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THE MÖBIUS BOUNDEDNESS OF THE SPACE Q_p

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Abstract

In this note, a characterization of the Möbius invariant space Q_p for the range $1 - 1/n < p \leq 1$ is given. As a special case $p = 1$, we get the Möbius boundedness of *BMOA* in the space H^2. This extends the corresponding result for 1-dimension.

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1. Introduction

Let B be the unit ball of $\mathbb{C}^n$ $(n \geq 1)$ with boundary S, ν the Lebesgue measure on B normalized so that $\nu(B) = 1$ and σ the normalized rotation invariant measure on S, that is $\sigma(S) = 1$. The class of all holomorphic functions with domain B will be denoted by $H(B)$.

Let f be in $H(B)$ with Taylor expansion $f(z) = \sum_{\alpha \geq 0} a_\alpha z^\alpha$. For $p \in \mathbb{R}$, f is said to be in the *Dirichlet type space* $\mathscr{D}_p$ provided that

$$\|f\|^2_{\mathscr{D}_p} = \sum_{\alpha \geq 0} (|\alpha| + n)^p \omega_\alpha |a_\alpha|^2 < \infty. \tag{1}$$

Here [Ru]

$$\omega_\alpha = \int_S |\zeta^\alpha|^2 d\sigma(\zeta) = \frac{(n-1)!\alpha!}{(n+|\alpha|-1)!}.$$

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The space $\mathscr{D}_1$ is called *Dirichlet space*. The spaces $\mathscr{D}_0$ and $\mathscr{D}_{-1}$ are just the Hardy space H^2 and the Bergman space $L_a^2(B)$, respectively.

For $a \in B$, φ_a is the Möbius transformation of B which satisfies $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a = \varphi_a^{-1}$, $\varphi_a \in \mathrm{Aut}(B)$. $\mathrm{Aut}(B)$ is the group of biholomorphic automorphisms of B [Ru].

Let $D_j = \partial/\partial z_j$, $j = 1, \dots, n$ and $\nabla f = (D_1 f, \dots, D_n f)$ denote the complex gradient of f, $\mathscr{R}f = \sum_{j=1}^n z_j D_j f$ denote the radial derivative of f. If we let $d\lambda(z) = d\nu(z)/(1-|z|^2)^{n+1}$, then $d\lambda$ is $\mathscr{M}$-invariant (see [Ru]), which means

$$\int_B f(z)d\lambda(z) = \int_B f \circ \psi(z)d\lambda(z) \tag{2}$$

for each $f \in L^1(\lambda)$ and $\psi \in \mathrm{Aut}(B)$. Let $\widetilde{\nabla} f(z) = \nabla(f \circ \varphi_z)(0)$ denote the invariant gradient of f. In [St], the invariant Green's function is defined as $G(z, a) = g(\varphi_a(z))$, where

$$g(z) = \frac{n+1}{2n} \int_{|z|}^1 (1-t^2)^{n-1} t^{-2n+1} dt. \tag{3}$$

We define (as in [OYZ]), for $0 < p < \infty$,

$$Q_p(B) = \left\{ f \in H(B) : \sup_{a \in B} \int_B \left|\widetilde{\nabla} f(z)\right|^2 G^2(z, a) d\lambda(z) < \infty \right\}.$$

Obviously, $Q_p(B)$ is $\mathscr{M}$-invariant.

In [OYZ], the authors proved that $Q_p(B) = \mathrm{Bloch}(B)$ (the Bloch space) for $1 < p < n/(n-1)$, $Q_1(B) = BMOA(S)$ and $Q_p(B)$ is trivial when $0 < p \le (n-1)/n$ or $p \ge n/(n-1)$. For the case of $(n-1)/n < p \le 1$, they proved that $Q_p(B) = \{f \in H(B) : \sup_{a \in B} \int_B \left|\widetilde{\nabla} f(z)\right|^2 (1 - |\varphi_a(z)|^2)^{np} d\lambda(z) < \infty\}$. In this note, a new characterization of $Q_p(B)$ for $(n-1)/n < p \le 1$ is given by using the Möbius boundedness in the space $\mathscr{D}_{n(1-p)}$. As a special case, we get a characterization of *BMOA*. These results in the setting of one dimension can be found in [ALXZ] and [Ba].

Our main result is the following theorem.

