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A CHARACTERIZATION OF THE FINITE SIMPLE GROUP $\mathrm{PSp}_4(3)$

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The aim of this paper is to characterize the finite simple group $\mathrm{PSp}_4(3)$ by the structure of the centralizer of an element of order three contained in the center of its Sylow 3-subgroup. More precisely, we shall prove the following results.

THEOREM 1. *Let α be an element of order* 3 *contained in the center of a Sylow* 3-*subgroup of* $\mathrm{PSp}_4(3)$. *Denote by H_0 the centralizer of α in* $\mathrm{PSp}_4(3)$. *Let G be a finite group with the following properties:*

(a) *G has no normal subgroup of index* 3.

(b) *G has an element α_1 of order* 3 *such that $C_G(\alpha_1)$ is isomorphic to H_0.*

(c) *$C_G(\alpha_1)$ has an elementary abelian subgroup M of order* 27 *for which $\mathscr{M}_G(M, 3')$ is trivial.* (*Refer to the structure of H_0 in Section* 1.)

Then G is isomorphic to $\mathrm{PSp}_4(3)$.

THEOREM 2. *Let G be a finite group satisfying* (*b*) *and* (*c*) *of Theorem* 1. *Then one of the following occurs:*

(i) *G has a normal subgroup of index* 3.

(ii) *G is isomorphic to* $\mathrm{PSp}_4(3)$.

Clearly, Theorem 2 is an immediate consequence of Theorem 1.

The main difficulty in proving this theorem is in showing that a group possessing properties (a), (b), and (c) has a 3-structure similar to that of $\mathrm{PSp}_4(3)$. Once this is obtained, the centralizer of an involution in the centre of a Sylow 2-subgroup is determined. Finally, G is identified with $\mathrm{PSp}_4(3)$ by use of [**6**]. It seems unfortunate that condition (c) is necessary, but its use in determining the 3-structure of G is indispensable in this method of proof.

1. Structure of H_0. We shall now study the structure of the centralizer of an element α of order 3 contained in the center of a Sylow 3-subgroup of $\mathrm{PSp}_4(3)$. Let F_3 be the finite field of three elements and V be a four-dimensional vector space over F_3 equipped with a non-singular skew-symmetric bilinear form $x \cdot y \in F_3$ $(x, y \in V)$. Then V has a "symplectic basis", i.e., a basis n_1, n_2, n_3, n_4 such that $n_1 n_4 = n_2 n_3 = 1$ and $n_1 n_2 = n_1 n_3 = n_2 n_4 = n_3 n_4 = 0$. The group of all linear transformations σ of V such that $\sigma(x) \cdot \sigma(y) = x \cdot y$ for all x, y in V is called the symplectic group $\mathrm{Sp}_4(3)$. This group has a centre

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of order 2 and the corresponding factor group is $PSp_4(3)$. $PSp_4(3)$ is a simple group of order $3^4 \cdot 2^6 \cdot 5$ (see Artin [**1**]).

This means that a linear transformation σ of V belongs to $Sp_4(3)$ if and only if

$$\sigma(n_1) \cdot \sigma(n_2) = \sigma(n_1) \cdot \sigma(n_3) = \sigma(n_2) \cdot \sigma(n_4) = \sigma(n_3) \cdot \sigma(n_4) = 0,$$
$$\sigma(n_1) \cdot \sigma(n_4) = \sigma(n_2) \cdot \sigma(n_3) = 1.$$

It follows that a linear transformation σ given by the matrix (t_{ij}), $i, j = 1, 2, 3, 4$ in terms of the basis n_1, n_2, n_3, n_4 where

$$\sigma(n_1) = t_{11}n_1 + t_{21}n_2 + t_{31}n_3 + t_{41}n_4 \quad \text{etc.},$$

belongs to $Sp_4(3)$ if and only if $(t_{ij})'J(t_{ij}) = J$ where $(t_{ij})'$ denotes the transpose matrix and J is the 4×4 matrix

$$J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix}.$$

Take

$$\alpha = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{bmatrix}$$

which is an element of order 3 in $Sp_4(3)$. If I is the identity transformation of $Sp_4(3)$, it is clear that $-I$ generates the centre of $Sp_4(3)$. Therefore, $PSp_4(3)$ can be obtained from $Sp_4(3)$ by identifying a matrix of $Sp_4(3)$ and its negative. A matrix (t_{ij}) of $Sp_4(3)$ centralizes α if and only if

$$(t_{ij}) = \begin{bmatrix} \epsilon & & & \\ a & & T & \\ b & & & \\ c & e & f & \epsilon \end{bmatrix} \tag{1.1}$$

where

$$T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$$

belongs to $SL(2, 3)$, $\epsilon = \pm 1$ and $e = \epsilon(bt_{11} - at_{21})$, $f = \epsilon(bt_{12} - at_{22})$.

We conclude that H_0 is the totality of all matrices of the type (1.1) and that any two such matrices are identified if they are negatives of each other. From (1.1) we calculate that H_0 has order $3^4 \cdot 2^3$. Several subgroups of H_0 are important in the proof of Theorem 1. We list them here for convenience.

(1.2) Let P be the totality of all matrices of H_0 of the form

$$\begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & d & 1 & \\ c & e & f & 1 \end{bmatrix} \quad \text{where} \quad \begin{aligned} e &= b - ad, \\ f &= -a. \end{aligned}$$

It is easily verified that P is a Sylow 3-subgroup of H_0 of order 81 and is isomorphic to a Sylow 3-subgroup of $\mathrm{PSp}_4(3)$. The centre of P is generated by α and the center of H_0 and P coincide.

(1.3) Let M be the subgroup of P which is the totality of all matrices

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ b & d & 1 & \\ c & b & & 1 \end{bmatrix}.$$

It is easily checked that M is the unique elementary abelian subgroup of P of order 27. M is a self-centralizing subgroup of H_0 which contains α.

(1.4) $N_{H_0}(P) = P\langle t\rangle$ where $\langle t\rangle$ is a complement of order 2 generated by the involution

$$t = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{bmatrix}.$$

(1.5) A Sylow 2-subgroup of H_0 is a quaternion group of order 8. Let Q be the quaternion group with t in its center generated by

$$q_1 = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & -1 & & \\ & & & 1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1 & & & \\ & 1 & 1 & \\ & 1 & -1 & \\ & & & 1 \end{bmatrix}.$$

The mapping θ of H_0 which sends each matrix of (1.1) onto its corresponding matrix T is a homomorphism of H_0 onto $\mathrm{PSL}(2, 3)$. The kernel of this homomorphism is the set of all matrices of (1.1) with $T = \pm I$ and is the semidirect product of a nonabelian subgroup of P of order 27 and $\langle t\rangle$. It is easily verified that $O_3(H_0) = \mathrm{Ker}\ (\theta) \cap P$ and $Z(H_0) = Z(O_3(H_0)) = \langle\alpha\rangle$.

