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ON UNICITY OF MEROMORPHIC SOLUTIONS TO DIFFERENCE EQUATIONS OF MALMQUIST TYPE

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Dedicated to Professor Hongxun Yi on the occasion of his 72nd birthday

Abstract

In this note, we prove a uniqueness theorem for finite-order meromorphic solutions to a class of difference equations of Malmquist type. Such solutions f are uniquely determined by their poles and the zeros of $f - e_j$ (counting multiplicities) for two finite complex numbers $e_1 \neq e_2$.

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1. Introduction and main result

We study finite-order meromorphic solutions f to the difference equation

$$\sum_{j=1}^{n} a_j f(z + c_j) = \frac{P(f)}{Q(f)} = \frac{b_p f^p + b_{p-1} f^{p-1} + \cdots + b_1 f + b_0}{d_q f^q + d_{q-1} f^{q-1} + \cdots + d_1 f + d_0} = \frac{\sum_{k=0}^{p} b_k f^k}{\sum_{l=0}^{q} d_l f^l} \tag{1.1}$$

of Malmquist type. Here, $a_j\,(\not\equiv 0), b_k, d_l$ are small functions of f, $c_j\,(\neq 0)$ are pairwise distinct constants, P and Q are coprime polynomials and n, p, q are integers such that $p \leq q = n$. In view of Heittokangas *et al.* [6, Proposition 8 and Theorem 12], the assumption $p \leq q = n$ is natural. We shall give several examples below to show that our hypotheses are best possible.

Some qualitative properties of meromorphic solutions to (1.1) are known (see for example [6, 11, 15] and the references therein). In this note, we investigate how a finite-order meromorphic solution f to (1.1) is uniquely determined by its poles and the zeros of $f - e_j$ for two distinct, finite complex numbers e_1, e_2. This is motivated by the famous *Nevanlinna five-value theorem* and its improvements when one considers meromorphic solutions to ordinary and partial differential equations.

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For example, Brosch [1] proved that a meromorphic solution to the Malmquist-type ordinary differential equation $(w')^n = \sum_{j=0}^{2n} a_j w^j$ in the complex plane $\mathbb{C}$ is uniquely determined by three values. For Malmquist-type partial differential equations, we refer the reader to, for instance, Hu and Li [7, 8] as well as some earlier work of Tu [13], Hu and Yang [9, 10] and Gao [3].

Define $I(z, f) := \sum_{j=1}^{n} a_j f(z + c_j)$ and $H(z, f) := Q(f)I(z, f) - P(f)$. Then (1.1) can be rewritten as

$$H(z, f) = 0. \tag{1.2}$$

By applying the main ideas in [1], we will prove the following result for difference equations.

Theorem 1.1. *Let f be a finite-order transcendental meromorphic solution of (1.2) and let e_1, e_2 be two distinct finite numbers such that $H(z, e_1), H(z, e_2) \neq 0$. If f and another meromorphic function g share the values e_1, e_2 and ∞* CM, *then $f = g$.*

We assume familiarity with the basics of Nevanlinna theory of meromorphic functions in $\mathbb{C}$, such as the *first* and *second* main theorems and the usual notation such as the *characteristic function* $T(r, f)$, the *proximity function* $m(r, f)$ and the *counting function* $N(r, f)$. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, except possibly on a set of finite logarithmic measure, not necessarily the same at each occurrence. Let a, f, g be meromorphic functions on $\mathbb{C}$. We say that a is a small function of f whenever $T(r, a) = S(r, f)$. Given a, a small function of both f and g or some value in $\mathbb{C} \cup \{\infty\}$, we say that f and g share a CM if $f - a$ and $g - a$ have the same zeros with the same multiplicities. Finally, the order of f, $\rho(f)$, is the quantity

$$\rho(f) := \limsup_{r \to +\infty} \frac{\log T(r, f)}{\log r}.$$

Example 1.2. Below, we provide some examples that are related to the assumptions of Theorem 1.1, which shows that our result is best possible.