THEOREM 1. *For $f \in H(B)$, if $1 - 1/n < p \le 1$, then $f \in Q_p$ if and only if* Möb (f) *is bounded in $\mathscr{D}_{n(1-p)}$, where*

$$\text{Möb}(f) = \{f_a(z) = f(\varphi_a(z)) - f(a) : a \in B\}. \tag{4}$$

The fact $\mathscr{D}_0 = H^2$ together with this theorem gives a corollary.

COROLLARY 1. *For $f \in H(B)$, the following are equivalent*:

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(i) $f \in BMOA$;
(ii) Möb (f) is bounded in H^2.

In the following C denotes a positive constant which may be different from one occurrence to the next.

2. The proof of the main result

In order to prove the theorem, we first give some lemmas.

LEMMA 1 ([OYZ]). *Let $0 < p \leq 1$ and $f \in H(B)$, then $f \in Q_p$ if and only if $\sup_{a\in B} \int_B |\widetilde{\nabla} f(z)|^2 (1 - |\varphi_a(z)|^2)^{np} d\lambda(z)$ is finite.*

LEMMA 2. *Let $p < 2$, then $f \in \mathscr{D}_p$ if and only if*

$$\int_B |\nabla f(z)|^2 (1 - |z|^2)^{1-p} d\nu(z) < \infty,$$

and $\|f\|^2_{\mathscr{D}_p} - |f(0)|^2 \sim \int_B |\nabla f(z)|^2 (1 - |z|^2)^{1-p} d\nu(z)$.

The notation '$A \sim B$' means that there exist constants C_1 and C_2 such that $C_1 B \leq A \leq C_2 B$.

PROOF. It is the direct result of calculation with integration in polar coordinates. □

LEMMA 3. *Let $f \in H(B)$ and $p > 1 - 1/n$, then the following are equivalent*:

(i) $\int_B |\widetilde{\nabla} f(z)|^2 (1 - |z|^2)^{-1-n+np} d\nu(z) < \infty$;
(ii) $\int_B |\nabla_T f(z)|^2 (1 - |z|^2)^{np-n} d\nu(z) < \infty$;
(iii) $\int_B |\nabla f(z)|^2 (1 - |z|^2)^{np-n+1} d\nu(z) < \infty$.

Here $|\nabla_T f(z)|^2 = 2(|\nabla f(z)|^2 - \mathscr{R} f(z)|^2)$, and $\nabla_T f(z)$ is called *the tangent gradient* of f.

PROOF. First we show that (i) is equivalent to (ii). This is a direct result of the equality in [JP]

$$|\widetilde{\nabla} f(z)|^2 = (1 - |z|^2)|\nabla_T f(z)|^2.$$

Next we show that (ii) implies (iii). This we can get from

$$|\nabla_T f(z)|^2 = |\nabla f(z)|^2 - |\mathscr{R} f(z)|^2 \geq (1 - |z|^2)|\nabla f(z)|^2.$$

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Now suppose (iii) holds, we show that (ii) is true. Then

$$|\mathscr{R}f(z)|^2 \le |z|^2|\nabla f(z)|^2 \le |\nabla f(z)|^2 \tag{5}$$

implies that

$$\int_B |\mathscr{R}f(z)|^2(1-|z|^2)^{np-n+1}d\nu(z) < \infty. \tag{6}$$

Since

$$|z|^2|\nabla_T f(z)|^2 = 2\left((1-|z|^2)|\mathscr{R}f(z)|^2 + \sum_{i<j}|T_{ij}f(z)|^2\right), \tag{7}$$

where $T_{ij} = \overline{z_i}D_j - \overline{z_j}D_i$. Since f is holomorphic, then by (6) and (7), we need only to prove that

$$\int_B |T_{ij}f(z)|^2(1-|z|^2)^{np-n}d\nu(z) < \infty \tag{8}$$

for all $1 \le i < j \le n$.

An integration by parts shows that

$$f(z) = \int_0^1 (\mathscr{R}f(tz) + f(tz))dt. \tag{9}$$

Then

$$T_{ij}f(z) = \int_0^1 \left(\sum_{k=1}^n tz_k T_{ij}D_k f(tz) + 2T_{ij}f(tz)\right)dt.$$

From this we conclude that it is sufficient to prove

$$\int_B \left(\int_0^1 |T_{ij}D_k f(tz)|dt\right)^2 (1-|z|^2)^{np-n}d\nu(z) < \infty \tag{10}$$

and

$$\int_B \left(\int_0^1 |T_{ij}f(tz)|dt\right)^2 (1-|z|^2)^{np-n}d\nu(z) < \infty. \tag{11}$$