2. The structure of $N_G(M)$. Let G be a finite group satisfying (a), (b) and (c) of Theorem 1. Let H be a subgroup of G isomorphic to H_0 and let α_1 be the generator of the center of H. Let P, M, t have the same meaning as in (1.2), (1.3), and (1.4) but now identified as subgroups of H.

(2.1) *P is a Sylow 3-subgroup of G.*

Proof. Let S be a Sylow 3-subgroup of G containing P. Then $Z(S)$ is a subgroup of $C_G(M) = C_H(M) = M$ so that $Z(S)$ centralizes P. It follows that $Z(S) = Z(P) = \langle \alpha_1 \rangle$ and $S \subseteq H$. Since P is a Sylow 3-subgroup of H, $P = S$.

(2.2) *$N_G(M)/M$ is isomorphic to a subgroup of* $\mathrm{GL}(3, 3)$ *and has a self-centralizing Sylow* 3-*subgroup of order* 3. *In fact*, $N_G(P) = P\langle t \rangle$.

Proof. The subgroup M contains α_1 so that $C_G(M) = C_H(M) = M$ and it follows that $N_G(M)/M$ is isomorphic to a subgroup of $\mathrm{GL}(3, 3)$. The commutator subgroup P' of P is elementary abelian of order 9, contains α_1 and is calculated to be $O_3(H) \cap M$. It follows that $C_G(P') = C_H(P') = M$ and that $N_G(P')/M$ is isomorphic to a subgroup of $\mathrm{GL}(2, 3)$. Since $N_G(P)/M$ is a Sylow 3-normalizer of $N_G(P')/M$ and since $\mathrm{GL}(2, 3)$ has a Sylow 3-normalizer of order 12, $N_G(P)/M$ has order 6 or 12.

Suppose that $N_G(P)/M$ has order 12 so that the Sylow 2-subgroup of $N_G(P)/M$ is elementary abelian of order 4. If $N_G(P)$ contained an element y of order 4, then $y^2 \in M$ which is impossible. It follows that $N_G(P) = P\langle t, \tau \rangle$ where $t \in N_H(P)$ and τ is an involution not contained in H such that $\langle t, \tau \rangle$ is a four group. Since $N_G(P)/M$ is isomorphic to a Sylow 3-normalizer of $\mathrm{GL}(2, 3)$ and since t inverts a generator of P/M, we may assume that τ centralizes P/M. From the structure of P, $O_3(H)$ is the unique nonabelian subgroup of P of order 27 and exponent 3. Let X be the unique subgroup of P of order 27 which is nonabelian of exponent 9. It follows that $\langle t, \tau \rangle$ normalizes M, $O_3(H)$ and X and that $X = C_X(t)C_X(\tau)C_X(t\tau)$. Since $C_X(t) = \langle \alpha_1 \rangle$, it follows that τ or $t\tau$ centralize an element of $X - O_3(H)$. This implies that τ or $t\tau$ centralizes $P/O_3(H)$. However, t centralizes $P/O_3(H)$ so that $\langle t, \tau \rangle$ centralizes $P/O_3(H)$. Let $m \in M - P'$. Then $m \in P - O_3(H)$ so that $[\tau, m] \in O_3(H) \cap M = P'$. Thus τ centralizes M/P'.

Since τ centralizes P/M and M/P' we have that τ stabilizes the normal series $P/P' \supset M/P' \supset \bar{1}$ of P/P'. It follows that τ acts trivially on P/P'. Since $P' = \Phi(P)$, τ acts trivially on P which is not the case. We conclude that $N_G(P)/M$ has order 6 so that $N_G(P) = P\langle t \rangle$. Since t inverts a generator of P/M, P/M is a self-centralizing Sylow 3-subgroup of $N_G(M)/M$.

(2.3) *$N_G(M)$ is not* 3-*closed.*

Proof. Suppose that $N_G(M)$ is 3-closed. Then $N_G(M) \subseteq N_G(P)$ so that $N_G(M) = N_G(P) = P\langle t \rangle$. Let $g \in G$ and suppose that $\alpha_1{}^g \in P$. Since $\alpha_1{}^g$ has order 3, $\alpha_1{}^{gh} \in M$ for some $h \in H$. As M is the unique maximal elementary abelian subgroup of P, α_1 and $\alpha_1{}^{gh}$ are conjugate in $N_G(M)$. This implies that $\alpha_1 = \alpha_1{}^{gh}$ so that $\alpha_1 = \alpha_1{}^g$. It follows that the centre of P is weakly closed in P and by a theorem of Grün, the largest abelian 3-quotient group of G and $N_G(Z(P))$ are isomorphic. Since α_1 and $\alpha_1{}^{-1}$ are not conjugate, $N_G(Z(P)) = C_G(Z(P)) = H$. From the structure of H we see that $H/O_3(H)$ is isomorphic to $\mathrm{SL}(2, 3)$ so that H has a normal subgroup of index 3. This implies that condition (a) of Theorem 1 is violated, a contradiction.

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(2.4) *$N_G(M) = ML$, $M \cap L = 1$ and L is isomorphic to S_4.*

Proof. By a theorem of Gaschütz [**7**], M has a complement L such that $N_G(M) = ML$, $M \cap L = 1$. From (2.2), L has a self-centralizing Sylow 3-subgroup of order 3 and a Sylow 3-normalizer of order 6. Since L is isomorphic to a subgroup of GL(3, 3), the order of L is a divisor of $2^5 \cdot 3^3 \cdot 13$ and it follows from a theorem of Feit and Thompson [**3**] that L contains a normal nilpotent subgroup N such that L/N is isomorphic to S_3 or A_3. Let L be chosen that $t \in L$ and let $L \cap P$ be generated by x. Then $txt = x^{-1}$ and since $|L : N|$ is divisible by 3, $L/N \cong S_3$.

Suppose that 13 is a divisor of $|N|$. Since N is nilpotent, the Sylow 13-subgroup S is centralized by N and $C_L(S) = N$. It follows that L/N is isomorphic to a subgroup of Aut(S). This is impossible as $L/N \cong S_3$ and Aut(S) is cyclic.

We conclude that N is a 2-group and that N has order 4 or 16. If $|N| = 16$, then $|L| = 2^5 \cdot 3$ and L contains a Sylow 2-subgroup W which is isomorphic to a Sylow 2-subgroup of GL(3, 3). By a result of [**2**], $W = W_1 \times C_2$ where

$$W_1 = \langle a, b \,|a^8 = b^2 = 1, b^{-1}ab = a^3\rangle$$

and C_2 is a cyclic group of order 2. If N contained an element of order 8, then N would contain exactly 3 or 6 elements or order 8. However, any subgroup of W containing an element of order 8 contains 4 or 8 such elements. Thus N is a maximal subgroup of W whose elements are of order 4 or 2. Since a maximal subgroup of W_1 is cyclic, quaternion or dihedral, it follows that N has a maximal subgroup which is quaternion or dihedral of order 8. It follows that $|Z(N)| = 4$ and that $Z(N) = Z(W)$. Let $W = \mathrm{N}\langle t\rangle$ so that $Z(N) = Z(N\langle t\rangle)$. Then $t \in C_L(Z(N))$ which implies that $x \in C_L(Z(N))$, a contradiction.