(a) The number of shared values cannot be reduced. For example, the function $f(z) = (e^{2\pi i z} + z + 1)^{-1}$ is a solution of the difference equation

$$H(z, f) = [f^2(z) - 1][f(z + 1) + f(z - 1)] + 2f(z) = 0,$$

while f and $g(z) = e^{2\pi i z} + z + 1$ share the values $1, -1$ CM with $H(z, 1) = 2$ and $H(z, -1) = -2$.

(b) The condition $H(z, e_1), H(z, e_2) \neq 0$ cannot be dropped. For example, the function $f(z) = \tan z$ is a solution of the difference equation

$$H(z, f) = [f^2(z) - 1]\Big[f\Big(z + \frac{\pi}{4}\Big) + f\Big(z - \frac{\pi}{4}\Big)\Big] + 4f(z) = 0,$$

and f and $g(z) = -\tan z$ share the values $\pm i$ (as Picard exceptional values) and ∞ CM with $H(z, \pm i) = 0$.

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(c) The condition $p \leq q$ cannot be extended to include $p > q$. For example, the function $f(z) = e^z$ is a solution of the difference equation

$$H(z, f) = [f(z+1) + f(z-1)] - (e + e^{-1}) f(z) = 0,$$

and f and $g(z) = e^{-z}$ share the values ± 1 and ∞ CM with $H(z, 1) = 2 - e - e^{-1}$ and $H(z, -1) = e + e^{-1} - 2$.

(d) The condition $(p \leq)\, q = n$ cannot be weakened to $(p \leq)\, q \leq n$. Notice that we only need to eliminate the possibility $q < n$. For example, $f(z) = e^z + 1$ satisfies the following difference equation

$$H(z, f) = [f(z+1) - e^2 f(z-1)] + (e^2 - 1) = 0,$$

and f and $g(z) = e^{-z} + 1$ share $0, 2$ and ∞ CM with $H(z, 0) = e^2 - 1$ and $H(z, 2) = 1 - e^2$.

The assumption that f is of finite order presents different issues. For example, the function $f(z) = e^{\sin z}$ of infinite order is a solution of the difference equation $f(z+\pi) f(z) = 1$, and f and $g(z) = e^{2-\sin z}$ share the values $0, e, \infty$ CM. The *hyper-order*

$$\rho_2(f) := \limsup_{r \to +\infty} \frac{\log \log T(r, f)}{\log r}$$

of f is 1. There is an extension of the difference analogue of the lemma on the logarithmic derivative for functions with $\rho_2(f) < 1$ (see [5]) and it seems that the hypotheses of our Theorem 1.1 may be weakened to include such functions. However, the finite-order assumption on f is essential in our proof (see the discussions following equation (2.4)) and we are not able to see whether or not our result holds for functions with small hyper-order of growth.

2. Proof of theorem 1.1

In this section, we shall prove Theorem 1.1. We use the following results, the first of which is Theorem 3.2 of Halburd and Korhonen [4] or Theorem 2.4 of Laine and Yang [12].

LEMMA 2.1. *Let $f(z)$ be a transcendental meromorphic solution of finite order to the difference equation (1.2). If $H(z, a) \not\equiv 0$ for a small function a of f, then*

$$m\left(r, \frac{1}{f - a}\right) = S(r, f). \tag{2.1}$$

LEMMA 2.2. *If f is a transcendental meromorphic solution of finite order to (1.2), then*

$$m(r, f) = S(r, f). \tag{2.2}$$

PROOF. By Laine and Yang [12, Theorem 2.3],

$$m(r, I(z, f)) = S(r, f),$$

where we recall that $I(z, f) = \sum_{j=1}^{n} a_j f(z + c_j)$. On the other hand, since $p \le q = n$,

$$T(r, I(z, f)) = T\Big(r, \frac{P(f)}{Q(f)}\Big) = nT(r, f) + S(r, f).$$

By Chiang and Feng [2, Theorem 2.2], it follows that

$$nN(r, f) \ge N(r, I(z, f)) + S(r, f) = T(r, I(z, f)) + S(r, f) = nT(r, f) + S(r, f).$$

