \lambda^1) < 0$. Hence there exists x_1^* such that $Q(x_1^*, \lambda^1) = 0$. Moreover we can choose λ^1 so that $\frac{\partial Q}{\partial x}(x_1^*, \lambda^1) > 0$. Indeed, $x\frac{\partial Q}{\partial x}(x, \lambda^1) - (2k-1)Q(x, \lambda^1) = 2(2k+2)p_{2k+2}x^{2k+1} + o(x^{2k+1})$.

We use the induction hypothesis for $k-1$, $n-1$ and $\epsilon = x_1^*$. By the implicit function theorem it is possible to choose the neighborhood $V_1 \subset V$ of λ_1 sufficiently small so that the (structurally stable) root x_1^* persists under perturbation of the system. ∎

The two theorems below summarize our preliminaries; the first one is straightforward from the Weierstrass-Malgrange Preparation Theorem [Po], and the second one is based on the now classic derivation-division method. Both have been recalled and described in [T].

FINITE ORDER BIFURCATION THEOREM [CJ]. *No more than k local critical periods can bifurcate from weak centres of finite order k at the parameter value λ_*. Moreover, if the family satisfies the condition (P) and if X_{λ^*} satisfies the condition (P_k) then, for any $0 \le n \le k$, there are perturbations with exactly n local critical periods.*

ISOCHRONE BIFURCATION THEOREM [CJ]. *If the vector field X has an isochronous centre at the origin for the parameter value λ_* and if for each $m \ge 1$ the period coefficient p_{2m} is in the ideal $\langle p_2, \dots, p_{2k}, p_{2k+2}\rangle$ over the ring $\mathbb{R}\{\lambda_1, \dots, \lambda_n\}_{\lambda_*}$ of convergent power series at λ_*, then at most k local critical periods bifurcate from the isochronous centre at λ_*. Moreover, there are perturbations with exactly $n \le k$ local critical periods, if the family satisfies the condition (P) and if X_{λ^*} satisfies the condition (P_k).*

3. **The reduced Kukles system.** Consider the Kukles system

$$(K) \qquad \begin{aligned} \dot{x} &= -y \\ \dot{y} &= x + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3, \end{aligned}$$

with $a_7 = 0$, which we call (K_0). In this reduced form, necessary and sufficient conditions for a centre are known since 1944 [Ku] [CL].

CONDITIONS OF CENTRE [CL]. *System (K_0) has a centre at the origin if and only if the parameter value $\lambda = (a_1, a_2, a_3, a_4, a_5, a_6)$ is in one the following strata:*

$$K_I\colon a_4 = a_5 = a_6 = a_1 + a_3 = 0$$
$$K_{II}\colon a_4 + a_3(a_1 + a_3) = a_5 - a_2(a_1 + a_3) = a_6(a_1 + 2a_3) - a_3^2(a_1 + a_3) = 0$$
$$K_{III}\colon a_2 = a_5 = 0$$
$$K_{IV}\colon a_1 = a_3 = a_5 = 0.$$

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DEFINITION 3.1. We say that system (K_0) has a centre of type I (respectively II, III, IV) if the system is nonlinear and $\lambda \in K_I$ (respectively K_{II}, K_{III}, K_{IV}).

DISCUSSION OF THE CENTRE CONDITIONS. (1) Systems corresponding to parameters in K_I are quadratic systems, which have been analysed in [CJ].

(2) Systems corresponding to parameters in K_{II} have two invariant lines yielding an integrating factor and an elementary first integral.

(3) Systems corresponding to parameters in K_{III} are symmetric with respect to the x-axis, *i.e.* invariant under the transformation $(x, y, t) \longmapsto (x, -y, -t)$. They have a Liouvillian first integral which is generically not elementary.

(4) Systems corresponding to parameters in K_{IV} are symmetric with respect to the y-axis, *i.e.* invariant under the transformation $(x, y, t) \longmapsto (-x, y, -t)$. Generically they have no Liouvillian first integral.

(5) Systems corresponding to parameters in $K_{III} \cap K_{IV}$ have no quadratic terms. They are particular cases of the families studied in [RT].

Details can be found in [RST].

3.1. *Period function of the reduced Kukles system.* First, changing to polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ in system (K_0) and eliminating time yields the following differential equation

$$\frac{dr}{d\theta} = \frac{r^2 f_1(\theta,\lambda) + r^3 f_2(\theta,\lambda)}{1 + r g_1(\theta,\lambda) + r^2 g_2(\theta,\lambda)}, \tag{3-1}$$

where

$$\begin{aligned} f_1(\theta,\lambda) &= a_1\cos^2\theta\sin\theta + a_2\cos\theta\sin^2\theta + a_3\sin^3\theta,\\ f_2(\theta,\lambda) &= a_4\cos^3\theta\sin\theta + a_5\cos^2\theta\sin^2\theta + a_6\cos\theta\sin^3\theta,\\ g_1(\theta,\lambda) &= a_1\cos^3\theta + a_2\cos^2\theta\sin\theta + a_3\cos\theta\sin^2\theta,\\ g_2(\theta,\lambda) &= a_4\cos^4\theta + a_5\cos^3\theta\sin\theta + a_6\cos^2\theta\sin^2\theta. \end{aligned} \tag{3-2}$$