Finally, $|N| = 4$ and N is not cyclic. Clearly t does not centralize N so that $N\langle t\rangle$ is a dihedral group of order 8. It follows that $L = N\langle x, t\rangle$ is isomorphic to S_4.

Let L be a complement for M such that $N_G(M) = ML$, $M \cap L = 1$, $L \cong S_4$. We may assume $t \in L$ since $\langle t\rangle$ is a complement for P in $N_G(P)$. From the structure of P, $L \cap P$ is generated by an element x of order 3 inverted by t so that $x \in O_3(H)$ and $P = M\langle x\rangle$. Let τ be the involution of L such that $\langle \tau, x^{-1}\tau x\rangle$ is the normal subgroup of L of order 4 and t centralizes τ. We have the relations

$$(2.5) \qquad txt = x^{-1}, \quad t\tau = \tau t, \quad x\tau x = \tau x^{-1}\tau \quad \text{or} \quad (x\tau)^3 = 1.$$

Choose a basis for M given by

$$\alpha_1 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 1 & & & 1 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ 1 & & 1 & \\ & 1 & & 1 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 1 & 1 & \\ & & & 1 \end{bmatrix}.$$

From the structure of H we see that $t\alpha_1 t = \alpha_1$, $t\alpha_2 t = \alpha_2^{-1}$, $t\alpha_3 t = \alpha_3$. Replacing x by x^{-1} if necessary, we have that x satisfies $x^{-1}\alpha_1 x = \alpha_1$, $x^{-1}\alpha_2 x = \alpha_1^{-1}\alpha_2$ and $x^{-1}\alpha_3 x = \alpha_1\alpha_2\alpha_3$. Relative to this basis,

$$x \to \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \qquad t \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Viewing τ as an involution of GL(3, 3) satisfying the relations (2.5) we calculate that

$$\text{(i) } \tau = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{(ii) } \tau = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(iii) } \tau = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

The fusion pattern is calculated:

(i) $\alpha_1 \sim \alpha_1\alpha_3 \sim \alpha_1^{-1}\alpha_2\alpha_3 \sim \alpha_1^{-1}\alpha_2^{-1}\alpha_3$

$\alpha_1^{-1} \sim \alpha_1^{-1}\alpha_3^{-1} \sim \alpha_1\alpha_2^{-1}\alpha_3^{-1} \sim \alpha_1\alpha_2\alpha_3^{-1}$

$\alpha_2 \sim \alpha_2^{-1} \sim \alpha_1^{-1}\alpha_2 \sim \alpha_1\alpha_2 \sim \alpha_1^{-1}\alpha_2^{-1} \sim \alpha_1\alpha_2^{-1} \sim \alpha_3 \sim \alpha_1\alpha_2\alpha_3 \sim$
$\alpha_1\alpha_2^{-1}\alpha_3 \sim \alpha_3^{-1} \sim \alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1} \sim \alpha_1^{-1}\alpha_2\alpha_3^{-1}$

$\alpha_1^{-1}\alpha_3 \sim \alpha_2\alpha_3 \sim \alpha_2^{-1}\alpha_3 \sim \alpha_2^{-1}\alpha_3^{-1} \sim \alpha_2\alpha_3^{-1} \sim \alpha_1\alpha_3^{-1}$.

(ii) $\alpha_1 \sim \alpha_1^{-1}\alpha_3 \sim \alpha_2\alpha_3 \sim \alpha_2^{-1}\alpha_3$

$\alpha_1^{-1} \sim \alpha_1\alpha_3^{-1} \sim \alpha_2^{-1}\alpha_3^{-1} \sim \alpha_2\alpha_3^{-1}$

$\alpha_2 \sim \alpha_2^{-1} \sim \alpha_1^{-1}\alpha_2 \sim \alpha_1\alpha_2 \sim \alpha_1^{-1}\alpha_2^{-1} \sim \alpha_1\alpha_2^{-1} \sim \alpha_1\alpha_3 \sim \alpha_1^{-1}\alpha_3^{-1} \sim$
$\alpha_1^{-1}\alpha_2\alpha_3 \sim \alpha_1^{-1}\alpha_2^{-1}\alpha_3 \sim \alpha_1\alpha_2^{-1}\alpha_3^{-1} \sim \alpha_1\alpha_2\alpha_3^{-1}$

$\alpha_3 \sim \alpha_1\alpha_2\alpha_3 \sim \alpha_1\alpha_2^{-1}\alpha_3 \sim \alpha_3^{-1} \sim \alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1} \sim \alpha_1^{-1}\alpha_2\alpha_3^{-1}$.

(iii) $\alpha_1 \sim \alpha_3 \sim \alpha_1\alpha_2\alpha_3 \sim \alpha_1\alpha_2^{-1}\alpha_3$

$\alpha_1^{-1} \sim \alpha_3^{-1} \sim \alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1} \sim \alpha_1^{-1}\alpha_2\alpha_3^{-1}$

$\alpha_2 \sim \alpha_2^{-1} \sim \alpha_1^{-1}\alpha_2 \sim \alpha_1\alpha_2 \sim \alpha_1^{-1}\alpha_2^{-1} \sim \alpha_1\alpha_2^{-1} \sim \alpha_1^{-1}\alpha_3 \sim \alpha_1\alpha_3^{-1}$
$\alpha_2\alpha_3 \sim \alpha_2^{-1}\alpha_3 \sim \alpha_2^{-1}\alpha_3^{-1} \sim \alpha_2\alpha_3^{-1}$

$\alpha_1\alpha_3 \sim \alpha_1^{-1}\alpha_3^{-1} \sim \alpha_1^{-1}\alpha_2\alpha_3 \sim \alpha_1^{-1}\alpha_2^{-1}\alpha_3 \sim \alpha_1\alpha_2^{-1}\alpha_3^{-1} \sim \alpha_1\alpha_2\alpha_3^{-1}$.

In any case we see that G has exactly four classes of elements of order 3 with representatives α_1, α_1^{-1}, $\alpha_1^{-1}\alpha_1^{\tau}$, $\alpha_1\alpha_1^{\tau}$. Furthermore, $\tau\alpha_2\tau = \alpha_2^{-1}$ in all cases. We have proven the following.

(2.6) *$N_G(M) = M\langle x, t, \tau\rangle$ where $\langle x, t, \tau\rangle$ is a complement for M satisfying the relations $t^2 = \tau^2 = x^3 = 1$, $txt = x^{-1}$, $t\tau = \tau t$, $(x\tau)^3 = 1$. G has exactly four classes of elements of order 3 with representatives α_1, α_1^{-1}, $\alpha_1^{-1}\alpha_1^{\tau}$, $\alpha_1\alpha_1^{\tau}$ where $\alpha_1^{\tau} \neq \alpha_1, \alpha_1^{-1}$.*

$$C_{N_G(M)}(\alpha_1) = P\langle t\rangle, \quad C_{N_G(M)}(\alpha_1^{-1}) = P\langle t\rangle, \quad C_{N_G(M)}(\alpha_1^{-1}\alpha_1^{\tau}) = M\langle t\rangle,$$

and $C_{N_G(M)}(\alpha_1\alpha_1^{\tau}) = M\langle t, \tau\rangle$.

