Note on the "sum" of an integral function

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Theorem 1 of a recent paper "On the asymptotic periods of integral functions"[1] can be replaced by the following more precise result.

If $f(z)$ *is an integral function of order* ρ *there is an integral function* $g(z)$, *of order* ρ, *such that*

$$g(z+1) - g(z) = f(z). \tag{1}$$

The improvement consists in showing that, if $\rho < 1$, $g(z)$ can be chosen to be of order ρ, not merely of order less than or equal to one.

It is known[2] that, if $\rho < 1$, a solution of (1) is given by

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} B_{n+1}(z),$$

where

$$f(z) = \Sigma\, a_n z^n. \quad \text{Hence}$$

$$g^{(k)}(0) = \sum_{n=k-1}^{\infty} \frac{a_n}{n+1} B_{n+1}^{(k)}(0) = \sum_{n=k-1}^{\infty} a_n\, n(n-1)\,..\,(n-k+2)\, B_{n-k+1}.$$

Now

$$B_{2m+1} = 0,\ B_{2m} = (-)^{m+1} \frac{2\,(2m)!}{(2\pi)^{2m}} \sum_{s=1}^{\infty} \frac{1}{s^{2m}},$$

so that

$$|B_n| \leqq 4n!\,(2\pi)^{-n} \qquad (n \geqq 0).$$

Moreover, if $1 < \alpha < 1/\rho$,

$$|a_n| < n^{-\alpha n} \qquad (n \geqq n_\alpha).$$

Hence

$$|g^{(k)}(0)| < 4 \sum_{n=k-1}^{\infty} n^{-\alpha n}\, n!\,(2\pi)^{k-n-1} \qquad (k \geqq k_\alpha).$$

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[1] *Proc. Edinburgh Math. Soc.*, 3 (1933), 241-258.

[2] *Cf.* Nörlund, *Sur la "somme" d'une fonction* (Paris, 1927), for this and other properties of Bernoulli polynomials.

Now

$$\frac{n^{-an}\, n!\, (2\pi)^{k-n-1}}{(n+1)^{-a(n+1)}\,(n+1)!\,(2\pi)^{k-n-2}} = 2\pi\left(1+\frac{1}{n}\right)^{an}(n+1)^{a-1}$$
$$> k^{a-1} \qquad (n \geqq k-1),$$

so that

$$|c_k| = \frac{1}{k!}\,|g^{(k)}(0)| \leqq \frac{4}{k!}(k-1)^{-a(k-1)}(k-1)!\sum_{s=0}^{\infty}\frac{1}{k^{s(a-1)}}$$
$$< (k-1)^{-a(k-1)} \qquad (k \geqq k_a).$$

Thus

$$\varliminf_{k\to\infty}\frac{\log|c_k|^{-1}}{k\log k} \geqq a.$$

This is true for every $a < 1/\rho$, so that the order of $g(z)$ cannot be greater than ρ. On the other hand it is evident that the order of $g(z)$ cannot be less than ρ.

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