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(ANTI)COMMUTATIVE ALGEBRAS WITH A MULTIPLICATIVE BASIS

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Abstract

A basis $\mathcal{B} = \{u_i\}_{i\in I}$ of a commutative or anticommutative algebra $\mathfrak{C}$, over an arbitrary base field $\mathbb{F}$, is called multiplicative if for any $i, j \in I$ we have that $u_i u_j \in \mathbb{F}u_k$ for some $k \in I$. We show that if a commutative or anticommutative algebra $\mathfrak{C}$ admits a multiplicative basis then it decomposes as the direct sum $\mathfrak{C} = \bigoplus_j \mathfrak{i}_j$ of well-described ideals each one of which admits a multiplicative basis. Also the minimality of $\mathfrak{C}$ is characterised in terms of the multiplicative basis and it is shown that, under a mild condition, the above direct sum is indexed by the family of its minimal ideals admitting a multiplicative basis.

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1. Introduction and previous definitions

Throughout this paper $\mathfrak{C}$ will denote a commutative or anticommutative algebra (in which further identities on the product are not supposed) of arbitrary dimension and over an arbitrary base field $\mathbb{F}$, whose product will be denoted by juxtaposition.

Definition 1.1. A basis $\mathcal{B} = \{u_i\}_{i\in I}$ of $\mathfrak{C}$ is said to be *multiplicative* if for any $i, j \in I$ we have either $u_i u_j = 0$ or $0 \neq u_i u_j \in \mathbb{F}u_k$ for some (unique) $k \in I$.

We can easily construct many examples of (anti)commutative algebras admitting multiplicative bases. Indeed, it is enough to fix an arbitrary (nonempty) set of indexes I, a symmetric mapping $\alpha : I \times I \to I \,\dot\cup\, \{0\}$ and an ϵ-symmetric map $\beta : I \times I \to \mathbb{F}$ in the sense $\alpha(i, j) = \alpha(j, i)$ for any $i, j \in I$ and $\beta(k, l) = \epsilon\beta(l, k)$ for any $(k, l) \in I \times I$ such that $\alpha(i, j) \neq 0$, and $\epsilon \in \{\pm 1\}$. Then the $\mathbb{F}$-linear space $\mathfrak{C}$ with basis $\{u_i\}_{i\in I}$ and product among the elements of the basis given by $u_i u_j = \beta(i, j)u_{\alpha(i,j)}$, where $u_0 := 0$, becomes a commutative or anticommutative algebra admitting $\mathcal{B}$ as a multiplicative basis according as $\epsilon = 1$ or $\epsilon = -1$. For instance:

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EXAMPLE 1.2. Let $\mathfrak{C}$ be the algebra where $\mathcal{B} = \{u_n : n \in \mathbb{Z}\}$ is a basis of $\mathfrak{C}$ and the nonzero products with respect to the elements in the basis $\mathcal{B}$ are $u_n u_m = nm u_{n+m}$. Then $\mathfrak{C}$ becomes a commutative algebra admitting $\mathcal{B}$ as a multiplicative basis.

Classical examples of commutative algebras with a multiplicative basis are the commutative group-algebras [4], the more general category of commutative twisted group-algebras which generalises a number of types of Banach algebras (see [14, 18] and the seminal work [3]), and generalised L^1 algebras and commutative group-algebras (see [18]). We also observe that since it is usual in the literature to describe an algebra by exhibiting a multiplicative table among the elements of a fixed basis, we can also find many examples of (anti)commutative algebras admitting multiplicative bases in the categories of Lie algebras, Malcev algebras, hom-Lie algebras, Jordan algebras and hom-Jordan algebras. For instance, semisimple finite-dimensional Lie algebras, semisimple separable L^*-algebras [17], semisimple locally finite split Lie algebras [19], Heisenberg algebras [15], twisted Heisenberg algebras [1], the split Lie algebras considered in [6, Section 3] and the Lie algebras shown in [10, 11] as examples of nonsemigroup gradings are classes of anticommutative algebras admitting a multiplicative basis. By looking at the multiplication table of the non-Lie Malcev algebra $\mathfrak{C}_0$ (seven-dimensional algebra over its centroid) [16, Section 6], we have another example of an anticommutative algebra with a multiplicative basis. In [13] we can find examples of hom-Jordan algebras admitting multiplicative bases and so of commutative algebras with multiplicative bases.