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3. The structure of $C_G(\alpha_1^{-1}\alpha_1^{\tau})$ and $C_G(\alpha_1\alpha_1^{\tau})$. Let τ be the involution of $N_G(M)$ given in (2.6).

(3.1) $C_G(\alpha_1^{-1}\alpha_1^{\tau}) \cap C_G(\alpha_2) = M.$

Proof. By (2.6), $C_{N_G(M)}(\alpha_1^{-1}\alpha_1^{\tau}) = M\langle t\rangle$ and $C_{N_G(M)}(\alpha_2) = M\langle t\tau\rangle$. It follows that M is a Sylow 3-subgroup of

$$C_G(\alpha_1^{-1}\alpha_1^{\tau}) \cap C_G(\alpha_2)$$

which is located in the center of its normalizer. By a theorem of Burnside, $C_G(\alpha_1^{-1}\alpha_1^{\tau}) \cap C_G(\alpha_2) = MV$, $M \cap V = 1$ and V is a normal complement of M. Thus $V \in И_G(M, 3')$ and it follows that $V = 1$ by (c) of Theorem 1.

(3.2) $C_G(\alpha_1^{-1}\alpha_1^{\tau}) = M\langle t\rangle$, $C_G(\alpha_2) = M\langle t\tau\rangle$.

Proof. $C_G(\alpha_1^{-1}\alpha_1^{\tau})$ has $M\langle t\rangle$ as a Sylow 3-normalizer so that $\langle\alpha_1\rangle \times \langle\alpha_1^{\tau}\rangle$ is the centre of the Sylow 3-normalizer. By a theorem of Grün, $C_G(\alpha_1^{-1}\alpha_1^{\tau})$ has a normal subgroup K of index 9 and $M \cap K = [M, t] = \langle\alpha_2\rangle$. By (3.1), $C_K(\alpha_2) = \langle\alpha_2\rangle$ so that K has a self-centralizing Sylow 3-subgroup of order 3. By [**3**], K is isomorphic to A_5 or $\mathrm{PSL}(2, 7)$ or has a normal nilpotent subgroup N such that K/N is isomorphic to A_3, S_3 or A_5.

Since K admits α_1 as an automorphism, we may assume that α_1 acts nontrivially on K as otherwise $K = \langle\alpha_2, t\rangle$ and we have our desired result. Assuming that α_1 acts nontrivially on K we have that α_1 centralizes $\langle\alpha_2, t\rangle$ so can not be the inner automorphism induced by α_2. It follows that α_1 is a nontrivial outer automorphism of K and since A_5 and $\mathrm{PSL}(2, 7)$ admit no such outer automorphism, K is not simple. We have that K has a normal nilpotent subgroup N.

If $K/N \cong A_5$, then $N^{\alpha_1} \lhd K$ and since $NN^{\alpha_1} \lhd K$, it follows that $NN^{\alpha_1} = N$ and α_1 normalizes N. Thus $N \in И_G(M, 3')$ and we have $N = 1$. This implies that K is simple contrary to our preceding discussion.

Since $\langle\alpha_2, t\rangle$ is a dihedral group of order 6, $K/N \cong S_3$. It follows that $N^{\alpha_1} = N$, so that $N \in И_G(M, 3')$. Thus, in any case $K = \langle\alpha_2, t\rangle$ and $C_G(\alpha_1^{-1}\alpha_1^{\tau}) = M\langle t\rangle$. Since α_2 and $\alpha_1^{-1}\alpha_1^{\tau}$ are conjugate in $N_G(M)$, $C_G(\alpha_2) = M\langle t\tau\rangle$.

To determine the structure of $C_G(\alpha_1\alpha_1^{\tau})$ it is necessary to prove the following lemma.

(3.3) *Let G be any finite group which satisfies the following conditions:*

(i) $T = \langle\alpha\rangle \times \langle\beta\rangle$ *is a Sylow 3-subgroup of order 9 with* $\alpha^3 = \beta^3 = 1$.

(ii) $N_G(T) = T\langle t, \tau\rangle$ *such that* $\alpha^t = \alpha^{\tau} = \alpha^{-1}$, $t\tau = \tau t$, $t^2 = \tau^2 = 1$ *and* $\beta^t = \beta$, $\beta^{\tau} = \beta^{-1}$.

(iii) $C_G(\beta) = T\langle t\rangle$, $C_G(\alpha) = T\langle t\tau\rangle$, $C_G(\alpha\beta) = T$.

Then $G = N_G(T) = T\langle t, \tau\rangle$.

Proof. Let $D = T^{\#} = T - 1$. From (ii), $T^{\#}$ is the union of 3 classes of elements of order 3 of $N_G(T)$ with representatives α, β, and $\alpha\beta$. Since T is abelian, these representatives are not conjugate in G, and by (iii), $N_G(T)$ contains the centralizer of each of α, β, and $\alpha\beta$. It follows that D is a closed set of special classes of G and that $M(D)$, the module of generalized characters of $N_G(T)$ with support on D, has a basis consisting of three generalized characters of $N_G(T)$ (see [**8**]).

The character table of $N_G(T)$ can be computed by computing an induced character form each of T, $T\langle t\tau\rangle$, and $T\langle t\rangle$.

	1	β^*	α^*	$\alpha\beta^*$	t	τ	$t\tau$	$t\tau\alpha$	$t\beta$
ϕ_1	1	1	1	1	1	1	1	1	1
ϕ_2	1	1	1	1	-1	1	-1	-1	-1
ϕ_3	1	1	1	1	-1	-1	1	1	-1
ϕ_4	1	1	1	1	1	-1	-1	-1	1
ϕ_5	4	-2	-2	1	0	0	0	0	0
ϕ_6	2	2	-1	-1	0	0	2	-1	0
ϕ_7	2	2	-1	-1	0	0	-2	1	0
ϕ_8	2	-1	2	-1	2	0	0	0	-1
ϕ_9	2	-1	2	-1	-2	0	0	0	1

$*$ = special classes

A basis for $M(D)$ is given by

$$\Phi_1 = \phi_5 - \phi_1 - \phi_2 - \phi_3 - \phi_4$$
$$\Phi_2 = \phi_5 - \phi_6 - \phi_7$$
$$\Phi_3 = \phi_5 - \phi_8 - \phi_9.$$

By [**8**], $(\Phi_i^*, \Phi_j^*)_G = (\Phi_i, \Phi_j)_N$ where Φ_i^* denotes the induced character and $(\ ,\ )_G$, $(\ ,\ )_N$ denote, respectively, the inner product in G and $N_G(T)$. Let 1_G denote the principal character of G and let $|\Phi_i^*| = (\Phi_i^*, \Phi_i^*)_G$.

