*Combinatorics, Probability and Computing* (2024), 1–11
doi:10.1017/S0963548324000269

ARTICLE

# A note on extremal constructions for the Erdős–Rademacher problem

Xizhi Liu and Oleg Pikhurko

Mathematics Institute and DIMAP, University of Warwick, Coventry, UK
**Corresponding author:** Oleg Pikhurko; Email: o.pikhurko@warwick.ac.uk

(Received 5 December 2023; revised 25 July 2024; accepted 5 August 2024)

**Abstract**

For given positive integers $r \geq 3$, $n$ and $e \leq \binom{n}{2}$, the famous Erdős–Rademacher problem asks for the minimum number of $r$-cliques in a graph with $n$ vertices and $e$ edges. A conjecture of Lovász and Simonovits from the 1970s states that, for every $r \geq 3$, if $n$ is sufficiently large then, for every $e \leq \binom{n}{2}$, at least one extremal graph can be obtained from a complete partite graph by adding a triangle-free graph into one part.

In this note, we explicitly write the minimum number of $r$-cliques predicted by the above conjecture. Also, we describe what we believe to be the set of extremal graphs for any $r \geq 4$ and all large $n$, amending the previous conjecture of Pikhurko and Razborov.

**Keywords:** Erdős–Rademacher problem; Lovász–Simonovits conjecture; Clique density theorem

**2020 Mathematics Subject Classification:** Primary: 05C35

## 1. Introduction

Given integers $n \geq r \geq 2$, let $T_r(n)$ denote the balanced complete $r$-partite graph on $n$ vertices, and let $t_r(n)$ denote the number of edges in $T_r(n)$. The celebrated Turán Theorem [24] (with the case $r = 3$ proved earlier by Mantel [13]) states that, for $n \geq r \geq 3$, every $n$-vertex graph with at least $t_{r-1}(n) + 1$ edges contains a copy of an *$r$-clique* $K_r$, that is, a complete graph on $r$ vertices. An unpublished result of Rademacher from 1941 (see [3]) states that, in fact, every $n$-vertex graph with $t_2(n) + 1$ edges contains at least $\lfloor n/2 \rfloor$ copies of $K_3$. The graph obtained from $T_2(n)$ by adding one edge to the larger part shows that the bound $\lfloor n/2 \lfloor$ is tight. Rademacher's theorem motivated Erdős [3] to consider the following more general question, now referred to as the *Erdős–Rademacher problem*: determine

$$g_r(n, e) := \min \Big\{ N(K_r, G) : \ G \text{ is an } (n, e)\text{-graph} \Big\}, \tag{1}$$

where an *$(n, e)$-graph* means a graph with $n$ vertices and $e$ edges and $N(K_r, G)$ denotes the number of $r$-cliques in $G$.

This problem has attracted a lot of attention and has been actively studied since it first appeared. Various results covering special ranges of $(n, e)$ were obtained (see e.g. [2, 4–7, 11, 12, 14, 18–20]) until Razborov [22] determined the asymptotic value of $g_3(n, e)$ using flag algebras. Later, using different methods, Nikiforov [17] determined the asymptotic value of $g_r(n, e)$ for $r = 4$ and Reiher [23] did this for all $r \geq 5$. For some further related results, we refer the reader to [1, 8, 10, 15, 16, 21, 25].

Determining the exact value of $g_r(n, e)$ seems very challenging due to multiple (conjectured) extremal constructions. Given $n$ and $e$ in $\mathbb{N} := \{1, 2, \ldots\}$ with $e \leq \binom{n}{2}$, let

$$k = k(n, e) := \min \{s \in \mathbb{N} : \; t_s(n) \geq e\}, \tag{2}$$

that is, $k$ is the smallest chromatic number that an $(n, e)$-graph can have. Let $\mathcal{H}_1(n, e)$ (resp. $\mathcal{K}(n, e)$) denote the family of $(n, e)$-graphs that can be obtained from a complete $(k-1)$-partite (resp. complete multipartite) graph by adding a triangle-free graph into one part. Note that the only difference between these two definitions is that we restrict the number of parts to $k-1$ when defining $\mathcal{H}_1(n, e)$; thus $\mathcal{H}_1(n, e) \subseteq \mathcal{K}(n, e)$. Lovász and Simonovits [11] conjectured that for every integer $r \geq 3$ there exists $n_0$ such that, for all positive integers $n \geq n_0$ and $e \leq \binom{n}{2}$, it holds that