REMARK 1.3. The definition of multiplicative basis given in Definition 1.1 is a little more general than the usual one in the literature [2, 4, 5, 12]. In fact, in these references, a basis $\mathcal{B} = \{u_i\}_{i \in I}$ is called multiplicative if for any $i, j \in I$ we have either $u_i u_j = 0$ or $0 \neq u_i u_j = u_k$ for some $k \in I$.

The present paper is devoted to the study of commutative or anticommutative algebras $\mathfrak{C}$ of arbitrary dimension over an arbitrary base field $\mathbb{F}$ admitting a multiplicative basis, focusing on their structure. The paper is organised as follows. In Section 2, inspired by the connection techniques developed for split algebras in [7–9], we introduce connection techniques on the set of indexes I of the multiplicative basis so as to get a powerful tool for the study of this class of algebras. By making use of these techniques, we show that any commutative or anticommutative algebra $\mathfrak{C}$ admitting a multiplicative basis is of the form $\mathfrak{C} = \bigoplus_k \mathfrak{i}_k$, where each $\mathfrak{i}_k$ is a well-described ideal of $\mathfrak{C}$ admitting also a multiplicative basis. In Section 3 we characterise minimality of $\mathfrak{C}$ in terms of the multiplicative basis and show that, in case the basis is $\star$-multiplicative, the above decomposition of $\mathfrak{C}$ is by means of the family of its minimal ideals.

2. Connections in the support. Decompositions

In what follows $\mathcal{B} = \{u_i\}_{i \in I}$ denotes the multiplicative basis of $\mathfrak{C}$, and $\mathcal{P}(I)$ the power set of I.

We begin this section by developing connection techniques among the elements in the set of indexes I as the main tool in our study.

For each $i \in I$, a new variable $\bar{i} \notin I$ is introduced and we denote by

$$\bar{I} := \{\bar{i} : i \in I\}$$

the set consisting of all these new symbols.

Next, we consider the following operation which recovers, in some sense, certain multiplicative relations among the elements of $\mathcal{B}$:

$$\star : I \times (I \dot{\cup} \bar{I}) \to \mathcal{P}(I).$$

This is given by:

- for $i, j \in I$,

$$i \star j = \begin{cases} \emptyset & \text{if } u_i u_j = 0, \\ \{k\} & \text{if } 0 \neq u_i u_j \in \mathbb{F} u_k; \end{cases}$$

- for $i \in I$ and $\bar{j} \in \bar{I}$,

$$i \star \bar{j} = \{k \in I : 0 \neq u_k u_j \in \mathbb{F} u_i\}.$$

Finally, we also define the mapping

$$\phi : \mathcal{P}(I) \times (I \dot{\cup} \bar{I}) \to \mathcal{P}(I)$$

as

- $\phi(\emptyset, I \dot{\cup} \bar{I}) = \emptyset$;
- for any $J \in \mathcal{P}(I)$ and $a \in I \dot{\cup} \bar{I}$,

$$\phi(J, a) = \bigcup_{x \in J} (x \star a).$$

From now on, given any $\bar{i} \in \bar{I}$, we will denote

$$\overline{(\bar{i})} := i.$$

Observe that for any $i, j \in I$ and $a \in I \dot{\cup} \bar{I}$, we have that $j \in i \star a$ if and only if $i \in j \star \bar{a}$. This fact implies that for any $J \in \mathcal{P}(I)$ and $a \in I \dot{\cup} \bar{I}$,

$$i \in \phi(J, a) \quad \text{if and only if} \quad \phi(\{i\}, \bar{a}) \cap J \neq \emptyset. \tag{2.1}$$

Definition 2.1. Let i and j be distinct elements in the set of indexes I. We say that i is *connected* to j if there exists a subset

$$\{i_1, i_2, \ldots, i_{n-1}, i_n\} \subset I \dot{\cup} \bar{I}$$

with $n \geq 2$ such that the following conditions hold:

(1) $i_1 = i$;

(2) $\phi(\{i_1\}, i_2) \neq \emptyset$,
$\phi(\phi(\{i_1\}, i_2), i_3) \neq \emptyset$,

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$\phi(\phi(\phi(\{i_1\}, i_2), i_3), i_4) \neq \emptyset,$
$\ldots$
$\phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-2}), i_{n-1}) \neq \emptyset;$
(3) $j \in \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-1}), i_n).$

The subset $\{i_1, i_2, \ldots, i_{n-1}, i_n\}$ is called a *connection* from i to j. We consider i to be connected to itself.