By the Frobenius Reciprocity Formula, $(\Phi_1^*, 1_G)_G = (\Phi_1, 1_N)_N = -1$. Since $|\Phi_1^*| = 5$, it follows that $\Phi_1^* = \epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + \epsilon_4 x_4 - 1_G$ where $\epsilon_i = \pm 1$, $1 \leqq i \leqq 4$ and x_i are all distinct irreducible characters of G. Since $|\Phi_2^*| = 3$, $(\Phi_2^*, \Phi_1^*)_G = (\Phi_2, \Phi_1)_N = 1$, it follows that

$$\Phi_2^* = \epsilon_1 x_1 + \epsilon_2 x_2 + -\epsilon_3 x_3 \text{ or } \Phi_2^* = \epsilon_1 x_1 + n_1 y_1 + \eta_2 y_2$$

where $\eta_i = \pm 1$ and y_1, y_2 are irreducible characters of G distinct from the x_i, $1 \leqq i \leqq 4$.

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Let us suppose that we are in the first case so that

$$\Phi_1^* = \epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + \epsilon_4 x_4 - 1_G$$
$$\Phi_2^* = \epsilon_1 x_1 + \epsilon_2 x_2 - \epsilon_3 x_3.$$

Since $(\Phi_2^*, \Phi_3^*) = 1$, $(\Phi_1^*, \Phi_3^*) = 1$, Φ_3^* has exactly 3 characters in common with Φ_1^* or exactly one character in common. If $\Phi_3^* = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3$, then $\mu_1\epsilon_1 + \mu_2\epsilon_2 + \mu_3\epsilon_3 = 1$ and $\mu_1\epsilon_1 + \mu_2\epsilon_2 - \mu_3\epsilon_3 = 1$ which is impossible. Since $(\Phi_3^*, \Phi_2^*) = 1$, Φ_3^* can not have x_4 and two other characters of Φ_1^* in common. Thus Φ_3^* has exactly one character of Φ_1^* in common and it must be x_1, x_2 or x_3. Since x_3 appears in Φ_1^* and Φ_2^* with opposite signs, x_3 can not be the common character. Since Φ_1^*, Φ_2^* are symmetric in x_1 and x_2, we may suppose x_1 is the common character and we have $\Phi_3^* = \epsilon_1 x_1 + \eta_1 y_1 + \eta_2 y_2$, $\eta_i = \pm 1$, y_1, y_2 irreducible characters distinct from x_i, $1 \leqq i \leqq 4$.

Case (i):
$$\Phi_1^* = \epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + \epsilon_4 x_4 - 1_G$$
$$\Phi_2^* = \epsilon_1 x_1 + \epsilon_2 x_2 - \epsilon_3 x_3$$
$$\Phi_3^* = \epsilon_1 x_1 + \eta_1 y_1 + \eta_2 y_2,$$

with x_i, y_i, 1_G distinct irreducible characters of G, $\epsilon_i = \pm 1$, $\eta_i = \pm 1$.

On the other hand, if $\Phi_2^* = \epsilon_1 x_1 + \eta_1 y_1 + \eta_2 y_2$,

$$\Phi_1^* = \epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + \epsilon_4 x_4 - 1_G,$$

let us suppose that Φ_3^* has three nonprincipal characters in common with Φ_1^*. We have that $\Phi_3^* = \epsilon_1 x_1 + \epsilon_k x_k - \epsilon_j x_j$ and we are back in case (i) except for a permutation of the subscripts of the Φ_i^* and the x_i. We may assume that Φ_3^* has exactly one irreducible character in common with Φ_1^* and $\Phi_3^* = \epsilon_1 x_1 + \zeta_1 z_1 + \zeta_2 z_2$; z_i, $1 \leqq i \leqq 2$ distinct from 1_G and the x_i. If Φ_3^* has y_1 in common, then $\Phi_3^* = \epsilon_1 x_1 + \eta_1 y_1 - \eta_2 y_2$ and $\Phi_3^*(1) = \epsilon_1 f_1 + \eta_1 g_1 - \eta_2 g_2 = 0$, $\Phi_2^*(1) = \epsilon_1 f_1 + \eta_1 g_1 + \eta_2 g_2 = 0$ where $f_1 = x_1(1)$, $g_i = y_i(1)$, $1 \leqq i \leqq 2$. This is impossible. We have proved,

Case (ii):
$$\Phi_1^* = \epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3 + \epsilon_4 x_4 - 1_G$$
$$\Phi_2^* = \epsilon_1 x_1 + \eta_1 y_1 + \eta_2 y_2$$
$$\Phi_3^* = \epsilon_1 x_1 + \zeta_1 z_1 + \zeta_2 z_2$$

with $\epsilon_i = \pm 1$, $\eta_i = \pm 1$, $\zeta_i = \pm 1$, 1_G, x_i, y_i, z_i distinct irreducible characters of G.

It is now possible to compute a fragment of the character table of G. In case (i) it is found that $x_1 = \epsilon_1 \phi_5$, $x_2 = -\epsilon_2(\phi_1 + \phi_6)$, $x_3 = \epsilon_3(-\phi_1 + \phi_6)$, $x_4 = -\epsilon_4 \phi_1$, $y_1 = \eta_1(\phi_1 + \phi_5 + \phi_6)$, $y_2 = \eta_2(\phi_1 + \phi_5 + \phi_6)$ when restricted to the special classes of $N_G(T)$.

Case (i):

	1	β	α	$\alpha\beta$
1_G	1	1	1	1
x_1	f_1	$-2\epsilon_1$	$-2\epsilon_1$	ϵ_1
x_2	f_2	$-3\epsilon_2$	0	0
x_3	f_3	ϵ_3	$-2\epsilon_3$	$-2\epsilon_3$
x_4	f_4	$-\epsilon_4$	$-\epsilon_4$	$-\epsilon_4$
y_1	g_1	η_1	$-2\eta_1$	η_1
y_2	g_2	η_2	$-2\eta_2$	η_2

In case (ii), $x_1 = \epsilon_1\phi_5$, $x_2 = -\epsilon_2\phi_1$, $x_3 = -\epsilon_3\phi_1$, $x_4 = -\epsilon_4\phi_1$, $y_1 = -\eta_1\phi_6$, $y_2 = -\eta_2\phi_6$, $z_1 = \zeta_1(\phi_1 + \phi_5 + \phi_6)$, $z_2 = \zeta_2(\phi_1 + \phi_5 + \phi_6)$ when restricted to the special classes of $N_G(T)$.

Case (ii):

	1	β	α	$\alpha\beta$
1_G	1	1	1	1
x_1	f_1	$-2\epsilon_1$	$-2\epsilon_1$	ϵ_1
x_2	f_2	$-\epsilon_2$	$-\epsilon_2$	$-\epsilon_2$
x_3	f_3	$-\epsilon_3$	$-\epsilon_3$	$-\epsilon_3$
x_4	f_4	$-\epsilon_4$	$-\epsilon_4$	$-\epsilon_4$
y_1	g_1	$-2\eta_1$	η_1	η_1
y_2	g_2	$-2\eta_2$	η_2	η_2
z_1	h_1	ζ_1	$-2\zeta_1$	ζ_1
z_2	h_2	ζ_2	$-2\zeta_2$	ζ_2

Now we prove a lemma which will be used in computing $|G|$.