$$g_r(n, e) = \min \Big\{ N(K_r, H) : \; H \in \mathcal{K}(n, e) \Big\}, \tag{3}$$

that is, at least one $g_r(n, e)$-extremal graph is in $\mathcal{K}(n, e)$. Note that (3) trivially holds for $e \leq t_{r-1}(n)$ when $g_r(n, e) = 0$.

Erdős in [3] (resp. [4]) showed that (3) is true for $r = 3$ when $e \leq t_2(n) + 3$ (resp. $e \leq t_2(n) + cn$ for some constant $c > 0$). Lovász and Simonovits [11] (see also Nikiforov and Khadzhiivanov [19]) extended the result of Erdős to all $e$ satisfying $e \leq t_2(n) + \lfloor n/2 \rfloor$. Later, Lovász and Simonovits [12] proved (3) for $r \geq 3$ when $e/\binom{n}{2}$ lies in a small upper neighbourhood of $1 - 1/m$ for some integer $m \geq r - 1$. More recently, Liu, Pikhurko and Staden [9] determined $g_3(n, e)$ for all positive integers $n$ when $e \leq (1 - o(1))\binom{n}{2}$. Determining the exact value of $g_r(n, e)$ for $r \geq 4$ is still wide open in general.

Given $n, e \in \mathbb{N}$ with $e \leq \binom{n}{2}$, let $a^* = a^*(n, e) \in \mathbb{N}^k$ be the unique vector such that

$$a_k^* := \min \left\{ a \in \mathbb{N} : \; a(n-a) + t_{k-1}(n-a) \geq e \right\},$$
$$a_1^* + \cdots + a_{k-1}^* = n - a_k^*, \quad \text{and} \quad a_1^* \geq \cdots \geq a_{k-1}^* \geq a_1^* - 1,$$

where $k = k(n, e)$ is as defined in (2). Thus $a_k^*$ is the smallest possible part size that a $k$-partite $(n, e)$-graph can have. Also, let

$$m^* = m^*(n, e) := \sum_{\{i,j\} \in \binom{[k]}{2}} a_i^* a_j^* - e, \quad \text{and}$$
$$h_r^*(n, e) := \sum_{I \in \binom{[k]}{r}} \prod_{i \in I} a_i^* - m^* \cdot \sum_{I' \in \binom{[k-2]}{r-2}} \prod_{j \in I'} a_j^*,$$

where $[k] := \{1, \ldots, k\}$ and $\binom{X}{k} := \{Y \subseteq X : |Y| = k\}$. Let $T := K[A_1^*, \ldots, A_k^*]$ be the complete $k$-partite graph with parts $A_1^*, \ldots, A_k^*$ where $|A_i^*| = a_i^*$ for $i \in [k]$. Let $H^* = H^*(n, e)$ be the graph obtained from $T$ by removing an $m^*$-edge star whose centre lies in $A_k^*$ and whose leaves lie in $A_{k-1}^*$. It is not hard to see (see e.g. the calculation in (10)) that $0 \leq m^* \leq a_{k-1}^* - a_k^*$, so the graph $H^*$ is well-defined. Also, let $\mathcal{H}_1^*(n, e)$ be the family defined as follows: If $m^* = 0$, take all graphs obtained from $T$ by replacing, for some $i \in [k-1]$, the bipartite graph $T[A_i^* \cup A_k^*]$ with an arbitrary triangle-free graph with $a_i^* a_k^*$ edges. If $m^* > 0$, take all graphs obtained from $T$ by replacing $T[A_{k-1}^* \cup A_k^*]$ with an arbitrary triangle-free graph with $a_{k-1}^* a_k^* - m^*$ edges. Observe that $\mathcal{H}_1^*(n, e) \subseteq \mathcal{H}_1(n, e)$ and every graph in $\mathcal{H}_1^*(n, e)$ has the same number of $r$-cliques (see Fact 2.2); also, the graph $H^* = H^*(n, e)$ is contained in $\mathcal{H}_1^*(n, e)$.