Proposition 2.2. *The relation $\sim$ on I, defined by $i \sim j$ if and only if i is connected to j, is an equivalence relation.*

Proof. By definition $i \sim i$, that is, the relation $\sim$ is reflexive.

Let us show the symmetric character of $\sim$. If $i \sim j$ with $i \neq j$ then there exist an $n \geq 2$ and a connection

$$\{i_1, i_2, \ldots, i_{n-1}, i_n\} \subset I \dot{\cup} \overline{I}$$

from i to j satisfying Definition 2.1. Let us verify that the set

$$\{j, \overline{i_n}, \overline{i_{n-1}}, \ldots, \overline{i_3}, \overline{i_2}\} \subset I \dot{\cup} \overline{I}$$

gives us a connection from j to i. Indeed, (2.1) together with the fact that

$$j \in \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-1}), i_n)$$

gives us

$$\phi(\{j\}, \overline{i_n}) \cap \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-2}), i_{n-1}) \neq \emptyset$$

and so $\phi(\{j\}, \overline{i_n}) \neq \emptyset$. By taking

$$k_1 \in \phi(\{j\}, \overline{i_n}) \cap \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-2}), i_{n-1}),$$

Equation (2.1) and the fact that $k_1 \in \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-2}), i_{n-1})$ imply that

$$\phi(\phi(\{j\}, \overline{i_n}), \overline{i_{n-1}}) \cap \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-3}), i_{n-2}) \neq \emptyset$$

and consequently $\phi(\phi(\{j\}, \overline{i_n}), \overline{i_{n-1}}) \neq \emptyset$.

By iterating this process we get

$$\phi(\phi(\cdots(\phi(\{j\}, \overline{i_n}), \cdots), \overline{i_{n-i+1}}), \overline{i_{n-i}}) \cap \phi(\phi(\cdots(\phi(\{i_1\}, i_2), \cdots), i_{n-i-2}), i_{n-i-1}) \neq \emptyset$$

for $0 \leq i \leq n-3$. Observe that for $i = n-3$ we have

$$\phi(\phi(\cdots(\phi(\{j\}, \overline{i_n}), \cdots), \overline{i_4}), \overline{i_3}) \cap \phi(\{i_1\}, i_2) \neq \emptyset.$$

This equation, together with the fact that $i_1 = i$ and (2.1), allows us to assert that

$$i \in \phi(\phi(\cdots(\phi(\{j\}, \overline{i_n}), \cdots), \overline{i_3}), \overline{i_2})$$

and conclude that $\sim$ is symmetric.

Finally, let us verify the transitive character of $\sim$. Suppose that $i \sim j$ and $j \sim k$. If $i = j$ or $j = k$ it is clear that $i \sim k$. Consider then $i \neq j$ and $j \neq k$ and write $\{i_1, \ldots, i_n\}$ for a connection from i to j and $\{j_1, \ldots, j_m\}$ for a connection from j to k. Then we clearly have that $\{i_1, \ldots, i_n, j_2, \ldots, j_m\}$ is a connection from j to k. We have shown that the connection relation is an equivalence relation. □

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By the above proposition we can introduce the quotient set

$$I/\sim = \{[i] : i \in I\},$$

where $[i]$ denotes the set of elements in I which are connected to i.

For any $[i] \in I/\sim$ we define the linear subspace

$$\mathfrak{C}_{[i]} := \bigoplus_{j\in[i]} \mathbb{F}u_j.$$

Lemma 2.3. *If* $\mathfrak{C}_{[i]}\mathfrak{C}_{[j]} \neq 0$ *for some* $[i], [j] \in I/\sim$, *then* $[i] = [j]$ *and* $\mathfrak{C}_{[i]}\mathfrak{C}_{[j]} \subset \mathfrak{C}_{[i]}$.