Let us denote by γ_ξ the closed trajectory through $(\xi, 0)$. The period function is then given by

$$\begin{aligned} P(\xi,\lambda) &= \int_{\gamma_\xi} dt\\ &= \int_0^{2\pi} \frac{d\theta}{1 + r(\theta,\xi,\lambda)g_1(\theta,\lambda) + r^2(\theta,\xi,\lambda)g_2(\theta,\lambda)}, \end{aligned} \tag{3-3}$$

where $\theta \longmapsto r(\theta,\xi,\lambda)$ is the solution of the differential equation (3-1) with initial condition $r(0,\lambda) = \xi$. It is known that $r(\theta,\xi,\lambda)$ may be locally represented as a convergent power series in ξ:

$$r(\theta,\xi,\lambda) = \sum_{k\geq 1} u_k(\theta,\lambda)\xi^k, \tag{3-4}$$

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Substituting (3-4) in (3-1) yields a sequence of linear differential equations satisfied by the coefficients $u_k(\theta,\lambda)$ in (3-4):

$$\begin{aligned} u_1'(\theta,\lambda) &= 0, \\ u_k'(\theta,\lambda) &= \sum_{i=1}^{k-1} u_i(f_1 u_{k-i} + f_2 v_{k-i}) - u_{k-i}'(g_1 u_i + g_2 v_i), \end{aligned} \tag{3-5}$$

with

$$v_1 = 0, \quad v_k = \sum_{i=1}^{k-1} u_i u_{k-i}, \tag{3-6}$$

and initial conditions $u_1(0,\lambda) = 1$, $u_i(0,\lambda) = 0$ for $i > 1$. This yields in particular $u_1(\theta,\lambda) \equiv 0$. The integrand in (3-3) is analytic for $\theta \in [0, 2\pi]$ and $|\xi|$ sufficiently small. Thus we may write locally:

$$P(\xi,\lambda) = 2\pi + \sum_{k\geq 1} p_k(\lambda)\xi^k = \sum_{k\geq 0}\Big(\int_0^{2\pi} A_k(\theta,\lambda)\, d\theta\Big)\xi^k. \tag{3-7}$$

Therefore the period coefficients $p_k(\lambda)$ can be obtained by integrating the terms $A_k(\theta,\lambda)$ given by:

$$\begin{aligned} A_0(\theta,\lambda) &\equiv 1 \\ A_k(\theta,\lambda) &= -\Big[g_1 u_k + \sum_{i=1}^{k-1}\big(g_2 u_i u_{k-i} + (g_1 u_i + g_2 v_i)A_{k-i}\big)\Big]. \end{aligned} \tag{3-8}$$

REMARK. Note that the coefficients $p_k(\lambda)$ can be calculated for a weak focus as well as for a centre. In the former case they represent the coefficients of the time to go from an initial condition $(\xi, 0)$ to the first return $Q(\xi, 0)$. We also call them the period coefficients.

3.2. *The first period coefficients.* From (3-1), (3-7) and (3-8) we derive the period coefficients $p_k(\lambda)$, and in particular the following lemma. We use a simple program on Maple V.

LEMMA 3.2. *The period coefficient p_2 for the system (K_0) is given by*

$$p_2(\lambda_*) = a_2^2 + 10a_1^2 + 10a_1a_3 + 4a_3^2 - 9a_4 - 3a_6. \tag{3-9}$$

PROOF. From (3-8) we derive $A_1(\theta,\lambda) = -g_1u_1 = -g_1$. Since g_1 is an odd homogeneous trigonometric polynomial we get that $p_1(\lambda) = 0$. To calculate $p_2(\lambda)$ we consider

$$\begin{aligned} A_2(\theta,\lambda) &= -g_1u_2 - g_2u_1^2 - g_1u_1A_1 - g_2v_1A_1 \\ &= -g_1u_2 - g_2 + g_1^2. \end{aligned} \tag{3-10}$$

Equation (3-5) yields

$$u_2'(\theta,\lambda) = f_1(\theta,\lambda), \tag{3-11}$$

yielding

$$u_2(\theta,\lambda) = \int_0^\theta f_1(\tau,\lambda)\,d\tau. \tag{3-12}$$

Hence $u_2(\theta)$ can be written as a homogenous trigonometric polynomial of degree 3. Then

$$p_2(\lambda) = \int_0^{2\pi} -g_1(\theta,\lambda)u_2(\theta,\lambda) - g_2(\theta,\lambda) + g_1^2(\theta,\lambda)\,d\theta. \tag{3-13}$$

■

3.3. *Centre of type I.* The corresponding system is quadratic; thus, using the results in [CJ], this centre is of order at most two; but from the expression of p_2 in this case, we can conclude that no critical period can bifurcate from this centre; let us note that the stratum K_I is entirely included in K_{II}.

3.4. *Centre of type II.* We prove the following result.

THEOREM 3.3. *(1) A centre of type II may be at the intersection of strata K_I and K_{II} as well as at the intersection of K_{II} and K_{IV}. In the first case, the centre is of order zero; in the second case, the order is less than or equal to two.*

(2) At most two critical periods can bifurcate from a weak centre of type II.

(3) Moreover, each centre of order two has perturbations with exactly two local critical periods.

PROOF. We consider a nonzero parameter value λ_* on the stratum K_{II}. We then can split the problem into two cases: $a_1 + 2a_3 = 0$ and $a_1 + 2a_3 \neq 0$. In each case we analyze the period coefficients reduced modulo the ideal of the previous coefficients.

CASE 1. Assuming $a_1 + 2a_3 = 0$ with $a_3 = 0$ leads to a substratum of K_{IV}:

$$a_1 = a_3 = a_5 = a_4 = 0.$$

The three first period coefficients are given by:

$$\begin{aligned} p_2 &= a_2^2 - 3a_6 \\ p_4 &= 7(a_2^2 - 3a_6)(a_2^2 - 9a_6) \\ p_6 &= 23088a_2^6. \end{aligned} \tag{3-14}$$

Hence $p_2 = 0$ if and only if $a_2^2 = 3a_6$. For $a_6 < 0$ the first period coefficient p_2 is nonzero; therefore the corresponding centre is of order zero and no local critical period can bifurcate from it.

For $p_2 = 0$ and $a_6 > 0$, we get $p_4 = 0$ and $p_6 > 0$, yielding a centre of order at most $k = 2$. A perturbation with two critical periods is shown in the case of centres of type K_{IV}.