Sharpening the Lovász–Simonovits Conjecture, Pikhurko and Razborov [21, Conjecture 1.4] conjectured that, for $r \geq 4$ and sufficiently large $n$, every $n$-vertex graph with $e \leq \binom{n}{2}$ edges and that contains the minimum number of $K_r$ is in $\mathcal{K}(n, e)$. However, we show here that this conjecture is false (see Theorem 1.1 and Proposition 1.2) and present an amended version (see Conjecture 1.3) as follows.

https://doi.org/10.1017/S0963548324000269 Published online by Cambridge University Press

First, we write explicitly the value of $g_r(n, e)$ predicted by the Lovász–Simonovits Conjecture. (We also refer the reader to [9, Proposition 1.5] where similar results are proved for $r = 3$.)

**Theorem 1.1.** *Suppose that $r, n, e \in \mathbb{N}$ satisfy $n \geq r \geq 3$ and $e \leq \binom{n}{2}$. Then*

$$\min \left\{ N(K_r, G) : \ G \in \mathcal{K}(n, e) \right\} = h_r^*(n, e). \tag{4}$$

*Moreover, if $r \geq 4$ and $e > t_{r-1}(n)$, then*

$$\left\{ G \in \mathcal{K}(n, e) : \ N(K_r, G) = h_r^*(n, e) \right\} = \mathcal{H}_1^*(n, e). \tag{5}$$

Note that, since $\mathcal{H}_1^*(n, e) \subseteq \mathcal{H}_1(n, e)$, Theorem 1.1 remains true if we replace $\mathcal{K}(n, e)$ by $\mathcal{H}_1(n, e)$. In fact, the later version of the Lovász–Simonovits Conjecture from [12] states that, for all sufficiently large $n \geq n_0(r)$, at least one $g_r(n, e)$-extremal graph is in $\mathcal{H}_1(n, e)$. By (4), these two conjectures are equivalent. One should be able to show with some extra work that (5) also holds for $r = 3$ (it is also implied by the results in [9] that (5) holds for most $e$, given $n$). Since our main focus is the case $r \geq 4$, we do not pursue this strengthening here.

Given integers $n, e \in \mathbb{N}$ with $e \leq \binom{n}{2}$, we define the family $\mathcal{H}_2^*(n, e)$ as follows (with $k, a^*, m^*$ being as before). Take those graphs in $\mathcal{H}_1^*(n, e)$ that are $k$-partite, along with the following family. Take disjoint sets $A_1, \ldots, A_k$ of sizes $a_1^*, \ldots, a_k^*$, respectively, and let $m := m^*$. If $m^* = 0$ and $a_1^* \geq a_k^* + 2$, then we also allow $(|A_1|, \ldots, |A_k|) = \left(a_2^*, \ldots, a_{k-1}^*, a_1^* - 1, a_k^* + 1\right)$ and let $m := a_1^* - a_k^* - 1$. Take all graphs obtained from $K[A_1, \ldots, A_k]$ by removing any $m$ edges, each connecting $B_i$ to $A_i$ for some $i \in I$, where $I := \left\{ i \in [k-1] : \ |A_i| = |A_{k-1}| \right\}$ and $\{B_i \colon i \in I\}$ are some pairwise disjoint subsets of $A_k$. Clearly, every graph in $\mathcal{H}_2^*(n, e)$ is an $(n, e)$-graph.

**Proposition 1.2.** *Suppose that $n \geq r \geq 4$ and $t_{r-1}(n) < e \leq \binom{n}{2}$ are integers. Then*

$$N(K_r, G) = h_r^*(n, e), \qquad \text{for every } G \in \mathcal{H}_2^*(n, e).$$

*Also, there are infinitely many pairs $(n, e) \in \mathbb{N}^2$ with $t_{r-1}(n) < e \leq \binom{n}{2}$ such that $\mathcal{H}_2^*(n, e) \setminus \mathcal{H}_1^*(n, e) \neq \emptyset$.*

We propose the following amended conjecture.