Thus, we see that $T(r, f) = N(r, f) + S(r, f)$, that is, $m(r, f) = S(r, f)$. □

Proof of theorem 1.1. Since f and g share e_1, e_2 and ∞ CM, Nevanlinna's second main theorem gives

$$\begin{aligned} T(r, f) &\le N(r, f) + N\Big(r, \frac{1}{f - e_1}\Big) + N\Big(r, \frac{1}{f - e_2}\Big) + S(r, f) \\ &= N(r, g) + N\Big(r, \frac{1}{g - e_1}\Big) + N\Big(r, \frac{1}{g - e_2}\Big) + S(r, f) \le 3T(r, g) + S(r, f). \end{aligned}$$

Similarly, we have $T(r, g) \le 3T(r, f) + S(r, g)$. Thus, $\rho(g) = \rho(f) < \infty$ and

$$T(r, f) = T(r, g) + S(r, f).$$

This follows from a result of Brosch [1]; see, for example, Yang and Yi [14, Section 5.5.2].

In addition, there exist two polynomials α, β such that

$$\frac{f - e_1}{g - e_1} = e^{\alpha} \quad \text{and} \quad \frac{f - e_2}{g - e_2} = e^{\beta}. \tag{2.3}$$

Thus, $T(r, e^{\alpha}) \le T(r, f) + T(r, g) + O(1) \le 2T(r, f) + S(r, f)$ and $T(r, e^{\beta}) \le 2T(r, f) + S(r, f)$.

When $e^{\alpha} = 1$, $e^{\beta} = 1$, or $e^{\alpha-\beta} = 1$, it is easy to see that $f = g$. We now suppose that $f \not\equiv g$ and aim to deduce a contradiction. Define $\gamma := \beta - \alpha$. By (2.3),

$$f = e_1 + (e_2 - e_1)\frac{e^{\beta} - 1}{e^{\gamma} - 1}. \tag{2.4}$$

Therefore, $T(r, f) \le T(r, e^{\alpha}) + T(r, e^{\beta}) + S(r, f)$, so that $\max\{\rho(e^{\alpha}), \rho(e^{\beta})\} = \rho(f)$.

Substituting the representation of f from (2.4) into (1.2) leads to

$$\begin{aligned} &\sum_{k=0}^{p} b_k \Big[e_1 + (e_2 - e_1)\frac{e^{\beta(z)} - 1}{e^{\gamma(z)} - 1}\Big]^k \\ &\quad = \Big\{\sum_{j=1}^{n} a_j \Big[e_1 + (e_2 - e_1)\frac{e^{\beta(z+c_j)} - 1}{e^{\gamma(z+c_j)} - 1}\Big]\Big\}\Big\{\sum_{l=0}^{q} d_l \Big[e_1 + (e_2 - e_1)\frac{e^{\beta(z)} - 1}{e^{\gamma(z)} - 1}\Big]^l\Big\}. \end{aligned}$$

Write $e^{\beta(z+c_j)} = e^{\beta(z)+s_j(z)}$ and $e^{\gamma(z+c_j)} = e^{\gamma(z)+t_j(z)}$. Here, s_j and t_j are polynomials of degrees at most $\deg\beta - 1$ and $\deg\gamma - 1$, respectively. Since $p \le q = n$, we can rewrite the above equation as

$$\sum_{\mu=0}^{p}\sum_{\nu=0}^{2n} a_{\mu,\nu}\, e^{\mu\beta+\nu\gamma} = \sum_{\mu=0}^{n+1}\sum_{\nu=0}^{2n} b_{\mu,\nu}\, e^{\mu\beta+\nu\gamma},$$

