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A NOTE ON THE NORMAL MOORE SPACE CONJECTURE

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F. B. Jones (1937) conjectured that every normal Moore space is metrizable. He also defined a particular kind of topological space (now known as *Jones' spaces*), proved that they were all non-metrizable Moore spaces, but was unable to decide whether or not Jones' spaces are normal. J. H. Silver (1967) proved that a positive solution to Jones' conjecture was not possible, and W. Fleissner (1973) obtained an alternative proof by showing that it is not possible to prove the non-normality of Jones' spaces. These results left open the possibility of resolving the questions from the GCH. In this paper we show that if CH be assumed, then Jones' spaces are not normal (Devlin, Shelah, independently) and that the GCH does not lead to a positive solution to the Jones conjecture (Shelah). A brief survey of the progress on the problem to date is also included.

1. Introduction. For topological background, we refer the reader to any standard text on general topology. The *metrization problem* asks for necessary and sufficient conditions on a topological space X in order that the topology on X be determined by a metric on X. (A metric is said to *metrize* a topological space if the open balls determined by the metric form a base for the topology on X.) An early solution to this problem was supplied by Alexandroff and Urysohn in 1923. They proved, in [**1**]:

THEOREM 1.1. [**1**]. *A Hausdorff space is metrizable if and only if there is a sequence* $\{G_n\}$ *of open covers of the space such that:*

(i) G_{n+1} *refines* G_n;

(ii) *If* U *is an open neighbourhood of the point* p, *then for some* n,

$\bigcup \{g \in G_n | p \in g\} \subseteq U$;

(iii) *If* $H, K \in G_{n+1}$ *and* $H \cap K \neq \emptyset$, *then for some* $G \in G_n$, $H \cup K \subseteq G$.

One half of the the proof of 1.1 is easy. Let G_n consist of all open balls of radius $1/2^n$ (relative to some metrization of the space). This indicates the motivation behind the formulation of 1.1.

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Shelah wishes to acknowledge Uri Avraham for helpful discussions concerning the proof of 2.6. Devlin wishes to acknowledge the fact that it was a letter from Shelah which stimulated his own discovery of Theorem 2.6 (and a realisation that a claim he had previously made to a result flatly contradicting Theorem 2.6 must be in error.).

Of course, this solution is not a "good" solution, since condition (iii) above is not a standard topological condition, and is, indeed, a way of obtaining a "triangle inequality" from which one may recover a metrization. A better solution was found by Bing, Nagata and Smirnov in 1951. This is the solution which is given in most texts. Bing proved that a topological space is metrizable if and only if it is T_3 and has a σ-discrete base. (A collection of subsets of a space X is *discrete* if every point of X has a neighbourhood which intersects at most one member of the collection, and is *σ-discrete* if it is the union of countably many discrete subcollections.)

However, 1.1 already suggests an alternative solution to the metrization problem. Let us call a sequence $\{G_n\}$ of open covers of a space satisfying property (ii) of 1.1 a *development* of the space. (One may also demand 1.1 (i) here, but this is not important, and so we shall not do so.) T_3 spaces which possess a development were studied extensively by R. L. Moore, [**12**] and have become known as *Moore spaces.* Moore spaces resemble metrizable spaces to some extent, but not every Moore space is metrizable. (We shall see a counterexample in § 2.) In 1937, F. B. Jones [**10**] proved the following result:

THEOREM 1.2. [**10**]. *Assume $2^{\aleph_0} < 2^{\aleph_1}$. Then every separable normal Moore space is metrizable.*

Jones conjectured that every normal Moore space is metrizable. This became known as the "normal Moore space conjecture". Perhaps the most significant partial solution to this problem is due to Bing in his 1951 paper [**2**].

A space X is *collectionwise normal* if, whenever $\mathfrak{F}$ is a discrete collection of closed sets there is a disjoint collection $\{\mathfrak{U}_A | A \in \mathfrak{F}\}$ of open sets such that $A \subseteq \mathfrak{U}_A$ for all $A \in \mathfrak{F}$. (We call $\{\mathfrak{U}_A | A \in \mathfrak{F}\}$ a *separation* of $\mathfrak{F}$).

THEOREM 1.3. [**2**]. *A topological space is metrizable if and only if it is a collectionwise normal Moore space.*

In view of 1.3, the normal Moore space conjecture may be reformulated as: every normal Moore space is collectionwise normal. Assuming $V = L$ (the *axiom of constructibility*), one may obtain a partial result in this direction. We require a definition.