**Conjecture 1.3.** *Let* $r \geq 4$ be fixed. *For every sufficiently large integer $n$ and every integer $e$ with $t_{r-1}(n) < e \leq \binom{n}{2}$, it holds that*

$$\left\{ G \colon G \text{ is an } (n, e)\text{-graph with } N(K_r, G) = g_r(n, e) \right\} = \mathcal{H}_1^*(n, e) \cup \mathcal{H}_2^*(n, e).$$

For comparison with the case $r = 3$, the exact result of Liu, Pikhurko and Staden [9] valid for $e \leq (1 - o(1))\binom{n}{2}$ states that the set of $g_3(n, e)$-extremal graphs is exactly $\mathcal{H}_0^*(n, e) \cup \mathcal{H}_2^*(n, e)$ for a certain explicit family $\mathcal{H}_0^*(n, e) \supseteq \mathcal{H}_1^*(n, e)$, where the inclusion is strict for infinitely many pairs $(n, e)$. However, for $r \geq 4$ and $e > t_{r-1}(n)$, every graph in $\mathcal{H}_0^*(n, e) \setminus \mathcal{H}_1^*(n, e)$ can be shown to have more $K_r$'s than $H^*(n, e)$. (Basically, each such graph is obtained from a complete $(k-1)$-partite graph by adding edges into more than one part and cannot minimise the number of $K_r$'s for $r \geq 4$ by Lemma 2.5.)

For the purposes of this paper (namely for Proposition 1.2), only the difference $\mathcal{H}_2^*(n, e) \setminus \mathcal{H}_1^*(n, e)$ matters; we use the current definitions merely so that the families $\mathcal{H}_i^*(n, e)$ and $\mathcal{H}_i(n, e)$ are the same as in [9].

The rest of the paper of organised as follows. In the next section, we present some definitions and preliminary results. As a step towards proving Theorem 1.1, we first find extremal graphs in a certain family $\mathcal{H}_0(n, e)$ in Section 3 (see Proposition 3.1 for the exact statement). We derive Theorem 1.1 in Section 4. The proof of Proposition 1.2 is presented in Section 5.

## 2. Preliminaries

Given $\ell$ pairwise disjoint sets $A_1, \ldots, A_\ell$, we use $K[A_1, \ldots, A_\ell]$ to denote the complete $\ell$-partite graph with parts $A_1, \ldots, A_\ell$; if we care only about the isomorphism type of this graph (i.e. only the sizes of the parts matter), we may instead write $K_{a_1,\ldots,a_\ell}$, where $a_i := |A_i|$ for $i \in [\ell]$.

Let $G = (V, E)$ be a graph. By $|G|$ we denote the number of edges in $G$. Let $\overline{G} := \left(V, \binom{V}{2} \setminus E\right)$ denote the *complement* of $G$. The subgraph of $G$ *induced* by a set $A \subseteq V$ is $G[A] := \left(A, \binom{A}{2} \cap E\right)$. For disjoint $A, B \subseteq V$, we use $G[A, B]$ to denote the induced bipartite graph with parts $A$ and $B$ (which consists of edges connecting $A$ to $B$).

In the remainder of this note, we assume unless it is stated otherwise that $r, n, e \in \mathbb{N}$ satisfy $r \geq 3$ and $e \leq \binom{n}{2}$ (and we minimise the number of $r$-cliques over $(n, e)$-graphs). Also, $k = k(n, e)$ is defined in (2).

Given a family $\mathcal{F}$ of $(n, e)$-graphs, we use $\mathcal{F}^{\min}$ to denote the collection of graphs $F \in \mathcal{F}$ with the minimum number of $K_r$'s (over all graphs in $\mathcal{F}$). For convenience, we set $N(K_0, G) := 1$ and $N(K_{-1}, G) := 0$ for all graphs $G$.

Let the family $\mathcal{H}_0(n, e)$ be the collection of all $(n, e)$-graphs that can be obtained from an $n$-vertex complete $(k-1)$-partite graph by adding a (possibly empty) triangle-free graph into each part. It is clear from the definition that $\mathcal{H}_1(n, e) \subseteq \mathcal{H}_0(n, e)$.