(3.4) *Let a, b, c be elements of the special classes of $N_G(T)$. If for some $x, y \in G$ we have $a^x b^y = c$ and we assume that a, b, c do not belong to the same class of $N_G(T)$, then a^x, b^y belong to T.*

Proof. Let $a^x b^y = c$ and suppose that a, b, c do not belong to the same class of $N_G(T)$. Let $S = \langle a^x, b^y \rangle$. By a result of [**3**], S has a normal abelian subgroup

N of index 3. If N has order prime to 3, S has a Sylow 3-subgroup of order 3 and a, b, c are in the same class of $N_G(T)$. It follows that S has a Sylow 3-subgroup of order 9 containing c. Since $C_G(c) \subseteq N_G(T)$, $T \subseteq S$ and $N \cap T$ has order 3. This implies that N centralizes an element of T so that $N \subseteq N_G(T)$. It follows that $S \subseteq N_G(T)$ so that a^x, $b^y \in T$.

Let k be the number of ways $a^x b^y = c$, where a, b, c are elements of the special classes of $N_G(T)$. If χ_j, $1 \leqq j \leqq n$ are the irreducible characters of G we have,

$$k = \frac{|G|}{|C_G(a)||C_G(b)|} \sum_{j=1}^{n} \frac{\chi_j(a)\chi_j(b)\overline{\chi_j(c)}}{\chi_j(1)}. \tag{3.5}$$

Using the fragment in case (i), (3.4) and (3.5),

$a = \alpha$, $b = \beta$, $c = \alpha\beta$

$$(1)\ 1 = \frac{|G|}{(18)^2}\left[1 + \frac{4\epsilon_1}{f_1} + \frac{4\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} - \frac{2\eta_1}{g_1} - \frac{2\eta_2}{g_2}\right]$$

$a = \alpha$, $b = \beta$, $c = \alpha$

$$(2)\ 0 = 1 - \frac{8\epsilon_1}{f_1} + \frac{4\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} + \frac{4\eta_1}{g_1} + \frac{4\eta_2}{g_2}$$

$a = \beta$, $b = \alpha\beta$, $c = \beta$

$$(3)\ 0 = 1 + \frac{4\epsilon_1}{f_1} - \frac{2\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} + \frac{\eta_1}{g_1} + \frac{\eta_2}{g_2}$$

$a = \alpha$, $b = \beta$, $c = \beta$

$$(4)\ 0 = 1 - \frac{8\epsilon_1}{f_1} - \frac{2\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} - \frac{2\eta_1}{g_1} - \frac{2\eta_2}{g_2}.$$

Subtracting (4) from (3), we have

$$0 = \frac{4\epsilon_1}{f_1} + \frac{\eta_1}{g_1} + \frac{\eta_2}{g_2}.$$

Since $\Phi_3^* = \epsilon_1 x_1 + \eta_1 y_1 + \eta_2 y_2$, $\epsilon_1 f_1 = -(\eta_1 g_1 + \eta_2 g_2)$. Substituting for f_1 we have that $2 = \eta_1\eta_2(g_2/g_1 + g_1/g_2)$. It follows that $g_1 = g_2$ and $\eta_1 = \eta_2$. Since $\epsilon_1 f_1 = -(\eta_1 g_1 + \eta_2 g_2)$, we find that $f_1 = 2g_1$, $\eta_1 = -\epsilon_1$.

Using the above new information, (4) becomes

$$0 = 1 - \frac{2\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4}$$

and (2) becomes

$$0 = 1 - \frac{24\epsilon_1}{f_1} + \frac{4\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4}.$$

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Subtracting (2) from (4), we find that $f_1 = \epsilon_1\epsilon_3 4f_3$ and it follows that $\epsilon_1 = \epsilon_3$, $f_1 = 4f_3$.

From $0 = 1 - 2\epsilon_3/f_3 - \epsilon_4/f_4$, we now find that $\epsilon_4/f_4 = 1 - 8\epsilon_1/f_1$. Substituting all this information into (1), we have that $|G| = 9\epsilon_1 f_1$. Since $|G|$ is a positive integer, $\epsilon_1 = 1$ and $f_1 = |G|/9$. From elementary character theory, $|G| \geqq f_1^2$ and we find that $|G| \leqq 81$. Since $|N_G(T)| = 36$, $|G| = 36$ or 72. In the latter case $72 \geqq f_1^2 + g_1^2 = 80$, a contradiction. Thus $|G| = 36$ and (3.3) holds in case (i).

We may now assume that case (ii) holds. Using the fragment in case (ii), (3.4) and (3.5),

$a = \alpha,\ b = \beta,\ c = \alpha\beta$

$$(1)\ 1 = \frac{|G|}{(18)^2}\left[1 + \frac{4\epsilon_1}{f_1} - \frac{\epsilon_2}{f_2} - \frac{\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} - \frac{2\eta_1}{g_1} - \frac{2\eta_2}{g_2} - \frac{2\zeta_1}{h_1} - \frac{2\zeta_2}{h_2}\right]$$

$a = \alpha,\ b = \beta,\ c = \alpha$

$$(2)\ 0 = 1 - \frac{8\epsilon_1}{f_1} - \frac{\epsilon_2}{f_2} - \frac{\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} - \frac{2\eta_1}{g_1} - \frac{2\eta_2}{g_2} + \frac{4\zeta_1}{h_1} + \frac{4\zeta_2}{h_2}$$

$a = \beta,\ b = \alpha\beta,\ c = \beta$

$$(3)\ 0 = 1 + \frac{4\epsilon_1}{f_1} - \frac{\epsilon_2}{f_2} - \frac{\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} + \frac{4\eta_1}{g_1} + \frac{4\eta_2}{g_2} + \frac{\zeta_1}{h_1} + \frac{\zeta_2}{h_2}$$

$a = \alpha,\ b = \alpha,\ c = \alpha\beta$

$$(4)\ 0 = 1 + \frac{4\epsilon_1}{f_1} - \frac{\epsilon_2}{f_2} - \frac{\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} + \frac{\eta_1}{g_1} + \frac{\eta_2}{g_2} + \frac{4\zeta_1}{h_1} + \frac{4\zeta_2}{h_2}.$$

Subtracting (4) from (2), we have

$$0 = \frac{4\epsilon_1}{f_1} + \frac{\eta_1}{g_1} + \frac{\eta_2}{g_2}.$$

Since $\Phi_2^*(1) = \epsilon_1 f_1 + \eta_1 g_1 + \eta_2 g_2 = 0$, we have that $\eta_1 = \eta_2$, $g_1 = g_2$. Thus $f_1 = -2\epsilon_1\eta_1 g_1$ and we have $f_1 = 2g_1$, $\epsilon_1 = -\eta_1$.