A subset Y of a space X is *closed discrete* if and only if $\{\{y\} | y \in Y\}$ is a discrete collection of subsets of X, i.e. if and only if every point of X has a neighbourhood which contains at most one member of Y. (Notice that if $Y \subseteq X$ is discrete, then Y can have no limit points, so any subset of Y will be closed. We shall make great use of this fact in what follows.) We say X is *collectionwise Hausdorff* if every discrete subset of X has a *separation* (i.e. a family $\{\mathfrak{U}_y | y \in Y\}$ of pairwise disjoint open sets $\mathfrak{U}_y$ such that $y \in \mathfrak{U}_y$). Collectionwise normality implies collectionwise Hausdorff.

Clearly, any Moore space is first countable (consider the neighbourhood bases provided by the development). Hence the following result goes part way to resolving the normal Moore space conjecture positively (see [**8**] for further details):

THEOREM 1.4. [**8**]. *Assume* $V = L$. *Then every first countable* T_4 *space is collectionwise Hausdorff.*

Some assumption such as $V = L$ is necessary here, in view of the following result (see [**14**] for details):

THEOREM 1.5. *Assume MA (Martin's Axiom) together with* $2^{\aleph_0} > \aleph_1$. *Then there is a normal Moore space which is not collectionwise Hausdorff.*

Of course, one immediate consequence of 1.5 is:

COROLLARY 1.6. *If* ZFC *is consistent, then so is* ZFC *together with "there is a normal Moore space which is not metrizable."*

Hence, in order to resolve the normal Moore space problem positively one must assume extra axioms of set theory. In view of 1.2, one might expect GCH to be enough. In § 3 we show that this is not the case.

A recent result of P. Nyikos shows that relative to the consistency assumption of the existence of a strongly compact cardinal, it is consistent to assume that every normal Moore space is metrizable (with $2^{\aleph_0}$ very large).

Let us go back to Jones now. Jones (in [**11**]) constructed an example of a non-metrizable Moore space which became known as *Jones' space.* (In fact, what we have here is a class of spaces: *Jones' spaces.*) Even assuming CH, Jones was unable to show whether these spaces were normal or not. (And, of course, normality of such a space at once resolves the normal Moore space conjecture negatively.) Without putting too fine a point on it, Jones' spaces are just special Aronszajn trees with the tree topology. In § 2 of this paper we investigate Jones spaces, and prove that if we assume $2^{\aleph_0} < 2^{\aleph_1}$, then no Jones space is normal. This complements the following result of Fleissner (which also provides us with an alternative proof of 1.5):

THEOREM 1.7. [**8**]. *Assume MA together with* $2^{\aleph_0} > \aleph_1$. *Then every Jones space is normal.*

Finally, a word about notation, etc. We shall work in Zermelo-Fraenkel set theory, inclusive of the axiom of choice, and denote this theory by ZFC. We use the usual notations and conventions of current set theory. A set $E \subseteq \omega_1$ is *stationary* if and only if $E \cap C \neq \emptyset$ for every closed unbounded set (*club*) $C \subseteq \omega_1$. *Fodor's theorem* says that if $E \subseteq \omega_1$ is stationary and $f : E \to \omega_1$ is regressive (i.e. $f(\alpha) < \alpha$ for all $\alpha \in E$), then there is a stationary set $E' \subseteq E$ such that f is constant on E'. If α is an ordinal, 2^α denotes $\{f | f : \alpha \to 2\}$ and

$$2^{\alpha} = \bigcup_{\beta<\alpha} 2^{\beta}$$

We set $\Omega = \{\alpha \in \omega_1 | \lim (\alpha)\}$.

2. $\aleph_1$-Trees and the Tree Topology. A *tree* is a poset $\tilde{T} = \langle T, \leqq_T \rangle$ such that for each $x \in T$, $\hat{x} = \{y \in T | y <_T x\}$ is well-ordered by $<_T$. The order-type of the set $\hat{x}$ under $<_T$ is called the *height* of x in $\tilde{T}$, $ht(x)$. The set $T_\alpha = \{x \in T | ht(x) = \alpha\}$ is the α'th *level* of $\tilde{T}$. We set $T \upharpoonright \alpha = \bigcup_{\beta<\alpha} T_\beta$, and write $\tilde{T} \upharpoonright \alpha$ for $\langle T \upharpoonright \alpha, \leqq_T \cap (T \upharpoonright \alpha)^2 \rangle$. (In practice, we often do not bother to distinguish between a tree $\tilde{T}$ and its domain T, writing just T for both, according to context. And if T is some fixed set, say ω_1, we even use T to refer to the ordering, $\leqq_T$, of the tree.)