CASE 2. Assuming $a_1 + 2a_3 \neq 0$, we may take $a_1 + 2a_3 = 1$ without loss of generality. From Lemma 3.2 we get

$$p_2 = 3a_3^3 + 12a_3^2 - 21a_3 + 10 + a_2^2, \tag{3-15}$$

Therefore

$$p_2 = 0 \text{ if and only if } a_2^2 = b(a_3) = -3a_3^3 - 12a_3^2 + 21a_3 - 10 \geq 0 \tag{3-16}$$

We compute the period coefficients p_4 and p_6 modulo p_2 and obtain

$$\begin{aligned} p_4(\lambda_*) &= 18(a_3 - 1)^2 h_4(a_3) \\ p_6(\lambda_*) &= 6912(a_3 - 1)^4 h_6(a_3) \end{aligned} \tag{3-17}$$

with

$$\begin{aligned} h_4(a_3) &= -9a_3^4 - 87a_3^3 + 121a_3^2 - 209a_3 + 160 \\ h_6(a_3) &= a_3^5 - 8a_3^4 + 508a_3^3 + 1130a_3^2 - 7225a_3 + 5450. \end{aligned} \tag{3-18}$$

From our assumptions, we necessarily have $a_1 \neq 1$. Also,

$$\text{resultant}\big(h_4(a_3), h_6(a_3)\big) = 3130572800 \neq 0.$$

Therefore, p_4 and p_6 modulo p_2 have no common zero, *i.e.*, p_6 is nonzero on the variety $V(p_2, p_4)$. By the Finite Order Bifurcation Theorem, at most two critical periods can bifurcate from the corresponding weak centre.

A perturbation with exactly two local critical periods is easily constructed in the following way. By the intermediate value theorem, h_4 has a zero a_3^* in the interval

$$J = \left]a_3^1, a_3^2\right[= \left]\frac{-110820}{10000}, \frac{-110815}{10000}\right[. \tag{3-19}$$

Moreover, $b(a_3^*) > 0$ on J; indeed, we have $b(a_3^1) > 0$ and $b(a_3^2) > 0$, and we prove, using Sturm's algorithm [K], that $b(a_3)$ is nonzero on the interval J. A similar argument leads to $h_6 < 0$ on J, and p_6 as well.

We perturb in two steps in the standard way. ∎

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3.5. *Centre of type III.*

THEOREM 3.4. *A weak centre of type III is of order at most 3. Moreover, any such centre of order 3 has a perturbation with exactly n local critical periods, for all* $n \leq 3$.

PROOF. The proof goes into two parts.

PART 1. Let us assume $a_1 = 0$, and denote by λ_*^1 the corresponding value of the parameter. Then, we have:

$$
\begin{aligned}
p_2(\lambda_*^1) &= 4a_3^2 - 9a_4 - 3a_6 \\
p_4(\lambda_*^1) &= 27a_4^2 + 21a_3^2a_4 - 32a_3^4 \\
p_6(\lambda_*^1) &= -7290a_4^3 + 118746a_3^2a_4^2 - 68157a_3^4a_4 + 34016a_3^6,
\end{aligned}
\tag{3-20}
$$

where p_4 and p_6 are reduced modulo p_2. We note that p_4 is a polynomial of degree 2 in a_4, with discriminant: $D = 3897a_3^4$.

(1) This discriminant vanishes at $a_3 = 0$, leading to $p_4 = 27a_4^2 \neq 0$, for a_4 is necessarily nonzero. Therefore, the corresponding centre, *i.e.*, $a_1 = a_3 = 0, \quad a_4 \neq 0$ is of order at most one. A perturbation with exactly one local critical period is easily constructed.

Let us note that such a centre is at the intersection of the strata K_{III} and K_{IV}.

(2) The above discriminant is strictly positive for $a_3 \neq 0$; looking for the roots of p_4, we get

$$r_\pm = \frac{-7\pm\sqrt{433}}{18}a_3^2.$$

But, we have resultant $(p_4, p_6, a_4) = a_3^{12}$, modulo a nonzero constant. The periods coefficients p_4 and p_6 have thus no common roots in a_3; in other words, for $a_4 = r_\pm(a_3)$, we obtain

$$p_4(a_3, a_4) = 0, \quad p_6(a_3, a_4) \neq 0.$$

Therefore, the corresponding centre is of order at most two and such a centre of order two can be perturbed in the standard way to produce exactly two local critical periods.

PART 2. Next, we study the case $a_1 \neq 0$; we may then, without loss of generality, assume $a_1 = 1$. Let us denote by λ_*^3 the associated value of the parameter. Using the relations (3-7) and (3-8) to compute the corresponding period coefficients modulo p_2, we get:

$$
\begin{aligned}
p_2(\lambda_*^3) &= 4a_3^2 + 10a_3 + 10 - 9a_4 - 3a_6 \\
p_4(\lambda_*^3) &= 108a_4^2 + c_4^1(a_3)a_4 + c_4^2(a_3) \\
p_6(\lambda_*^3) &= -7290a_4^3 + c_6^1(a_3)a_4^2 + c_6^2(a_3)a_4 + c_6^3(a_3) \\
p_8(\lambda_*^3) &= 75246796800a_4^4 + c_8^1(a_3)a_4^3 + c_8^2(a_3)a_4^2 + c_8^3(a_3)a_4 + c_8^4(a_3),
\end{aligned}
\tag{3-21}
$$

with

$$\begin{aligned} c_4^1(a_3) &= 57a_3^2 - 402a_3 - 519 \\ c_4^2(a_3) &= -114a_3^4 - 457a_3^3 - 645a_3^2 - 195a_3 + 115 \\ c_6^1(a_3) &= 118746a_3^2 + 217404a_3 + 176094 \\ c_6^2(a_3) &= -68157a_3^4 - 232704a_3^3 - 438210a_3^2 - 404640a_3 - 176265 \\ c_6^3(a_3) &= 34016a_3^6 + 215880a_3^5 + 585210a_3^4 + 852920a_3^3 \\ &\quad + 723900a_3^2 + 340800a_3 + 73850; \end{aligned}$$

and finally

$$\begin{aligned} c_8^1(a_3) &= 572961807360a_3^2 - 1284204533760a_3 - 1829799797760 \\ c_8^2(a_3) &= -1285543180800a_3^4 - 4177321583616a_3^3 - 3345808011264a_3^2 \\ &\quad + 3848748880896a_3 + 3797890398720 \\ c_8^3(a_3) &= 384439336620a_3^6 + 3525901880472a_3^5 + 12693308746644a_3^4 \\ &\quad + 22123313129040a_3^3 + 18546928501140a_3^2 + 5004955583640a_3 \\ &\quad - 653908329300 \\ c_8^4(a_3) &= -45591948115a_3^8 - 485200347316a_3^7 - 2196106467040a_3^6 \\ &\quad - 6007981449460a_3^5 - 10914080051470a_3^4 - 12847208098300a_3^3 \\ &\quad - 8891765452600a_3^2 - 2876423451100a_3 - 223379079175. \end{aligned}$$