Using this information (4) becomes

$$0 = 1 - \frac{\epsilon_2}{f_2} - \frac{\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} + \frac{4\zeta_1}{h_1} + \frac{4\zeta_2}{h_2}$$

and (3) becomes

$$0 = 1 - \frac{12\epsilon_1}{f_1} - \frac{\epsilon_2}{f_2} - \frac{\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4} + \frac{\zeta_1}{h_1} + \frac{\zeta_2}{h_2}.$$

Subtracting (3) from (4), we have

$$0 = \frac{4\epsilon_1}{f_1} + \frac{\epsilon_1}{n_1} + \frac{\epsilon_2}{h_2}.$$

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Using $\Phi_3^*(1) = \epsilon_1 f_1 + \zeta_1 h_1 + \zeta_2 h_2 = 0$ and substituting the value of f_1, we find that $\zeta_1 = \zeta_2$, $h_1 = h_2$. Since $f_1 = -\epsilon_1(\zeta_1 h_1 + \zeta_2 h_2)$, $f_1 = -2\epsilon_1\zeta_1 h_1$ and it follows that $\epsilon_1 = -\zeta_1, f_1 = 2h_1$.

We now have that $f_1 = 2g_1$, $f_1 = 2h_1$, $g_1 = g_2$, $\eta_1 = \eta_2$, $h_1 = h_2$, $\zeta_1 = \zeta_2$, $\epsilon_1 = -\zeta_1$ and $\epsilon_1 = -\eta_1$. Using this information, (2) becomes

$$0 = 1 - \frac{16\epsilon_1}{f_1} - \frac{\epsilon_2}{f_2} - \frac{\epsilon_3}{f_3} - \frac{\epsilon_4}{f_4}. \tag{2}$$

Equation (1) becomes,

$$1 = \frac{|G|}{(18)^2}\left[1 + \frac{20\epsilon_1}{f_1} - \frac{\epsilon_2}{f_1} - \frac{\epsilon_3}{f_1} - \frac{\epsilon_4}{f_4}\right]. \tag{1}$$

Substituting (2) into (1), we have

$$1 = \frac{|G|}{(18)^2}\left[\frac{36\epsilon_1}{f_1}\right] \quad \text{or} \quad f_1 = \frac{|G|}{9}\epsilon_1.$$

It follows that $\epsilon_1 = 1$ and $f_1 = |G|/9$. Since $|G| \geqq f_1^2$, $81 \geqq |G|$ so that $|G| = 36$ or 72. In the latter case, $f_1 = 8$, $g_1 = 4$, $g_2 = 4$ and $72 \geqq 8^2 + 4^2 + 4^2 = 96$, a contradiction. Finally, $|G| = 36$ so that $G = N_G(T)$ and (3.3) holds in case (ii).

We are now in a position to prove the following theorem.

(3.6) *G has precisely* 4 *classes of elements of order* 3 *given by the representatives* $\alpha_1, \alpha_1^{-1}, \alpha_1^{-1}\alpha_1^{\tau}, \alpha_1\alpha_1^{\tau}$. *We have* $C_G(\alpha_1) = C_G(\alpha_1^{-1}) = H$ *and* $C_G(\alpha_1^{-1}\alpha_1^{\tau}) = M\langle t\rangle$, $C_G(\alpha_1\alpha_1^{\tau}) = M\langle t, \tau\rangle$.

Proof. The first part of this theorem follows from (2.6) and (3.2). It remains to determine the structure of $C_G(\alpha_1\alpha_1^{\tau})$. Since $\alpha_1\alpha_1^{\tau}$ is not conjugate to α_1 or α_1^{-1}, $C_G(\alpha_1\alpha_1^{\tau})$ has $N = M\langle t, \tau\rangle$ as a Sylow 3-normalizer. Since $Z(N) \cap M = \langle\alpha_1\alpha_1^{\tau}\rangle$, $C_G(\alpha_1\alpha_1^{\tau})$ has a normal subgroup K of index 3 and $K \cap M = [M, \langle t, \tau\rangle] = \langle\alpha_2\rangle \times \langle\alpha_1^{-1}\alpha_1^{\tau}\rangle$. Let $T = K \cap M$. From (3.2), $C_K\langle\alpha_2\rangle = T\langle\tau t\rangle$ and $C_K\langle\alpha_1^{-1}\alpha_1^{\tau}\rangle = T\langle t\rangle$. Since $\alpha_1\alpha_1^{\tau}$ centralizes K, $C_K\langle\alpha_1^{-1}\alpha_2\alpha_1^{\tau}\rangle \subseteq C_K(\alpha_1\alpha_2)$. However, $\alpha_1\alpha_2$ and α_2 are conjugate so that $C_G(\alpha_1\alpha_2) \subseteq M\langle t, \tau, x\rangle = N_G(M)$. It follows that $C_K(\alpha_1^{-1}\alpha_2\alpha_1^{\tau}) = T$. We have shown that K is a group satisfying the hypothesis of (3.3) so that $K = T\langle t, \tau\rangle$. If follows that $C_G(\alpha_1\alpha_1^{\tau}) = M\langle t, \tau\rangle$.

4. Structure of $C_G(t)$. It is now possible to determine the structure of $C_G(t)$. The first result of this section is a characterization of the centralizer of a central involution in $PSp_4(3)$.

(4.1) *Let G be a finite group with an involution t in its center. If G has a Sylow* 3-*subgroup T which is elementary abelian of order* 9 *and a Sylow* 3-*normalizer* $T\langle t, \tau\rangle$ *such that*

(i) $T = \langle\alpha\rangle \times \langle\beta\rangle, \alpha^{\tau} = \beta, \tau^2 = 1$;

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(ii) $C_G(\alpha) = TQ$, $C_G(\beta) = TQ^\tau$ *where* Q *is a quaternion group of order* 8 *normalized by* T *and not centralized by* β;

(iii) $C_G(\alpha^{-1}\beta) = T\langle t\rangle$, $C_G(\alpha\beta) = T\langle t, \tau\rangle$.

Then $G = S_1S_2\langle\tau\rangle$ *where* $S_1 = \langle\beta\rangle Q$, $S_2 = \langle\alpha\rangle Q^\tau$, $[S_1, S_2] = 1$, $S_1 \cap S_2 = \langle t\rangle$ *and* $S_1{}^\tau = S_2$.

Proof. Let $N = T\langle t, \tau\rangle$. By a theorem of Grün, G has a normal subgroup K of index 3 such that $K \cap T = [T, \langle t, \tau\rangle] = \langle\alpha^{-1}\beta\rangle$. By (iii), $C_K(\alpha^{-1}\beta) = \langle\alpha^{-1}\beta, t\rangle$. Let $X = K/\langle t\rangle$. Then X has a self-centralizing Sylow 3-group of order 3 and a Sylow 3-normalizer of order 6. Since X admits T as a group of automorphisms and since $\alpha\beta$ does not centralize Q, $\alpha\beta$ induces a nontrivial automorphism of X. Since $\alpha\beta$ centralizes τ and $\alpha^{-1}\beta$ is inverted by τ, it follows that $\alpha\beta$ and $\alpha^{-1}\beta$ induce distinct automorphisms of X. Furthermore, the only inner automorphism of X that centralizes $\langle\alpha^{-1}\beta, t\rangle/\langle t\rangle$ is induced by $\alpha^{-1}\beta$. It follows that $\alpha\beta$ induces a nontrivial outer automorphism of X of order 3. We conclude that X is not isomorphic to PSL(2, 5) or PSL(2, 7).