Proof. For any $k \in [i]$ and $h \in [j]$ such that $u_k u_h \neq 0$, we have $0 \neq u_k u_h \in \mathbb{F}u_l$ for some $l \in I$. Hence, $l \in \phi(\{k\}, h)$ and so the set $\{k, h\}$ is a connection from k to l which, together with the transitivity of the connection relation, gives us $[i] = [l]$. Consequently $\mathfrak{C}_{[i]}\mathfrak{C}_{[j]} \subset \mathfrak{C}_{[i]}$. Now observe that by (anti)commutativity $0 \neq u_h u_k \in \mathbb{F}u_l$ and then $h \in l \star \overline{k}$. Hence $\{l, k\}$ is a connection from l to h and we conclude that $[i] = [j]$. □

Definition 2.4. Let $\mathfrak{C}$ be an (anti)commutative algebra with a multiplicative basis $\mathcal{B}$. It is said that a subalgebra $\mathfrak{a}$ of $\mathfrak{C}$ admits a multiplicative basis $\mathcal{B}_{\mathfrak{a}}$ *inherited* from $\mathcal{B}$ if $\mathcal{B}_{\mathfrak{a}}$ is a multiplicative basis of $\mathfrak{a}$ satisfying $\mathcal{B}_{\mathfrak{a}} \subset \mathcal{B}$.

Theorem 2.5. *Let* $\mathfrak{C}$ *be an (anti)commutative algebra with a multiplicative basis. Then*

$$\mathfrak{C} = \bigoplus_{[i]\in I/\sim} \mathfrak{C}_{[i]},$$

where each $\mathfrak{C}_{[i]}$ *is an ideal of* $\mathfrak{C}$ *admitting a multiplicative basis inherited from that of* $\mathfrak{C}$ *and satisfying*

$$\mathfrak{C}_{[i]}\mathfrak{C}_{[j]} = 0$$

whenever $[i] \neq [j]$.

Proof. Since we can write

$$\mathfrak{C} = \bigoplus_{i\in I} \mathbb{F}u_i,$$

we have

$$\mathfrak{C} = \bigoplus_{[i]\in I/\sim} \mathfrak{C}_{[i]}.$$

Hence, Lemma 2.3 gives us that for any $[i] \in I/\sim$,

$$\mathfrak{C}_{[i]}\mathfrak{C} = \mathfrak{C}_{[i]}\Big(\mathfrak{C}_{[i]} \oplus \Big(\bigoplus_{[j]\in I/\sim, [j]\neq[i]} \mathfrak{C}_{[j]}\Big)\Big) \subset \mathfrak{C}_{[i]}.$$

That is, any $\mathfrak{C}_{[i]}$ is actually an ideal of $\mathfrak{C}$ satisfying $\mathfrak{C}_{[i]}\mathfrak{C}_{[j]} = 0$ whenever $[j] \neq [i]$ by Lemma 2.3. □

Corollary 2.6. *If* $\mathfrak{C}$ *is simple, then there exists a connection between any two elements of* I.

Proof. The simplicity of $\mathfrak{C}$ means that $\mathfrak{C}_{[i]} = \mathfrak{C}$ for some $[i] \in I/\sim$. Hence $[i] = I$ and so any pair of elements in I are connected. □

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3. The minimal components

In this section, our target is to characterise the minimality of the ideals which give rise to the decomposition of $\mathfrak{C}$ in Theorem 2.5 in terms of connectivity properties in the set of indexes I. We begin by introducing the concept of minimality.

DEFINITION 3.1. An anti(commutative) algebra $\mathfrak{C}$ admitting a multiplicative basis $\mathcal{B}$ is called *minimal* if its only nonzero ideal admitting a multiplicative basis inherited from $\mathcal{B}$ is $\mathfrak{C}$.

Let us introduce the notion of $\star$-multiplicativity in the framework of anti(commutative) algebras with multiplicative bases, in a similar way to the concept of closed multiplicativity for Poisson algebras, split Leibniz algebras, or split colour algebras among other classes of algebras (see [7–9] for these notions and examples).

DEFINITION 3.2. We say that an (anti)commutative algebra $\mathfrak{C}$ admits a $\star$*-multiplicative basis* $\mathcal{B} = \{u_i\}_{i\in I}$ if it is multiplicative and given $i, j \in I$ such that $j \in i \star a$ for some $a \in I \dot\cup \bar{I}$ then $u_j \in u_i\mathfrak{C}$.

Examples of (anti)commutative algebras admitting $\star$-multiplicative bases are the semisimple finite-dimensional Lie algebras, the semisimple separable L^*-algebras, the semisimple locally finite split Lie algebras, the split Lie algebras considered in [6, Section 3], the non-Lie Malcev algebra $\mathfrak{C}_0$ (see Section 1) and the algebra $\mathfrak{C}$ in Example 1.2.