We then analyse the following equations:

$$p_4(a_3, a_4) = 0; \quad p_6(a_3, a_4) = 0; \quad p_8(a_3, a_4) = 0.$$

Let (C_4), (C_6), (C_8) be the algebraic curves defined respectively by the previous equations. Using the theory of resultants, we show that the previous system has no solution; indeed, denote

(3-22) $$R_{46} = \text{resultant}(p_4, p_6, a_4), \quad R_{68} = \text{resultant}(p_6, p_8, a_4),$$

with p_4, p_6 and p_8 in $\mathbb{R}[a_3][a_4]$. Hence R_{46} and R_{68} are in $\mathbb{R}[a_3]$. We obtain resultant $(R_{46}, R_{68}, a_3) \neq 0$. From this, it follows that the polynomials p_4, p_6 and p_8 have no common root; the corresponding centre is of order at most three. We show that there exist centres of order $k = 3$. It suffices to prove that the system $p_4 = p_6 = 0$ has at least one real root at which the curves (C_4) and (C_6) have a transversal intersection; therefore the corresponding parameter value may be perturbed into a certain $\tilde{\lambda}_*^3$ satisfying

$$\begin{aligned} p_2(\tilde{\lambda}_*^3) &= 0 \\ p_4(\tilde{\lambda}_*^3) \times p_6(\tilde{\lambda}_*^3) &< 0 \\ p_6(\tilde{\lambda}_*^3) \times p_8(\tilde{\lambda}_*^3) &< 0, \end{aligned}$$

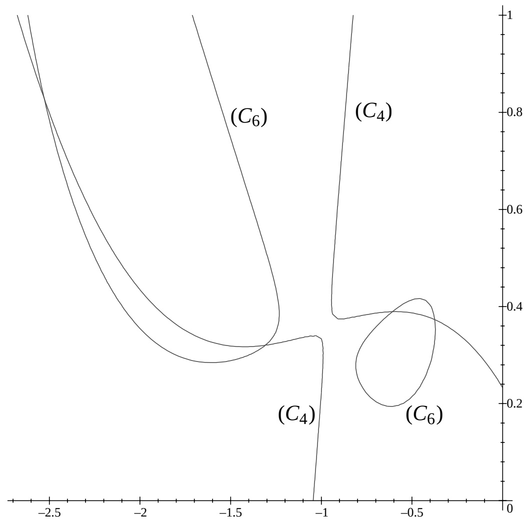

FIGURE 1: Transversal intersection of curves (C_4) and (C_6)

and leading to condition (P_3). This is done as a computer assisted proof. First, using a computer algebra system such as *Mathematica*, we represent the algebraic curves (C_4) and (C_6) in Figure 1 below.

This representation suggests the existence of real roots in some rectangles, one of them being

$$R_1^0 = \left[-\frac{4}{10}, -\frac{3}{10}\right] \times \left[\frac{35}{100}, \frac{4}{10}\right]. \tag{3-23}$$

We will show (using Maple V) that the two curves (C_4) and (C_6) intersect transversally in R_1^0 and that $p_8 > 0$ on R_1^0. For this purpose we will do an extensive use of Sturm's algorithm and discriminants. Moreover the different polynomial functions will only be evaluated at rational points.

The proof goes as follows: we define the following one variable polynomials:

$$f_m^1(a_3) = p_m\left(a_3, \frac{35}{100}\right), \quad f_m^2(a_3) = p_m\left(a_3, \frac{4}{10}\right);$$
$$g_m^1(a_4) = p_m\left(-\frac{4}{10}, a_4\right), \quad g_m^2(a_4) = p_m\left(-\frac{3}{10}, a_4\right),$$

for $m = 4, 6, 8$. By Sturm's algorithm, the polynomials f_4^i, $i = 1, 2$ in the variable a_3 have no real root in the interval $(-\frac{4}{10}, -\frac{3}{10}]$; but the polynomials f_6^i, $i = 1, 2$ have exactly one real root in the same interval. On the other hand the polynomials g_4^i, $i = 1, 2$ have one unique real root in $(\frac{35}{100}, \frac{4}{10}]$, while the polynomials g_6^i, $i = 1, 2$ are nonzero.

Finally, there will exist at least one real intersection denoted by $M_0 = (a_3^0, a_4^0)$ in R_1^0, if we show that the curves (C_4) and (C_6) are non singular, *i.e.* that the systems

$$p_j(a_3, a_4) = 0; \quad \frac{\partial p_j}{\partial a_3}(a_3, a_4) = 0; \quad \frac{\partial p_j}{\partial a_4}(a_3, a_4) = 0, \tag{3-24}$$

for $j = 4, 6$, have no real solution. We only need to show that

$$\text{(3-25)} \qquad \begin{aligned} \mathrm{Disc}_4(a_4) &= \text{discriminant}\,(p_4, a_3) \neq 0 \text{ on } \Big(-\frac{4}{10}, -\frac{3}{10}\Big] \\ \mathrm{Disc}_6(a_3) &= \text{discriminant}\,(p_6, a_4) \neq 0 \text{ on } \Big(\frac{35}{100}, \frac{4}{10}\Big]. \end{aligned}$$

This is again checked by Sturm's algorithm.