The following fact follows from some simple calculations (with the argument for Part (i) being the same as in (10)).

**Fact 2.1.** *Let $k$, $a^*$, $m^*$, $H^*$, and $h_r^*(n, e)$ be as defined in Section 1. Then it holds for all $r \geq 3$ that*

*(i)* $0 \leq m^* \leq a^*_{k-1} - a^*_k$,

*(ii)* $|K_{a^*_1,\ldots,a^*_k}| - |K_{a^*_1,\ldots,a^*_{k-2},a^*_{k-1}+1,a^*_k-1}| = a^*_{k-1} - a^*_k + 1$,

*(iii)* $N(K_r, H^*) = h_r^*(n, e) \geq g_r(n, e)$.

We also need the following simple facts for counting $r$-cliques in some special classes of graphs.

**Fact 2.2.** *Let $G$ be a graph, $S \subseteq V(G)$ be a vertex set, and $\overline{S} := V(G) \setminus S$. Suppose that the induced subgraph $G[S]$ is triangle-free, and the induced bipartite graph $G[S, \overline{S}]$ is complete. Then*

$$N(K_r, G) = |G[S]| \cdot N(K_{r-2}, G[\overline{S}]) + |S| \cdot N(K_{r-1}, G[\overline{S}]) + N(K_r, G[\overline{S}]).$$

**Fact 2.3.** *Suppose that $G$ is a graph obtained from $K[V_1, \ldots, V_\ell]$ by adding a triangle-free graph. Let $S := V_1 \cup V_2$ and $\overline{S} := V(G) \setminus S$. Then*

$$\begin{aligned} N(K_r, G) &= |G[V_1]| \cdot |G[V_2]| \cdot N(K_{r-4}, G[\overline{S}]) \\ &\quad + (|G[V_1]| \cdot |V_2| + |G[V_2]| \cdot |V_1|) \cdot N(K_{r-3}, G[\overline{S}]) \\ &\quad + |G[S]| \cdot N(K_{r-2}, G[\overline{S}]) + |S| \cdot N(K_{r-1}, G[\overline{S}]) + N(K_r, G[\overline{S}]). \end{aligned}$$

**Fact 2.4.** *Let $G$ be a graph, $S \subseteq V(G)$, and $\overline{S} := V(G) \setminus S$. Suppose that the induced subgraph $G[S]$ is 3-partite, and the induced bipartite subgraph $G[S, \overline{S}]$ is complete. Then*

$$\begin{aligned} N(K_r, G) = N(K_3, G[S]) \cdot N(K_{r-3}, G[\overline{S}]) + |G[S]| \cdot N(K_{r-2}, G[\overline{S}]) \\ + |S| \cdot N(K_{r-1}, G[\overline{S}]) + N(K_r, G[\overline{S}]). \end{aligned}$$

We will also use the following results.

**Lemma 2.5.** *Let $r \geq 4$ and let $n, e \in \mathbb{N}$ satisfy $t_{r-1}(n) < e \leq \binom{n}{2}$. Suppose that $G \in \mathcal{H}_0^{\min}(n, e)$ is a graph with a vertex partition $V(G) = B_1 \cup \ldots \cup B_{k-1}$ such that $G$ is the union of $K[B_1, \ldots, B_{k-1}]$ with a triangle-free graph. Then $G$ contains at most one part $B_i$ which is partially full, meaning that $0 < |G[B_i]| < t_2(|B_i|)$.*

https://doi.org/10.1017/S0963548324000269 Published online by Cambridge University Press

**Proof.** Suppose to the contrary that $G$ contains two partially full parts $B_i$ and $B_j$ for some $1 \le i < j \le k-1$. Let $x := |G[B_i]|$, $\sigma := |G[B_i]| + |G[B_j]|$ and $H := G[V(G) \setminus (B_i \cup B_j)]$. Observe from Fact 2.3 that there exist constants $C_2, C_3, C_4$ depending on $|B_i|, |B_j|$ and $H$ (but not on $x$) such that

$$N(K_r, G) = N(K_{r-4}, H) \cdot x(\sigma - x) + C_2 x + C_3(\sigma - x) + C_4 =: P(x).$$