THEOREM 3.3. *Let $\mathfrak{C}$ be an (anti)commutative algebra admitting a $\star$-multiplicative basis $\mathcal{B} = \{u_i\}_{i\in I}$. Then $\mathfrak{C}$ is minimal if and only if the set of indexes I has all of its elements connected.*

PROOF. The first implication is similar to Corollary 2.6. To prove the converse, consider a nonzero ideal $\mathfrak{i}$ of $\mathfrak{C}$ admitting a basis inherited by $\mathcal{B}$. Then we can write $\mathfrak{i} = \oplus_{j\in I_{\mathfrak{i}}}\mathbb{F}u_j$ for a certain $\emptyset \neq I_{\mathfrak{i}} \subset I$. Fix some $i_0 \in I_{\mathfrak{i}}$ whence

$$0 \neq u_{i_0} \in \mathfrak{i}. \tag{3.1}$$

Given now any $k \in I$, since I has all of its elements connected, there exists a connection

$$\{i_0, i_2, \ldots, i_{n-1}, i_n\} \subset I \dot\cup \bar{I} \tag{3.2}$$

from i_0 to k. Hence

$$\phi(\{i_0\}, i_2) \neq \emptyset,$$

and so for any $b_1 \in \phi(\{i_0\}, i_2)$ we have $b_1 \in i_0 \star i_2$. Taking into account (3.1) and the $\star$-multiplicativity of $\mathcal{B}$,

$$u_{b_1} \in u_{i_0}\mathfrak{C} \subset \mathfrak{i}.$$

Hence we can assert that

$$\bigoplus_{j\in\phi(\{i_0\},i_2)} \mathbb{F}u_j \subset \mathfrak{i}. \tag{3.3}$$

Since

$$\phi(\phi(\{i_0\}, i_2), i_3) \neq \emptyset,$$

we can argue as above, taking into account (3.3), that

$$\bigoplus_{j \in \phi(\phi(\{i_0\}, i_2), i_3)} \mathbb{F} u_j \subset \mathfrak{i}.$$

By iterating this process with the connection (3.2), we obtain

$$\bigoplus_{j \in \phi(\phi(\cdots(\phi(i_0, i_2), \cdots), i_{n-1}), i_n)} \mathbb{F} u_j \subset \mathfrak{i}$$

and so, since $k \in \phi(\phi(\cdots(\phi(i_0, i_2), \cdots), i_{n-1}), i_n)$, we get $u_k \in \mathfrak{i}$. Hence $\mathfrak{i} = \mathfrak{C}$ and $\mathfrak{C}$ is minimal. □

Theorem 3.4. *Let $\mathfrak{C}$ be an (anti)commutative algebra admitting a $\star$-multiplicative basis $\mathcal{B} = \{u_i\}_{i \in I}$. Then*

$$\mathfrak{C} = \bigoplus_k \mathfrak{i}_k$$

is the direct sum of the family of its minimal ideals, each summand of which admits a $*$-multiplicative basis inherited from $\mathcal{B}$.

Proof. By Corollary 2.6 we have that $\mathfrak{C} = \bigoplus_{[i] \in I/\sim} \mathfrak{C}_{[i]}$ is the direct sum of the ideals $\mathfrak{C}_{[i]}$.

We wish to apply Theorem 3.3 to any summand $\mathfrak{C}_{[i]}$, so we have to verify that $\mathfrak{C}_{[i]}$ admits a $\star$-multiplicative basis and that the basis $\{u_i : i \in [i]\}$ of $\mathfrak{C}_{[i]}$ is such that all of the elements in the set of indexes $[i]$ are $[i]$-connected (connected through connections contained in $[i] \dot\cup \overline{[i]}$).

Clearly, $\mathfrak{C}_{[i]}$ admits a $\star$-multiplicative basis as a consequence of having a basis inherited from $\mathcal{B}$ and the fact that $\mathfrak{C}_{[i]}\mathfrak{C}_{[j]} = 0$ when $[i] \neq [j]$. Taking into account the anti(commutativity) of $\mathfrak{C}$, it is easy to verify that $[i]$ has all of its elements $[i]$-connected. So we can apply Theorem 3.3 to any $\mathfrak{C}_{[i]}$ to conclude that $\mathfrak{C}_{[i]}$ is minimal. It is clear that the decomposition $\mathfrak{C} = \bigoplus_{[i] \in I/\sim} \mathfrak{C}_{[i]}$ satisfies the assertions of the theorem. □

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