Moreover, the polynomial p_8 has no root on the rectangle R_1^0. Indeed, f_8^i and g_8^i do not vanish on the sides of this rectangle and p_8, for a_3 then a_4 fixed, has no multiple zero respectively in the intervals $(\frac{35}{100}, \frac{4}{10}]$ and $(-\frac{4}{10}, -\frac{3}{10}]$. From $p_8(-\frac{35}{100}, \frac{38}{100}) > 0$ it follows that $p_8 > 0$ on R_1^0. So we have a parameter value $\tilde{\lambda}_*^3$ associated to the zero M_0 such that

$$\text{(3-26)} \qquad p_2(\tilde{\lambda}_*^3) = p_4(\tilde{\lambda}_*^3) = p_6(\tilde{\lambda}_*^3) = 0; \quad p_8(\tilde{\lambda}_*^3) > 0.$$

We may conclude by showing that the curves $p_4(a_3, a_4) = 0$; $p_6(a_3, a_4) = 0$ intersect transversally at the real root M_0; in other words, the system

$$\text{(3-27)} \qquad \begin{aligned} p_4(a_3, a_4) = 0; \quad p_6(a_3, a_4) = 0 \\ T(a_3, a_4) = \frac{\partial p_4}{\partial a_3}\frac{\partial p_6}{\partial a_4} - \frac{\partial p_4}{\partial a_4}\frac{\partial p_6}{\partial a_3} = 0 \end{aligned}$$

has no real solution. In fact, one must only prove:

(1) $T(a_3, a_4)$ does not vanish on the boundary of the previous rectangle R_1^0; this amounts to showing, using, once again, Sturm's algorithm, that the polynomials $T_1(a_3) = T(a_3, \frac{35}{100})$ and $T_2(a_3) = T(a_3, \frac{4}{10})$ are nonzero on $(-\frac{4}{10}, -\frac{3}{10}]$; similarly, $T_3(a_4) = T(-\frac{4}{10}, a_4)$ and $T_4(a_4) = T(-\frac{3}{10}, a_4)$ are nonzero on $(\frac{35}{100}, \frac{4}{10}]$.

(2) The curve $T(a_3, a_4) = 0$ has no closed component in R_1^0: $\mathrm{Disc}(a_4) =$ discriminant (T, a_3), and $\mathrm{Disc}(a_3) =$ discriminant (T, a_4) have no real zeros respectively on $(\frac{35}{100}, \frac{4}{10}]$ and on $(-\frac{4}{10}, -\frac{3}{10}]$.

We prove in fact more than the transversality of the curves (C_4) and (C_6). We have shown that

$$\text{(3-28)} \qquad \frac{\partial p_4}{\partial a_4} \neq 0; \quad \frac{\partial p_6}{\partial a_3} \neq 0 \text{ and } \frac{\partial p_2}{\partial a_4} \neq 0.$$

The combination of the previous results leads to the following successive perturbations: first, perturb $\tilde{\lambda}_*^3$ into $\hat{\lambda}_*^3$ to get

$$p_2(\hat{\lambda}_*^3) = p_4(\hat{\lambda}_*^3) = 0; \quad p_6(\hat{\lambda}_*^3) < 0; \quad p_8(\hat{\lambda}_*^3) > 0.$$

For that we choose δ_3 sufficiently small,

$$\hat{a}_3 = a_3 + \delta_3; \quad \hat{a}_4 = a_4 + \delta_4(\delta_3); \quad \hat{a}_6 = a_6 + \delta_6(\delta_3, \delta_4),$$

to stay on the surfaces $p_4 = 0$ and $p_2 = 0$. We then take a perturbation $\overline{\lambda_*^3}$ of $\hat{\lambda}_*^3$ with components

$$\overline{a_4} = \hat{a}_4 + \epsilon_4; \quad \overline{a_6} = \hat{a}_6 + \epsilon(\epsilon_4),$$

with $|\epsilon_4| \ll |\delta_3|$ such that

$$p_2(\overline{\lambda_*^3}) = 0; \quad p_4(\overline{\lambda_*^3}) > 0; \quad p_6(\overline{\lambda_*^3}) < 0; \quad p_8(\overline{\lambda_*^3}) > 0.$$

Finally, in the neighborhood of $\overline{\lambda_*^3}$ we may choose $\check{\lambda}_*^3$ with

$$\check{a}_6 = \overline{a_6} + \eta_6, \quad |\eta_6| \ll \min(|\overline{a_6}|, |\epsilon_4|), \quad \eta_6 > 0,$$

for $a_6 > 0$ on the defined rectangle, allowing to realize condition (P_3): three local critical periods are then present at the weak centre associated to $\check{\lambda}_*^3$. ■

3.6. *Centre of type IV.* This case leads to another interesting result: the appearance of a nonlinear isochronous centre and the use of a linearizing transformation to establish the isochronicity.

The corresponding system is written in the form:

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x + a_2xy + a_4x^3 + a_6xy^2. \end{aligned} \tag{3-29}$$

The associated period coefficients are derived from formulas (3-7) and (3-8) via Maple V. The results are given in the following lemma.

LEMMA 3.5. *The first three periods coefficients associated to a weak centre of type IV are:*

$$\begin{aligned} p_2(\lambda_*) &= a_2^2 - 9a_4 - 3a_6 \\ p_4(\lambda_*) &= 7a_2^4 - 324a_2^2a_4 + 2349a_4^2 - 84a_2^2a_6 + 918a_4a_6 + 189a_6^2 \\ p_6(\lambda_*) &= -2579a_2^6 - 29493a_2^4a_4 + 2192346a_2^2a_4^2 - 1546090a_4^3 \\ &\quad + 8841a_2^4a_6 + 764316a_2^2a_4a_6 - 7899930a_4^2a_6 \\ &\quad + 79434a_2^2a_6^2 - 1888110a_4a_6^2 - 225150a_6^3. \end{aligned} \tag{3-30}$$

With this lemma, we prove the following result.

THEOREM 3.6. *The reduced Kukles system has an isochronous centre at the origin if and only if*

$$a_1 = a_3 = a_5 = a_6 = 0 = a_2^2 - 9a_4, \tag{3-31}$$

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in other words, if and only if the system is of the form

(3-32)
$$\dot{x} = -y$$
$$\dot{y} = x + a_2xy + \frac{a_2^2}{9}x^3.$$

A first integral of (3-32) is given by

(3-33)
$$F(x,y) = \frac{(a_2^2x^2 + 3a_2y + 9)^2}{a_2^2x^2 + 6a_2y + 9}.$$

PROOF. We have seen already that there exist no nonlinear isochrones in the strata of centres of type I, II or III. The proof is based on the reduction of coefficients $p_{2n}(\lambda_*)$ for centres of type IV. They have been calculated in Lemma 3.5. Modulo p_2 given in (3-30), p_4 is reduced to $p_4 = a_4a_6$, up to a multiplicative constant. We then reduce p_6 in the Gröbner basis of p_2, p_4. We obtain $p_6 = -a_6^3$, up to a multiplicative constant. Analyzing the system

$$p_2 = p_4 = 0,$$

amounts to the study of only two cases:

CASE 1.