Let $\tilde{T}$ be a tree. A *branch* of $\tilde{T}$ is a totally ordered initial segment of $\tilde{T}$. If α is the order-type of a branch b of $\tilde{T}$, we say b is an *α-branch*. An *antichain* of $\tilde{T}$ is a pairwise incomparable subset of $\tilde{T}$. (For instance, every level of $\tilde{T}$ is an antichain of $\tilde{T}$.)

Let $\lambda \leqq \omega_1$. A tree $\tilde{T}$ is a *λ-tree* if and only if the following conditions are met:

(i) $|T_0| = 1$; (by convention, 0 always denotes the unique element of T_0)
(ii) $(\forall \alpha < \lambda)(T_\alpha \neq \emptyset)$;
(iii) $T_\lambda = \emptyset$;
(iv) $(\forall \alpha < \lambda)(|T_\alpha| \leqq \aleph_0)$;
(v) $(\forall \alpha, \beta < \lambda)(\forall x \in T_\alpha)[(\alpha < \beta) \rightarrow (\exists y \in T_\beta)(x <_T y)]$;
(vi) $(\forall \alpha < \lambda)(\forall x \in T_\alpha)[(\alpha + 1 < \lambda) \rightarrow (\exists y_1, y_2 \in T_{\alpha+1}) (y_1 \neq y_2 \wedge x <_T y_1 \wedge x <_T y_2)]$;
(vii) $(\forall \alpha < \lambda)(\forall x, y \in T_\alpha)[\lim(\alpha) \rightarrow (x = y \leftrightarrow \hat{x} = \hat{y})]$.

Let T be an ω_1-tree. If $a, b \in T$, $a <_T b$, we set:

$[a, b] = \{x \in T | a \leqq_T x \leqq_T b\}$ (we call $[a, b]$ a closed interval);
$[a, b) = \{x \in T | a \leqq_T x <_T b\}$ (we call $[a, b)$ a half-open interval);
$(a, b] = \{x \in T | a <_T x \leqq_T b\}$ (we call $(a, b]$ a half-open interval);
$(a, b) = \{x \in T | a <_T x <_T b\}$ (we call (a, b) an open interval).

We make T into a topological space by taking as an open basis all sets of the form (a, b) for $a <_T b$ and all sets of the form $[0, a)$ for $a \in T$. This topology is the *tree topology* on T.

Let T be an ω_1-tree with the tree topology. The following facts are easily verified:

1. T_0 is open.
2. If $x \in T_{\alpha+1}$, then $\{x\}$ is open.
3. Any branch of T is open.
4. Let $A \subseteq T$. A point $x \in T$ will be a limit point of A if and only if $ht(x)$ is a limit ordinal and $A \cap \hat{x}$ is cofinal in $\hat{x}$ under $<_T$.
5. Any maximal branch of T is closed.
6. An antichain of T is a closed, discrete subset of T.
7. T is first countable.
8. T is a T_3 space.

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By a *tree space* we shall mean an ω_1-tree with the tree topology. An *Aronszajn tree* (AT) is an ω_1-tree with no ω_1-branch. An Aronszajn tree is *special* if it is the union of a countable collection of antichains. We write SAT for "special Aronszajn tree". The following result is proved in [**9**]:

THEOREM 2.1. *There is a* SAT.

Indeed, it is known from work of Kunen, Baumgartner, et. al. that a non-special AT cannot be constructed in ZFC. And by later work of Jensen, even if GCH be assumed, a non-special AT cannot be constructed. (See [**5**] for details.)

Suppose T is an ω_1-tree, $A \subseteq \omega_1$. We set

$$T \upharpoonright A = \bigcup_{\alpha \in A} T_\alpha.$$

THEOREM 2.2. *Let T be an ω_1-tree. T is special if and only if for some club set $A \subseteq \omega_1$, $T \upharpoonright A$ is special.*

Proof: $(\rightarrow)$ Take $A = \omega_1$.

$(\leftarrow)$ Let $\langle \alpha_\nu | \nu < \omega_1 \rangle$ enumerate $A \cup \{0\}$ in order of magnitude. For each ν, let $\langle \alpha_n{}^\nu | n < \lambda^\nu \rangle$ enumerate $\{\xi | \alpha_\nu \leqq \xi < \alpha_{\nu+1}\}$, where $\lambda^\nu \leqq \omega$. Let A_n, $n < \omega$, be antichains of $T \upharpoonright A$ such that $T \upharpoonright A = \bigcup_{n<\omega} A_n$. For $n, m < \omega$, let

$$A_{n,m} = \{y \in T | (\exists\, x \in A_n)(\exists\, \nu < \omega_1)[x \in T_{\alpha_\nu} \text{ and } y \in T_{\alpha_m^\nu} \text{ and } x \leqq_T y]\}$$

(where we set $\alpha_m{}^\nu = 0$ if $m \geqq \lambda^\nu$, for convenience). Clearly, the $A_{n,m}$ sets are antichains of T, and $\bigcup_{n,m<\omega} A_{n,m} = T$.