To prove (10), we note that for any $s > 0$, [Je]

$$\int_0^1 |T_{ij}D_k f(tz)|dt \le C\int_B \frac{(1-|w|^2)^s|D_k f(w)|}{|1-\langle z, w\rangle|^{n+s+\frac{1}{2}}}d\nu(w) \tag{12}$$

Since $p > 1 - 1/n$, then there exists $\delta > 0$ such that $p - \delta > 1 - 1/n$. Using Hölder's inequality, Fubini's theorem and [Ru, Proposition 1.4.10], (12) we obtain

$$\begin{aligned}
&\int_B \left(\int_0^1 |T_{ij} D_k f(tz)| dt \right)^2 (1 - |z|^2)^{np-n} d\nu(z) \\
&\quad \le C \int_B \left(\int_B \frac{(1 - |w|^2)^s |D_k f(w)|}{|1 - \langle z, w \rangle|^{n+s+\frac{1}{2}}} d\nu(w) \right)^2 (1 - |z|^2)^{np-n} d\nu(z) \\
&\quad \le C \int_B \left(\int_B \frac{(1 - |w|^2)^s |D_k f(w)|^2}{|1 - \langle z, w \rangle|^{n+s-n\delta}} d\nu(w) \right) \\
&\qquad \cdot \left(\int_B \frac{(1 - |w|^2)^s}{|1 - \langle z, w \rangle|^{n+s+1+n\delta}} d\nu(w) \right) (1 - |z|^2)^{np-n} d\nu(z) \\
&\quad \le C \int_B (1 - |z|^2)^{np-n-n\delta} \int_B \frac{(1 - |w|^2)^s |D_k f(w)|}{|1 - \langle z, w \rangle|^{n+s-n\delta}} d\nu(w) d\nu(z) \\
&\quad = C \int_B (1 - |w|^2)^s |D_k f(w)|^2 \int_B \frac{(1 - |z|^2)^{np-n-n\delta}}{|1 - \langle z, w \rangle|^{n+s-n\delta}} d\nu(z) d\nu(w) \\
&\quad \le C \int_B |D_k f(w)|^2 (1 - |w|^2)^{np-n+1} d\nu(w). \qquad (13)
\end{aligned}$$

This gives (10).

In order to get (11), we first prove

$$\int_B |f(z)|^2 (1 - |z|^2)^{np-n+1} d\nu(z) < \infty. \qquad (14)$$

From [Je], for $s > 1$,

$$|f(z)|^2 \le C \int_B \frac{|\nabla f(w)|^2 (1 - |w|^2)^s}{|1 - \langle w, z \rangle|^{n+1+s}} d\nu(w). \qquad (15)$$

Fubini's theorem and [Ru, Proposition 1.4.10], and (15) gives

$$\begin{aligned}
&\int_B |f(z)|^2 (1 - |z|^2)^{np-n+1} d\nu(z) \\
&\quad \le C \int_B |\nabla f(w)|^2 (1 - |w|^2)^s \int_B \frac{(1 - |z|^2)^{np-n+1}}{|1 - \langle w, z \rangle|^{n+1+s}} d\nu(z) d\nu(w) \\
&\quad \le C \int_B |\nabla f(w)|^2 (1 - |w|^2)^{np-n+1} d\nu(w). \qquad (16)
\end{aligned}$$

Then (14) is valid. The similar method used in the proof of (10) gives (11). So the proof is complete. □

PROOF OF THEOREM 1. By Lemma 1, Lemma 2 and Lemma 3, and the invariance of $\widetilde{\nabla}$ for $1 - 1/n < p \leq 1$, we have

$$\begin{aligned} f \in Q_p &\Longleftrightarrow \sup_{a \in B} \int_B \left|\widetilde{\nabla} f(z)\right|^2 (1 - |\varphi_a(z)|^2)^{np} d\lambda(z) < \infty \\ &\Longleftrightarrow \sup_{a \in B} \int_B \left|\widetilde{\nabla}(f \circ \varphi_a)(z)\right|^2 (1 - |z|^2)^{np} d\lambda(z) < \infty \\ &\Longleftrightarrow \sup_{a \in B} \int_B |\nabla(f \circ \varphi_a)(z)|^2 (1 - |z|^2)^{np-n+1} d\nu(z) < \infty \\ &\Longleftrightarrow \text{Möb}\,(f) \text{ is bounded in the space } \mathscr{D}_{n(1-p)}. \end{aligned}$$

This completes the proof. □

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