(3-34)
$$a_4 = 0; \quad a_2^2 = 3a_6, \quad a_6 > 0.$$

Let us denote by λ_*^2 the induced parameter value. Then

(3-35)
$$p_2(\lambda_*^2) = p_4(\lambda_*^2) = 0; \quad p_6(\lambda_*^2) = -a_6^3 \neq 0.$$

Hence there are no nonlinear isochrones in this case.

CASE 2.

(3-36)
$$a_6 = 0, \quad a_2^2 = 9a_4, \quad a_4 > 0.$$

Denoting by λ_*^3 the induced parameter value we obtain:

(3-37)
$$p_2(\lambda_*^3) = p_4(\lambda_*^3) = p_6(\lambda_*^3) = 0.$$

We show the isochronicity of the centre of the corresponding system (3-36). The method described in [MRT] shows that the knowledge of a first integral of a system suggests linearizing changes of coordinates. Hence, in this case, we first look for a first integral. This first integral suggests a change of coordinates, which transforms our system into a well known quadratic isochronous system.

Before looking for a first integral we first simplify the system (3-32), using the transformation $(u,v) \longmapsto (a_2x, a_2y)$, followed by $(u^2, v, t) \longmapsto (x_1, y_1, ut)$. The system (3-32) is transformed into

$$\begin{aligned} \dot{u} &= -v \\ \dot{v} &= u + uv + \frac{u^3}{9}. \end{aligned} \tag{3-38}$$

and

$$\begin{aligned} \dot{x}_1 &= -2y_1 \\ \dot{y}_1 &= 1 + y_1 + \frac{x_1}{9}. \end{aligned} \tag{3-39}$$

The last one has two invariant lines, yielding a Darboux first integral of the form ([RS]):

$$f(x_1, y_1) = (x_1 + 3y_1 + 9)^2(x_1 + 6y_1 + 9)^{-1}. \tag{3-40}$$

Therefore, a first integral for system (3-38) is written:

$$G(u,v) = (u^2 + 3v + 9)^2(u^2 + 6v + 9)^{-1}. \tag{3-41}$$

This form suggests the following change of coordinates $(X,Y) = (\frac{u}{3}, \frac{u^2+3v}{9})$, under which system (3-38) may be rewritten as:

$$\begin{aligned} \dot{X} &= -Y + X^2 \\ \dot{Y} &= X + XY. \end{aligned} \tag{3-42}$$

which, in polar coordinates, is transformed to $\dot{\theta} = 1$. The original system is then isochronous. ■

THEOREM 3.7. *A weak centre of type IV is, either of order at most two, or a nonlinear isochrone. Moreover:*

(1) A centre of order two may be perturbed to produce exactly two local critical periods.

(2) A centre of type IV may be at the intersection of different strata: on $K_{II} \cap K_{IV}$, it is of order at most two; on $K_{III} \cap K_{IV}$, the order is at most one and one local critical period may bifurcate from it.

PROOF. The previous study has shown (Theorem 3.4) that the centre corresponding to $a_1 = a_2 = a_3 = a_5 = 0$ is at the intersection of the strata III and IV. It is of order at most one and can be perturbed to produce exactly one local critical period.

We have shown in Theorem 3.6 that weak centres of order greater or equal to two can only be found in Case 1, *i.e.* $a_4 = 0 = a_2^2 - 3a_6$, $a_6 > 0$, in which case we have (3-35).

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It is easily checked that the weak centre is of order exactly two and we can construct a perturbation with exactly two local critical periods. ∎

We start the study of the bifurcations of critical periods from the isochrones with a few comments.

REMARKS. (1) $K_I, K_{II}, K_{III}, K_{IV}$ are closed sets and intersect at the origin of $\mathbb{R}^6$.

(2) K_I is included in K_{II}. Strata K_{II} and K_{IV} intersect at $a_1 = a_3 = a_5 = a_4 = 0$; and the intersection of K_{III} and K_{IV} give $a_1 = a_2 = a_3 = a_5 = 0, \quad a_4 \neq 0, \quad a_6 \neq 0$. But respectively K_I and K_{IV}, K_{II} and K_{III} have no nonzero intersection.

(3) Any perturbation $\tilde{\lambda}_*$ of $\lambda_* \in K_i \cap K_j$, stays either in K_i or in K_j. We must then consider perturbations of the linear isochrone into each of the strata K_j, $j = I, II, III, IV$, keeping in mind that the nonlinear isochronous point lies uniquely in the stratum K_{IV}.

We then prove the following results.

THEOREM 3.8. *Any perturbation of the linear isochrone may produce at most three local critical periods.*

The proof uses Propositions 3.9 and 3.10 below.

PROPOSITION 3.9. *(1) Any perturbation of the linear isochrone into centres of type I produces no critical period.*

(2) At most two (respectively three) local critical periods can bifurcate from a perturbation of the linear isochrone into centres of types II (respectively of type III), but outside of the stratum K_{IV}.

PROOF. These results come from the bifurcation of a linear isochrone into a stratum with no other isochrones and from the study of centres of type I, II, III exclusively. ∎

The above remarks stated the possible perturbations in the stratum K_{IV}, which is the only stratum to contain nonlinear isochrones. We prove the following.

PROPOSITION 3.10. *(1) The ideal $\mathcal{M}$ of the period coefficient p_{2k}, $k \geq 1$ is finitely generated by the first three coefficients: $\mathcal{M} = (p_2, p_4, p_6)$ over the noetherian ring $\mathbb{R}[a_2, a_4, a_6]$ of polynomials in the variables a_2, a_4, a_6.*

(2) At most two critical periods can bifurcate from the origin in a perturbation of the linear system into the family K_{IV}; moreover, there exist perturbations leading to the maximum number of critical periods.