THEOREM 2.3. *Let T be an ω_1-tree. T is a* SAT *if and only if T is a Moore space.*

Proof. Let $T = \bigcup_{n<\omega} A_n$, where each A_n is an antichain of T. For each n, each $x \in T$, set $U_n(x) = \{y \leqq_T x | [y, x) \cap [\bigcup_{k \leqq n} A_k] = \emptyset\}$. Clearly, $x \in U_n(x)$. And it is easily seen that each $U_n(x)$ is open. Set $G_n = \{U_n(x) | x \in T\}$, an open cover of T, for each n. We show that $\{G_n\}_{n<\omega}$ is a development of T. Let U be open in T, $p \in U$. Pick $n < \omega$ with $p \in A_n$. Clearly, if $m \geqq n$ and $p \in U_m(x)$, then $x = p$. Pick $q <_T p$ so that $(q, p] \subseteq U$. Let $q \in A_k$. Set $m = \max(n, k)$. Then $\bigcup \{g \in G_m | p \in g\} = U_m(p) \subseteq (q, p] \subseteq U$.

Conversely, Let $\{G_n\}_{n<\omega}$ be a development of T. Clearly, if G_n' is an open cover of T which refines G_n for each n, then $\{G_n\}_{n<\omega}$ will also be a development of T. Hence we may assume that the elements of each G_n consist of basic open sets. Indeed, we may assume that for each n, G_n consists of all sets $\{x\}$ for x a member of a successor level of T together with sets $(f_n(x), x]$ for x a member of a limit level of T, where $f_n(x) <_T x$. By taking refinements again if necessary we may assume further that for each x on a limit level of T, $f_0(x) <_T f_1(x) <_T f_2(x) <_T \ldots <_T f_n(x) <_T \ldots$.

Recall that $\Omega = \{\alpha \in \omega_1 | \lim (\alpha)\}$. Define $h : T \upharpoonright \Omega \rightarrow \omega$ thus: $h(x)$ is the least n such that wherever $y \in T \upharpoonright \Omega$ and $x <_T y$, it is the case that $x \leqq_T f_n(y)$. We show that h is well-defined. Let U $= [0, x]$, an open set containing x. Pick n

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so that $\bigcup \{g \in G_n | x \in g\} \subseteq U$. Suppose that for some $y >_T x$ it is not the case that $x \leqq_T f_n(y)$. Since $f_n(y) <_T y$, it must be the case that $f_n(y) <_T x$. Thus $x \in (f_n(y), y]$. So we must have $(f_n(y), y] \subseteq U$, giving $y \in U$, which is absurd. It follows that h is well-defined. For $n < \omega$, set

$$E_n = \{x \in T \restriction \Omega | h(x) = n\}.$$

Claim. No E_n contains a chain of type $\omega + 1$.

Proof. Suppose $\langle a_\nu | \nu \leqq \omega \rangle$ is a $<_T$-chain in E_n. Set $\gamma = \sup_{\nu<\omega} \mathrm{ht}(a_\nu)$, and let x be the unique predecessor of a_ω on T_γ. For each $\nu < \omega$, $h(a_\nu) = n$ and $a_\nu <_T x$, so $a_\nu \leqq_T f_n(x)$. Hence $\mathrm{ht}(f_n(x)) \geqq \gamma$, contrary to $f_n(x) <_T x \in T_\gamma$. The claim is proved.

By Theorem 2.2, we are done if we can show that $T \restriction \Omega$ is special. Well-order $T \restriction \Omega$ as $\langle x_\nu | \nu < \omega_1 \rangle$ so that $\nu < \tau \rightarrow \mathrm{ht}(x_\nu) \leqq \mathrm{ht}(x_\tau)$. We define disjoint antichains $E_{n,m}$ of T by recursion. At stage ν we decide which antichain will receive x_ν. We pick n with $x_\nu \in E_n$. By the claim we can find an m such that $[\tau < \nu$ and $x_\tau \in E_n$ and $x_\tau <_T x_\nu] \rightarrow [x_\tau \notin E_{n,m}]$. We put x_ν into $E_{n,m}$ for such an m. Clearly, $T \restriction \Omega = \bigcup_{n,m<\omega} E_{n,m}$ partitions $T \restriction \Omega$ into disjoint antichains.