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where $a_{\mu,\nu}, b_{\mu,\nu}$ are either 0 or polynomials in a_j, b_k, d_l and e^{s_j}, e^{t_j} whose coefficients are polynomials in e_1, e_2. By combining terms, this yields

$$\sum_{\mu=0}^{n+1}\sum_{\nu=0}^{2n} A_{\mu,\nu}\, e^{\mu\beta+\nu\gamma} = 0,$$

where $A_{\mu,\nu}$ are completely determined by $a_{\mu,\nu}, b_{\mu,\nu}$ or 0. In particular, we observe that

$$\begin{aligned} A_{0,2n} &= \Big(\prod_{m=1}^{n} e^{t_m}\Big)\Big[\Big(\sum_{j=1}^{n} a_j e_1\Big)\Big(\sum_{l=0}^{q} d_l e_1^l\Big) - \Big(\sum_{k=0}^{p} b_k e_1^k\Big)\Big] = \Big(\prod_{j=1}^{n} e^{t_j}\Big) H(z, e_1) \neq 0, \\ A_{0,0} &= \Big(\sum_{j=1}^{n} a_j e_2\Big)\Big(\sum_{l=0}^{q} d_l e_2^l\Big) - \Big(\sum_{k=0}^{p} b_k e_2^k\Big) = H(z, e_2) \neq 0. \end{aligned} \tag{2.5}$$

Next, we will prove that

$$\deg\beta = \deg\gamma = \deg(\mu\beta + \nu\gamma) = \deg(\mu\beta - \nu\gamma) \tag{2.6}$$

for any $\mu, \nu \geq 0$ such that $(\mu, \nu) \neq (0, 0)$.

First, we claim that, for an integer $d \geq 0$,

$$\deg\alpha = \deg\beta = \deg\gamma = d. \tag{2.7}$$

Suppose that $e^{\beta} - 1$ and $e^{\gamma} - 1$ have a largest common factor ξ, so that $e^{\beta} - 1 = \xi\beta_1$ and $e^{\gamma} - 1 = \xi\gamma_1$, where ξ, β_1, γ_1 are entire functions such that β_1, γ_1 have no common nonconstant factor. From (2.4), $f = e_1 + (e_2 - e_1)\beta_1\gamma_1^{-1}$ and, from (2.1) and (2.2),

$$\begin{aligned} T(r, f) &= m\Big(r, \frac{1}{f - e_1}\Big) + N\Big(r, \frac{1}{f - e_1}\Big) + O(1) = N\Big(r, \frac{1}{\beta_1}\Big) + S(r, f), \\ T(r, f) &= m(r, f) + N(r, f) + O(1) = N\Big(r, \frac{1}{\gamma_1}\Big) + S(r, f). \end{aligned}$$

Furthermore,

$$\begin{aligned} T(r, e^{\beta}) &= N\Big(r, \frac{1}{e^{\beta} - 1}\Big) + S(r, f) = N\Big(r, \frac{1}{\beta_1}\Big) + N\Big(r, \frac{1}{\xi}\Big) + S(r, f) \\ &= T(r, f) + N\Big(r, \frac{1}{\xi}\Big) + S(r, f) \end{aligned}$$

and

$$\begin{aligned} T(r, e^{\gamma}) &= N\Big(r, \frac{1}{e^{\gamma} - 1}\Big) + S(r, f) = N\Big(r, \frac{1}{\gamma_1}\Big) + N\Big(r, \frac{1}{\xi}\Big) + S(r, f) \\ &= T(r, f) + N\Big(r, \frac{1}{\xi}\Big) + S(r, f). \end{aligned}$$

Combining the preceding equalities yields

$$T(r, e^{\beta}) = T(r, e^{\gamma}) + S(r, f).$$

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On the other hand, it is easy to see that

$$f = e_2 + (e_2 - e_1)\Big(\frac{e^{\beta} - 1}{e^{\gamma} - 1} - 1\Big) = e_2 + (e_2 - e_1)\frac{e^{\alpha} - 1}{e^{\gamma} - 1}e^{\gamma},$$

so, by applying the same analysis to $e^{\alpha} - 1$ and $e^{\gamma} - 1$,

$$T(r, e^{\alpha}) = T(r, e^{\gamma}) + S(r, f).$$

This proves (2.7) and, as a result, $\rho(e^{\alpha}) = \rho(e^{\beta}) = \rho(e^{\gamma}) = \rho(f)$.