By [**3**], K has a normal nilpotent subgroup R such that K/R is isomorphic to S_3 or PSL(2, 5). For any $\sigma \in T$, RR^σ is a normal subgroup of K of order prime to 3. From the structure of K/R, $RR^\sigma = R$ so that $R^\sigma = R$. Thus R is a T invariant nilpotent subgroup of K. Let P be a Sylow p-subgroup of R, $p \neq 2$, $p \neq 3$. Then

$$P = C_P(\alpha)C_P(\beta)C_P(\alpha\beta)C_P(\alpha^{-1}\beta) = 1$$

and we conclude that R is a 2-group such that $R = C_R(\alpha)C_R(\beta)$. Since $R \neq \langle t\rangle$, we may assume without loss of generality that $C_R(\alpha) \neq \langle t\rangle$. Then $Q \cap R \neq \langle t\rangle$ and since β acts regularly on the nonidentity elements of $Q/\langle t\rangle$, $Q \subseteq R$. Since $R \lhd K$, $Q^\tau \subseteq R$ and we have $R = QQ^\tau$.

Now let us suppose that $K/R \cong$ PSL(2, 5). Then K has a subgroup F such that F/R is an elementary abelian group of order 4 which is normalized by $\alpha^{-1}\beta$. If $\alpha\beta$ induces a trivial automorphism of K/R, then T leaves F/R invariant and $F = C_F(\alpha)C_F(\beta)C_F(\alpha\beta)C_F(\alpha^{-1}\beta)$. This implies that $F = QQ^\tau = R$ which is not the case. We conclude that $\alpha\beta$ induces a nontrivial outer automorphism of K/R of order 3, a contradiction.

Finally, $K/R \cong S_3$ so that $K = R\langle\alpha^{-1}\beta, \tau\rangle = QQ^\tau\langle\alpha^{-1}\beta, \tau\rangle$. Since R is a 2-group, $Q^\tau \cap N_G(Q) \neq \langle t\rangle$. This implies that $Q^\tau \subseteq N(Q)$ since α acts regularly on the nonidentity elements of $Q^\tau/\langle t\rangle$. It follows that R is a 2-group of $N_G(Q)$ of order 32 and since $N_G(Q)/C_G(Q)$ is isomorphic to a subgroup of S_4, $Q^\tau \subseteq C_G(Q)$. Let $S_1 = \langle\beta\rangle Q$, $S_2 = \langle\alpha\rangle Q^\tau$. Then $S_1 \cap S_2 = \langle t\rangle$, $[S_1, S_2] = 1$, $S_1{}^\tau = S_2$ and $G = S_1S_2\langle\tau\rangle$.

We can now determine the structure of $C_G(t)$. Let t, τ, Q have the same meaning as in (1.4), (1.5) and (2.6).

(4.2) $C_G(t) = S_1S_2\langle\tau\rangle$, $S_1 = \langle\alpha_1{}^\tau\rangle Q$, $S_2 = \langle\alpha_1\rangle Q^\tau$, $[S_1, S_2] = 1$, $S_1 \cap S_2 = \langle t\rangle$ *and* $S_1{}^\tau = S_2$.

Proof. From the structure of H, $C_M(t) = \langle\alpha_1\rangle \times \langle\alpha_1^\tau\rangle$. Let $T = \langle\alpha_1\rangle \times \langle\alpha_1^\tau\rangle$. Then $C_G(T) = C_H(T) = M\langle t\rangle$ and it follows that $N_G(T) \subseteq N_G(M)$. From the structure of $N_G(M)$, $T\langle t, \tau\rangle$ is a Sylow 3-normalizer of $C_G(t)$. From the structure of H, $C_G(t) \cap C_G(\alpha_1) = TQ$ and $C_G(t) \cap C_G(\alpha_1^\tau) = TQ^\tau$. By (3.6), $C_G(\alpha_1\alpha_1^\tau) = M\langle t, \tau\rangle$ and $C_G(\alpha_1^{-1}\alpha_1^\tau) = M\langle t\rangle$ so that $C(t) \cap C_G(\alpha_1\alpha_1^\tau) = T\langle t, \tau\rangle$ and $C_G(t) \cap C(\alpha_1^{-1}\alpha_1^\tau) = T\langle t\rangle$. Applying (4.1), the result follows.

(4.3) *Let $S = QQ^\tau\langle\tau\rangle$ be a Sylow 2-group of $C_G(t)$. Then S is a Sylow 2-group of G.*

Proof. Let S_1 be a Sylow 2-group of G containing S. Since $Z(S) = \langle t\rangle$, $N_{S_1}(S) \subseteq C(t)$. This implies that $N_{S_1}(S) = S$ and we have $S_1 = S$.

From (4.2) we see that condition (b) of [**6**] has been established.

By the structure of $C_G(t)$, $C_G(t)$ has exactly 4 classes of involutions with representatives t, τ, $t\tau$, and qq^τ where $q \in Q$ is some element of order 4. The involution t is conjugate in G to one of τ, $t\tau$ or qq^τ since otherwise $G = O_{2'}(G)C_G(t)$ by [**4**] and G has a normal subgroup of index 3, a contradiction to (a) of Theorem 1. The proof of Theorem 1 now follows from [**6**].

It is perhaps interesting to notice that Sections 2, 3, and 4 of this paper together with the first three sections of [**6**] determine the local 2 and 3 structure of G. Letting $B = N_G(P)$ and $N = \langle\tau, \tau q\rangle$, $N/\langle t\rangle$ is a dihedral group of order 8 and one may show that $G = BNB \cong PSp_4(3)$ directly without the use of the last sections of [**6**] and particularly without the aid of the characterization of $PSp_4(3)$ in [**9**].

References

1. E. Artin, *Geometric algebra* (Interscience, New York, 1957).

2. R. Carter and P. Fong, *The Sylow 2-subgroup of the finite classical groups*, J. Algebra *1* (1964), 139–151.

3. W. Feit and J. Thompson, *Finite groups which contain a self-centralizing subgroup of order 3*, Nagoya Math. J. *21* (1962), 185–197.

4. G. Glauberman, *Central elements in core-free groups*, J. Algebra *4* (1966), 403–420.

5. D. Gorenstein, *Finite groups* (Harper and Row, New York, 1968).

6. Z. Janko, *A characterization of the finite simple group* $PSp_4(3)$, Can. J. Math. *19* (1967), 872–894.

7. W. Scott, *Group theory* (Prentice-Hall, New York, 1968).

8. M. Suzuki, *Applications of group characters*, Proc. Symp. Pure Math., Amer. Math. Soc. *1* (1959), 88–99.

9. J. G. Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable*. II, Pacific J. Math. *33* (1970), 451–536.

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