Fleissner proved that if we assume $V = L$ (or more generally, if we assume $\blacklozenge$), then no SAT space is normal. Devlin and Shelah (independently) strengthened this to obtain the result from $2^{\aleph_0} < 2^{\aleph_1}$. The key to the result lies in the following two results, proved in [**6**].

THEOREM 2.4. *Assume* $2^{\aleph_0} < 2^{\aleph_1}$. *Let* $F : 2^{\underset{\smile}{\omega_1}} \rightarrow 2$. *Then there is a* $g \in 2^\omega$ *such that for any* $f \in 2^{\omega_1}$, $\{\alpha \in \omega_1 | F(f \restriction \alpha) = g(\alpha)\}$ *is stationary in* ω_1.

THEOREM 2.5. *Let* I *be the set of all sets* $S \subseteq \omega_1$ *such that there exists an* $F : 2^{\underset{\smile}{\omega_1}} \rightarrow 2$ *such that for all* $g \in 2^{\omega_1}$ *there is an* $f \in 2^{\omega_1}$ *for which*

$$\{\alpha \in S | F(f \restriction \alpha) = g(\alpha)\}$$

is not stationary. Then I *is a normal ideal on* ω_1.

Using these results, we may now prove the main new result of this section:

THEOREM 2.6. *Assume* $2^{\aleph_0} < 2^{\aleph_1}$. *Let* T *be a* SAT (*with the tree topology*). *Then* T *is not normal.*

Proof. Let T be identified with ω_1 so that $\alpha <_T \beta \rightarrow \alpha < \beta$ and $\lim(\alpha) \rightarrow \mathrm{ht}_T(\alpha) = \alpha$. Let A_n be disjoint antichains of T with $T = \bigcup_{n\in\omega} A_n$. Let I be as in Theorem 2.5. Suppose $A_n \in I$ for all n. Then, as I is a countably complete ideal, $T \in I$, i.e. $\omega_1 \in I$. This contradicts Theorem 2.4. Hence for some n, $A_n \notin I$. Let $A = A_n$ for such an n. Let $E = \{\alpha \in A | \lim(\alpha)\}$. Clearly, $E \notin I$.

Define $F : 2^{\underset{\smile}{\omega}} \rightarrow 2$ as follows. Given $f \in 2^\alpha$, $\alpha < \omega_1$, set $F(f) = 0$ if and only if there is any $\gamma <_T \alpha$ such that $f(\delta) = 0$ for all δ such that $\gamma <_T \delta <_T \alpha$; and let $F(f) = 1$, otherwise.

Since $E \notin I$, there is a $g \in 2^{\omega_1}$ such that for all $f \in 2^{\omega_1}$, the set

$$\{\alpha \in E | F(f \restriction \alpha) = g(\alpha)\}$$

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is stationary. Set

$$H = \{\alpha \in E | g(\alpha) = 0\}, K = \{\alpha \in E | g(\alpha) = 1\}.$$

Since E is an antichain of T, it is closed discrete. Hence H and K are disjoint, closed subsets of T. We claim that H and K cannot be separated. Suppose, on the contrary, that there are disjoint open sets $U, V \subseteq T$ with $H \subseteq U, K \subseteq V$. Define $f \in 2^{\omega_1}$ by

$$f(\alpha) = \begin{cases} 0 & \text{if } \alpha \in V, \\ 1 & \text{otherwise.} \end{cases}$$

By choice of g, $E' = \{\alpha \in E | F(f \upharpoonright \alpha) = g(\alpha)\}$ is stationary, hence non-empty. Pick $\alpha \in E'$.

Suppose first that $g(\alpha) = 0$. Thus $F(f \upharpoonright \alpha) = 0$. So for some $\gamma <_T \alpha$, $f(\delta) = 0$ wherever $\gamma <_T \delta <_T \alpha$. So, by definition of f, $(\gamma, \alpha)^T \subseteq V$. But $g(\alpha) = 0$, so $\alpha \in H \subseteq U$, so for some $\gamma' <_T \alpha$, $(\gamma', \alpha)^T \subseteq U$. This is a contradiction.