Next, we will prove that, when $\mu\nu \neq 0$,

$$\deg(\mu\beta + \nu\gamma) = d. \tag{2.8}$$

On the contrary, suppose that $\deg(\mu\beta + \nu\gamma) < d$. For brevity, write $\Xi_1 := e^{\mu\beta+\nu\gamma}$. Then Ξ_1 is a small function of $e^{-\alpha}$ by (2.7), so that

$$T(r, \Xi_1\, e^{-\mu\alpha}) = T(r, e^{-\mu\alpha}) + S(r, f) = T(r, e^{\mu\alpha}) + S(r, f) = \mu\, T(r, e^{\alpha}) + S(r, f).$$

On the other hand, using (2.7) again,

$$\begin{aligned} T(r, \Xi_1\, e^{-\mu\alpha}) &= T(r, e^{(\mu+\nu)(\beta-\alpha)}) = T(r, e^{(\mu+\nu)\gamma}) \\ &= (\mu + \nu)T(r, e^{\gamma}) + S(r, f) = (\mu + \nu)T(r, e^{\alpha}) + S(r, f). \end{aligned}$$

That is, $\nu = 0$, which is a contradiction, so that (2.8) is confirmed.

Finally, we will prove that, when $\mu\nu \neq 0$,

$$\deg(\mu\beta - \nu\gamma) = d. \tag{2.9}$$

On the contrary, suppose that $\deg(\mu\beta - \nu\gamma) < d$. For brevity, write $\Xi_2 := e^{\mu\beta-\nu\gamma}$. Then Ξ_2 is a small function of $e^{-\alpha}$ by (2.7). If $\mu \geq \nu$,

$$T(r, \Xi_2\, e^{-\mu\alpha}) = T(r, e^{-\mu\alpha}) + S(r, f) = T(r, e^{\mu\alpha}) + S(r, f) = \mu\, T(r, e^{\alpha}) + S(r, f).$$

On the other hand, using (2.7) again,

$$\begin{aligned} T(r, \Xi_2\, e^{-\mu\alpha}) &= T(r, e^{(\mu-\nu)(\beta-\alpha)}) = T(r, e^{(\mu-\nu)\gamma}) \\ &= (\mu - \nu)T(r, e^{\gamma}) = (\mu - \nu)T(r, e^{\alpha}) + S(r, f). \end{aligned}$$

That is, $\nu = 0$, which is a contradiction. If $\nu \geq \mu$, we use $\Xi_2^{-1} = e^{-(\mu\beta-\nu\gamma)}$ instead to observe that

$$T(r, \Xi_2^{-1} e^{\nu\alpha}) = T(r, e^{\nu\alpha}) + S(r, f) = \nu\, T(r, e^{\alpha}) + S(r, f).$$

On the other hand, using (2.7),

$$\begin{aligned} T(r, \Xi_2^{-1} e^{\nu\alpha}) &= T(r, e^{-(\mu\beta-\nu\gamma)} e^{\nu\alpha}) = T(r, e^{(\nu-\mu)\beta}) \\ &= (\nu - \mu)T(r, e^{\beta}) + S(r, f) = (\nu - \mu)T(r, e^{\alpha}) + S(r, f). \end{aligned}$$

That is, $\mu = 0$, which is a contradiction again. Hence, (2.9) and thus (2.6) follow.

By definition, we easily notice that

$$T(r, A_{\mu,\nu}) = S(r, e^{\mu\beta+\nu\gamma}) \quad \text{and} \quad T(r, A_{\mu,\nu}) = S(r, e^{\mu\beta-\nu\gamma}).$$

Thus, by Borel's lemma (see for example [14, Theorem 1.51]), we have $A_{\mu,\nu} \equiv 0$. This clearly contradicts (2.5), and so completes the proof. □

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