The proof uses the lemma below.

LEMMA 3.11. *For every $k \geq 1$, the exponents of a_2 in the coefficients p_{2k} are even.*

The proof is straightforward using the transformation $(x, y) \longmapsto (a_2 x, y a_2 y), \quad a_2 \neq 0$.

PROOF OF PROPOSITION 3.10. Denote $\mathcal{I} = \langle p_2, p_4, p_6 \rangle$ the ideal generated by p_2, p_4, p_6.

Every period coefficient p_{2k}, $k \geq 1$ is a polynomial in the variables $b_2 = a_2^2$, a_4, a_6 of degree k; it then may be written:

$$p_{2k} = R_1(b_2, a_4, a_6) + R_2(a_4, a_6), \tag{3-43}$$

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where R_1 is the sum of all terms containing b_2 and R_2 is the sum of the remaining terms. From $p_2 = a_2^2 - 9a_4 - 3a_6$, *i.e.*, $a_2^2 = b_2 = p_2 + 9a_4 + 3a_6$, we may rewrite p_{2k} as

$$
\begin{aligned}
p_{2k} &= p_2 S_2(b_2, a_4, a_6) + T_4(a_4, a_6) + R_2(a_4, a_6) \\
&= p_2 S_2(b_2, a_4, a_6) + R_4(a_4, a_6).
\end{aligned} \tag{3-44}
$$

We can write $R_4(a_4, a_6)$ as

$$R_4(a_4, a_6) = a_4 a_6 S_4(a_4, a_6) + R_8(a_4, a_6),$$

where R_8 is the sum of all terms in only a_4^i, a_6^i, $1 \leq i \leq k$, with no term containing $a_4 a_6$. Therefore, we obtain

$$p_{2k} = p_2 S_2(a_2, a_4, a_6) + a_4 a_6 S_4(a_4, a_6) + R_8(a_4, a_6). \tag{3-45}$$

We know that the system is isochronous, *i.e.*, $p_{2k} = 0$, $k \geq 1$, for $a_6 = p_2 = 0$. Thus a_6 divides $R_8(a_4, a_6)$ yielding that R_8 is a polynomial in a_6 only.

It is only interesting to prove that p_{2k} is in the ideal $\langle p_2, p_4, p_6 \rangle$ for $k \geq 4$. In that case R_8 is of degree greater or equal to 4 and may then be written:

$$
\begin{aligned}
R_8(a_4, a_6) = R_8(a_6) &= a_6^3 S_6(a_6) \\
&= -p_6 S_6(a_6).
\end{aligned} \tag{3-46}
$$

Finally, we get:

$$p_{2k} = p_2 S_2(a_2, a_4, a_6) + p_4 S_4(a_4, a_6) - p_6 S_6(a_6). \tag{3-47}$$

In other words, for every $k \geq 1$, p_{2k} is in the ideal I over the ring $\mathbb{R}[a_2, a_4, a_6]$.

We have $p_{2k} \in I = \langle p_2, p_4, p_6 \rangle$; by the Isochrone Bifurcation Theorem, we may conclude that at most two critical periods can bifurcate from a perturbation of a linear isochrone into the family K_{IV}. As in the previous case, one may construct a perturbation giving birth to exactly two critical periods. ■

We next move to the case of a perturbation of a nonlinear isochrone and prove the following theorem.

THEOREM 3.12. *Any perturbation of a nonlinear isochrone into the family K_{IV} can produce at most one critical period; in addition, there exist perturbations with exactly one critical period.*

PROOF. Denote $\tilde{\lambda}_*^4 = \lambda_*^4 + \delta$ the perturbation into K_{IV}, where λ_*^4 corresponds to a nonlinear isochronous centre, *i.e.*, meets the conditions

$$a_1 = a_3 = a_5 = a_6 = 0 \text{ with } a_2^2 = 9a_4 > 0.$$

$\tilde{\lambda}_*^4$ has components $\tilde{a}_6 = \delta_6$; $\tilde{a}_4 = a_4 + \delta_4$; $\tilde{a}_2 = a_2 + \delta_2$.

Let us call $\tilde{p}_{2k}(\tilde{\lambda}_*^4)$ the perturbed period coefficients. The result will be straightforward from the Isochrone Bifurcation Theorem if we show that $\tilde{p}_6(\tilde{\lambda}_*^4)$ is in the ideal $\tilde{I} = \langle \tilde{p}_2, \tilde{p}_4 \rangle$

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over the noetherian ring $\mathbb{R}\{a_2, a_4, a_6\}_{\lambda_*^4}$ of convergent power series at λ_*^4; for, then, the ideal M will be finitely generated by $\tilde{I}$, from the part (1) of Proposition 3.10.

Using the expressions of the period coefficients related to K_{IV}, we get:

$$\begin{aligned} \tilde{p}_2(\tilde{\lambda}_*^4) &= \tilde{a}_2^2 - 9\tilde{a}_4 - 3\tilde{a}_6 \\ &= (a_2+\delta_2)^2 - 9(a_4+\delta_4) - 3(a_6+\delta_6) \\ &= \delta_2^2 + 2a_2\delta_2 - 9\delta_4 - 3\delta_6. \end{aligned} \tag{3-48}$$

Modulo $\tilde{p}_2$, *i.e.*, $\tilde{a}_2^2 = 9\tilde{a}_4 + 3\tilde{a}_6$, we obtain:

$$\begin{aligned} \tilde{p}_4(\tilde{\lambda}_*^4) &= -432\tilde{a}_4\tilde{a}_6 \\ &= -432(a_4+\delta_4)\delta_6 \\ &= -432a_4\Big(1+\frac{\delta_4}{a_4}\Big)\delta_6, \end{aligned} \tag{3-49}$$

since $a_4 \neq 0$. Therefore, we have $\delta_6 = -\frac{\tilde{p}_4}{432}\frac{1}{a_4} \times \frac{1}{(1+\frac{\delta_4}{a_4})}$. Modulo $\tilde{p}_2$, and $\tilde{p}_4$, we write

$$p_6(\tilde{\lambda}_*^4) = -6912\delta_6^3. \tag{3-50}$$

Finally, by (3-50) and for $\delta_4 < |a_4|$, we obtain:

$$\begin{aligned} \tilde{p}_6(\tilde{\lambda}_*^4) &= \frac{6912}{432^3}\frac{1}{a_4^3}\frac{1}{(1+\frac{\delta_4}{a_4})^3}\tilde{p}_4^3 \\ &= \frac{6912}{432^3a_4^3}\Big[\sum_{n\geq 0}(-1)^n\Big(\frac{\delta_4}{a_4}\Big)^n\Big]^3\tilde{p}_4^3. \end{aligned} \tag{3-51}$$