Now suppose that $g(\alpha) = 1$. Thus $\alpha \in K \subseteq V$. So for some $\gamma <_T \alpha$,

$$(\gamma, \alpha)^T \subseteq V.$$

Thus $\gamma <_T \delta <_T \alpha$ implies $f(\delta) = 0$. Hence by definition of F, $F(f \upharpoonright \alpha) = 0$. But then $g(\alpha) \neq F(f \upharpoonright \alpha)$. This is a contradiction and the theorem is proved.

Hence, assuming CH, Jones' spaces are not normal. For completeness we prove finally that they are not collectionwise Hausdorff. We need a simple lemma, whose proof we leave to the reader.

Lemma 2.7. *Let T be a* SAT. *Then T has an antichain A such that*

$$E = \{\text{ht}(a) | a \in A\}$$

is stationary in ω_1.

Theorem 2.8. *No* SAT *space is collectionwise Hausdorff.*

Proof. Let T be a SAT. By Lemma 2.7, we can choose an antichain A of T such that, if $A = \{a_\alpha | \alpha \in E\}$ and $a_\alpha \in T_\alpha$, then E is stationary in ω_1. We may assume that $E \subseteq \Omega$.

Since A is an antichain of T, it is clearly a discrete subset of T. Suppose we could separate the elements of A by disjoint open sets. Then we could separate the elements of A by disjoint basic open sets. Indeed, we may define a function $f : A \to T$ such that $f(a) <_T a$ for all a and $\{(f(a), a] | a \in A\}$ is a separating set. Define $h : E \to \omega_1$ by $h(\alpha) = \text{ht}(f(a_\alpha))$. Then h is regressive, so by Fodor's theorem there is a stationary set $E' \subseteq E$ on which h is constant, say with value δ. Let $A' = \{a_\alpha | \alpha \in E'\}$. Now, for each $a \in A'$, $f(a) \in T_\delta$, so there is a unique member of $(f(a), a]$ on $T_{\delta+1}$, say $g(a)$. But A' is uncountable and $T_{\delta+1}$ is countable, so for some $a_1, a_2 \in A'$, $a_1 \neq a_2$, we must have $g(a_1) = g(a_2)$. Thus $(f(a_1), a_1] \cap (f(a_2), a_2] \neq \emptyset$, a contradiction.

We make a final remark. The reader may well wonder, in view of Theorem 2.6, if any Aronszajn tree space is normal. This question is investigated fully in [**7**]: it turns out that the normality of tree spaces is connected with the degree to which the tree is "Souslin". In particular, any Souslin tree space is normal. (But since any Souslin tree space is also collectionwise Hausdorff, this does not help us with the normal Moore space problem.)

3. Ladder Systems and *LS* Topologies. As before, Ω denotes the set of all countable limit ordinals. If $\delta \in \Omega$, a *ladder* on δ is a strictly increasing ω-sequence cofinal in δ. A *2-colouring* of a ladder η on δ is a function $k : \omega \to 2$. We might say that k assigns to $\eta(n)$ the "colour" $k(n)$. If $E \subseteq \Omega$, a *ladder system* on E is a sequence $\eta = \langle \eta_\delta | \delta \in E \rangle$ such that η_δ is a ladder on δ for each $\delta \in E$. A *2-colouring* of η is a sequence $k = \langle k_\delta | \delta \in E \rangle$ such that k_δ is a 2-colouring of η_δ for each $\delta \in E$. A *uniformisation* of k is a function $h : \omega_1 \to 2$ such that for each $\delta \in E$ there is an $n \in \omega$ such that $m \geqq n$ implies $k_\delta(m) = h(\eta_\delta(m))$. The basic question is: given a ladder system on a set E, does every 2-colouring possess a uniformisation?

The above notions were introduced by Shelah in connection with the famous Whitehead Problem in group theory. (At least, the above is a simple modification of Shelah's concepts.) The terminology is due to Devlin. It has turned out that the concept has other applications. We present one here. (Another is given in [**3**].)

The following results are fundamental, and quite easy to prove.

THEOREM 3.1. *Assume* MA *together with* $2^{\aleph_0} > \aleph_1$. *Let* $E \subseteq \Omega$. *Then every 2-colouring of every ladder system on* E *is uniformisable.*

THEOREM 3.2. *Assume* $V = L$. *Let* $E \subseteq \Omega$ *be stationary. Then every ladder system on* E *has a non-uniformisable 2-colouring.*

Both of the above results are proved in [**3**]. This is also our reference for the following result, whose proof led to the formulation (and proof) of Theorems 2.4 and 2.5.

THEOREM 3.3. *Assume* $2^{\aleph_0} < 2^{\aleph_1}$. *Let* $E \subseteq \Omega$ *be* club. *Then every ladder system on* E *has a non-uniformisable colouring.*

The assumption that E be club in Theorem 3.3 is essential. In [**13**] the following result is proved.

THEOREM 3.4. *Let* M *be a countable transitive model of* ZFC *together with* GCH. *In* M, *let* E *be a stationary, costationary subset of* ω_1, *with* $E \subseteq \Omega$, *and let* $\eta = \langle \eta_\delta | \delta \in E \rangle$ *be a ladder system on* E. *Then there is a generic extension* N *of* M *such that:*

(i) *M and N have the same cardinals and cofinality function;*

(ii) $M^\omega \cap M = M^\omega \cap N$;

(iii) *the* GCH *holds in* N;

(iv) *in* N, *both* E *and* ω_1-E *are stationary;*
(v) *in* N, *every* 2-*colouring of* η *is uniformisable.*

The idea behind the proof of Theorem 3.4 is to iterate (in an obvious manner) the obvious algebra for uniformising a given 2-colouring of η. What is not easy is proving that E remains stationary. The proof is given in [**13**].

Let $E \subseteq \Omega$ now, and let $\eta = \langle \eta_\delta | \delta \in E \rangle$ be a ladder system on E. We may define a topology on ω_1 as follows.

(a) If $\delta \in \omega_1 - E$, $\{\delta\}$ is an open neighbourhood of δ.

(b) If $\delta \in E$, each set of the form $N_n(\delta) = \{\eta_\delta(m) | n \leqq m < \omega\} \cup \{\delta\}$ is an open neighbourhood of δ.

This clearly defines a 1st countable topology on ω_1, the η-topology. Any such topology is called a *ladder system topology*, or LS-*topology* for short.

LEMMA 3.5. *Any* LS-*topology is* T_3.

Proof. Let $\eta = \langle \eta_\delta | \delta \in E \rangle$ be a ladder system on $E \subseteq \Omega$, and let X be the η-topology on ω_1.

I. X is T_1. To see this, let $\alpha \in X$. We show that $\{\alpha\}$ is closed. Let $\delta \neq \alpha$. If $\delta \notin E$, then $\{\delta\}$ is an open neighbourhood of δ disjoint from $\{\alpha\}$. Now suppose $\delta \in E$. If $\delta < \alpha$ then $N_0(\delta)$ is an open neighbourhood of δ disjoint from $\{\alpha\}$. Otherwise, if $\alpha < \delta$, then for some n, $\eta_\delta(n) > \alpha$, and $N_n(\delta)$ is then an open neighbourhood of δ disjoint from $\{\alpha\}$. Hence $X - \{\alpha\}$ is open. Thus $\{\alpha\}$ is closed.

II. X is regular. For let δ be any member of X, A any closed set not containing δ. We find disjoint open sets U, V with $\delta \in U$, $A \subseteq V$.

Suppose first that $\delta \notin E$. Let $U = \{\delta\}$. For each $\alpha \in A$, let $V_\alpha = \{\alpha\}$ if $\alpha \notin E$, and let $V_a = N_n(\alpha)$ if $\alpha \in E$, where n is least such that $m \geqq n$ implies $\eta_\alpha(m) \neq \delta$. (Such an n always exists, of course.) Let $V = \bigcup_{\alpha \in A} V_a$. Clearly U and V are as required.

Suppose now that $\delta \in E$. Since A is closed and $\delta \notin A$, there is an $n_0 \in \omega$ such that $N_{n_0}(\delta) \cap A = \emptyset$, i.e. $m \geqq n_0 \rightarrow \eta_\delta(m) \notin A$. Set

$$U = N_{n_0}(\delta),$$

an open neighbourhood of δ. Suppose $\alpha \in A$. If $\alpha \notin E$, let $V_\alpha = \{\alpha\}$. Now suppose $\alpha \in E$. If $\alpha > \delta$, then for some $n \in \omega$, $\eta_\alpha(n) > \delta$, and we set $V_\alpha = N_n(\alpha)$. Now suppose $\alpha < \delta$. If $\eta_\delta(0) > \alpha$, set $V_\alpha = N_0(\alpha)$. Otherwise, let n_1 be largest with $\eta_\delta(n_1) < \alpha$, and let n_2 be least with $\eta_\alpha(n_2) > \eta_\delta(n_1)$. Set

$$V_\alpha = N_{n_2}(\alpha)$$

in this case. Let $V = \bigcup_{\alpha \in A} V_\alpha$. Clearly, U, V are as required.