Consequently, $\tilde{p}_6(\tilde{\lambda}_*^4)$ is in the ideal $\tilde{I}$ over the ring $\mathbb{R}\{a_2, a_4, a_6\}_{\lambda_*^4}$; hence $\tilde{p}_{2k}(\tilde{\lambda}_*^4) \in \tilde{I}$, for every $k \geq 1$.

A perturbation with one critical period may be constructed by perturbing the linear isochrone to a weak centre of order 1. ■

4. **The study of the isochrone and concluding remarks.** We study the isochrone in its reduced form, namely

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x + xy + \frac{x^3}{9}. \end{aligned} \tag{4-1}$$

PROPOSITION 4.1. *The phase portrait of (4-1) appears in Figure 2.*

PROOF. The system (4-1) has a unique finite singular point at the origin. The two invariant lines used in the construction of the Darboux first integral (3-40) yield two invariant parabolas $x^2 + 3y + 9 = 0$ and $x^2 + 6y + 9 = 0$. To derive the complete phase portrait we

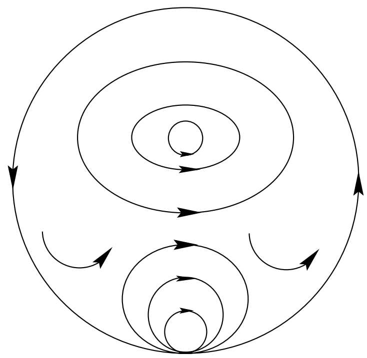

FIGURE 2: Phase portrait of (4-1).

study the points at infinity. There is a unique singular point at infinity in the direction of the y-axis, which is analysed through the change of coordinates $(V, Z) = (x/y, 1/y)$. After multiplication by z^2 the system (4-1) becomes

$$\dot{V} = -Z^2 - V^2Z - \frac{1}{9}V^4 - V^2Z^2 \quad (4\text{-}2)$$
$$\dot{Z} = -VZ(Z + Z^2 + \frac{1}{9}V^2),$$

with a unique singular point at $V = Z = 0$. Its nature is determined by blow-up. In this case we use a weighted blow-up $(V, Z) = (\epsilon, \epsilon^2 Z_1)$ [BM]. The system (4-2) becomes, after division by ϵ^3

$$\dot{\epsilon} = -\epsilon\left(\frac{1}{9} + Z_1 + Z_1^2 + \epsilon^2 Z_1^2\right) \quad (4\text{-}3)$$
$$\dot{Z}_1 = Z_1\left(\frac{1}{9} + Z_1 + 2Z_1^2 + \epsilon^2 Z_1^2\right).$$

On $\epsilon = 0$ the system (4-3) has three singular points. The points $(\epsilon, Z_1) = (0, 0)$ and $(\epsilon, Z_1) = (0, -\frac{1}{6})$ are saddles, while the point $(\epsilon, Z_1) = (0, -\frac{1}{3})$ is a repelling node. The two points $\epsilon = 0$, $Z_1 = -\frac{1}{6}, -\frac{1}{3}$ correspond precisely to the two invariant parabolas. This yields the phase portrait of Figure 2. ∎

PROPOSITION 4.2. *The trajectories of the system (4-1) are quartic curves. The quartic passing through the origin factors as two conics.*

PROOF. The system has the first integral

$$H(x, y) = \frac{(x^2 + 3y + 9)^2}{x^2 + 6y + 9}. \quad (4\text{-}4)$$

The trajectory $H(x, y) = 9$ passes through the origin. It is equivalent to

$$9x^2 + 9y^2 + 6x^2y + x^4 = (3x + 3iy + ix^2)(3x - 3iy - ix^2) = 0. \quad (4\text{-}5)$$

The two conics $3x \pm 3iy \pm ix^2 = 0$ are the two separatrices of the origin. ∎

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PROPOSITION 4.3. *A linearizing change of coordinates of (4-1) is given by*

$$u = \frac{3x}{x^2 + 3y + 9} \qquad v = \frac{3y + x^2}{x^2 + 3y + 9}. \tag{4-6}$$

PROOF. The linearizing change of coordinates (4-6) comes from the linearizing change of coordinates $(u, v) = (\frac{X}{1+Y}, \frac{Y}{1+Y})$ for the system (3-42). ∎

The reduced Kukles family is not sufficiently rich to draw deep conclusions or conjectures about isochronous systems. Several interesting small conclusions can however be drawn. Indeed:

(1) In opposition to the quadratic and cubic cases, not all systems with centers in the reduced Kukles system have Darboux first integrals. It has been shown in [RST] that generically centres of type IV have no elementary, nor Liouvillian first integrals. However, it is precisely in that stratum that the nonlinear isochrones lie. Moreover they have a rational first integral and a rational linearizing change of coordinates.

(2) Here again the two separatrices of the origin are different algebraic curves. In [MRT] are found other examples of isochronous systems which do not necessarily have a rational first integral, nor are reversible. However, all these examples have at least two invariant algebraic curves which are the two separatrices of the origin. This leads us to formulate the following conjecture which should probably be attacked by looking at the geometry of the complex underlying system:

CONJECTURE 4.4. *For a system with a centre at the origin and a rational first integral, the irreducibility of the algebraic curve passing through the origin is an obstruction to isochronicity.*

COMMENTS. After we finished this work we learned that the full Kukles system has been examined in [ChD] via a Lyapunov-type method to obtain necessary and sufficient conditions for an isochronous centre. They obtained no other isochrones than the ones we have found.

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