Fix now some SAT T. We may identify T with ω_1 so that:

(i) $\alpha <_T \beta \rightarrow \alpha < \beta$;
(ii) $\alpha \in \Omega \rightarrow \text{ht}(\alpha) = \alpha$;
(iii) $\alpha \in \Omega \rightarrow \sup\{\beta \in \alpha | \beta <_T \alpha\} = \alpha$.

Let A_n be disjoint antichains of T such that $T = \bigcup_{n\in\omega} A_n$. Define $h : \Omega \rightarrow \omega$ by $h(\alpha) = n$ if and only if $\alpha \in A_n$. Since h is regressive there is a stationary set $E \subseteq \Omega$ such that h is constant on E, say with value n. Since $E \subseteq A_n$, E is an antichain of T.

By thinning down E if necessary, we may assume that $\omega_1 - E$ is also stationary.

Define a ladder system $\eta = \langle \eta_\delta | \delta \in E\rangle$ now by picking η_δ an ω-sequence cofinal in δ in the sense of $<_T$. Let X be the η-space.

LEMMA 3.6. *X is a Moore space.*

Proof. Now, $T = \bigcup_{n\in A} A_n$, where each A_n is an antichain of T. Let $n \in \omega$. If $\delta \in \omega_1 - E$, let $g_n(\delta) = \{\delta\}$. Now suppose $\delta \in E$. Pick m so that

$$\eta_\delta(m) \leqq_T \gamma <_T \delta \rightarrow \gamma \notin U_{k\leqq n} A_k,$$

and set $g_n(\delta) = N_m(\delta)$. Let $G_n = \{g_n(\delta) | \delta \in \omega_1\}$, an open cover of X. We show that $\{G_n\}$ is a development of X. Let U be open in X, $\delta \in U$. Pick n with $\delta \in A_n$. If $\delta \notin E$, then $\bigcup \{g \in G_n | \delta \in g\} = g_n(\delta) = \{\delta\} \subseteq U$. Suppose that $\delta \in E$. Now, if $m \geqq n$ and $\delta \in g_m(\gamma)$, then $\gamma = \delta$. Pick m so that $p \geqq m \rightarrow \eta_\delta(p) \in U$. Let $\eta_\delta(m) \in A_k$. Set $p = \max(n, k)$. Then $\bigcup \{g \in G_p | \delta \in g\} = g_p(\delta) \subseteq U$. The proof is complete.

LEMMA 3.7. *X is not collectionwise Hausdorff.*

Proof. Clearly, E is an uncountable, closed discrete subset of X. But since E is stationary, an argument much as in Theorem 2.8 shows that E has no separation.

LEMMA 3.8. *If every 2-colouring of η is uniformisable, then X is normal.*

Proof. Let A, B be disjoint, closed subsets of X. Define a 2-colouring, k, of η as follows. If $\delta \in E \cap A$, let $k_\delta(n) = 0$, for all $n \in \omega$. If $\delta \in E - A$, let $k_\delta(n) = 1$ for all $n \in \omega$. Let $h : \omega_1 \rightarrow 2$ uniformise k.

Let $\alpha \in A$. If $\alpha \notin E$, set $U_\alpha = \{\alpha\}$. If $\alpha \in E$, there is an $n \in \omega$ such that $m \geqq n \rightarrow [\eta_\alpha(m) \notin B$ and $h(\eta_\alpha(m)) = 0]$, so pick the least such n and set $U_\alpha = N_n(\alpha)$.

Set $U = \bigcup_{\alpha\in A} U_\alpha$, an open set containing A. Let $\beta \in B$. If $\beta \notin E$, set $V_\beta = \{\beta\}$. If $\beta \in E$, let $n \in \omega$ be least such that $m \geqq n \rightarrow [\eta_\beta(m) \notin A$ and $h(\eta_\beta(m)) = 1]$, and set $V_\beta = N_n(\beta)$.

Set $V = \bigcup_{\beta\in B} V_\beta$, an open set containing B. Clearly, $U \cap V = \emptyset$, so the lemma is proved.

We now have everything we need to prove:

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THEOREM 3.9. *If* ZFC *is consistent, then so is* ZFC *together with* GCH *and "There is an* LS-*space X which is a non-collectionwise Hausdorff, non-metrizable, normal Moore space".*

Proof. Let M be a countable transistive model of ZFC and GCH. Define T, E, η, X as above inside the model M. Obtain N from M as in Theorem 3.4. Since M and N have the same cardinals and since E is stationary in N, X retains all its relevant properties: i.e. in N, X is an LS-space which is a non-collectionwise Hausdorff Moore space. But by Lemma 3.8, X is normal in N, so we are done.

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