Let $G_i$ be the graph obtained from $G$ by moving one edge from $G[B_j]$ to $G[B_i]$ and rearranging the latter graph to be still $K_3$-free, which is possible by Mantel's theorem. Similarly, let $G_j$ be the graph obtained from $G$ by moving one edge from $G[B_i]$ to $G[B_j]$. Note that $N(K_r, G_i) = P(x+1)$ and $N(K_r, G_j) = P(x-1)$. Since $e > t_{r-1}(n)$, we have

$$P(x+1) + P(x-1) - 2P(x) = -2N(K_{r-4}, H) < 0. \tag{6}$$

Thus $\min\left\{N(K_r, G_i), N(K_r, G_j)\right\} < N(K_r, G)$, contradicting the minimality of $G$. □

The following simple inequality from [9] will be useful. For completeness, we include its short proof here.

**Lemma 2.6** ([9, Lemma 4.5]). *For all integers $a \ge 1$, $k \ge 2$, and $n \ge ak$, we have*

$$a(n-a) + t_{k-1}(n-a) > (a-1)(n-a+1) + t_{k-1}(n-a+1). \tag{7}$$

**Proof.** Let $a_1 \ge \cdots \ge a_{k-1}$ denote the part sizes of $T_{k-1}(n-a)$. If we increase its number of vertices by one, then the part sizes of the new Turán graph, up to reordering, can be obtained by increasing $a_{k-1}$ by one. Thus the difference between the expressions in (7) is

$$|K_{a_1,\ldots,a_{k-1},a}| - |K_{a_1,\ldots,a_{k-2},a_{k-1}+1,a-1}| = a_{k-1}a - (a_{k-1}+1)(a-1) = a_{k-1} - a + 1, \tag{8}$$

which is positive since $a_{k-1} \ge \lfloor (n-a)/(k-1) \rfloor \ge \lfloor (ak-a)/(k-1) \rfloor = a$. □

## 3. Extremal graphs in $\mathcal{H}_0(n, e)$

As an intermediate step towards Theorem 1.1, we will first prove the following result, which determines the extremal graphs in $\mathcal{H}_0(n, e)$.

**Proposition 3.1.** *For all integers $n \ge r \ge 4$ and $t_{r-1}(n) < e \le \binom{n}{2}$, we have that $\mathcal{H}_0^{\min}(n, e) = \mathcal{H}_1^*(n, e)$.*

We will use this result later to prove Theorem 1.1 by induction on the number of parts in a graph in $\mathcal{K}(n, e)$. Note that, in general, neither $\mathcal{K}(n, e)$ nor $\mathcal{H}_0(n, e)$ is a subfamily of the other. However, when we work on the structure of extremal graphs in $\mathcal{K}(n, e)$ in the proof of Theorem 1.1, some intermediate graphs may be in $\mathcal{H}_0(n, e)$.

We need some further preliminaries before we can prove Proposition 3.1.

Given a graph $G \in \mathcal{H}_0^{\min}(n, e)$ with partition $B_1, \ldots, B_{k-1}$, we apply the following modification to $G$ to obtain a new graph $H' = H'(G) \in \mathcal{H}_0^{\min}(n, e)$. Note that, in fact, these steps do not depend on $r$.

Step 1: If there is a part $B_i$ that is partially full in $G$, then let $B := B_i$ (by Lemma 2.5, such $B_i$ is unique if it exists). Otherwise, take an arbitrary $i \in [k-1]$ with $|G[B_i]| = t_2(|B_i|)$ and let $B := B_i$. Since $|G| > t_{k-1}(n)$, $|G[B_i]|$ cannot be 0 for all $i \in [k-1]$. Thus, the set $B$ is well-defined.

Step 2: Note that $G - B$ is a complete multipartite graph. Let $A_1, \ldots, A_{t-2}$ denote its parts. Let $a_i := |A_i|$ for $i \in [t-2]$ and assume that $a_1 \ge \cdots \ge a_{t-2}$. Note that each original part $B_\ell$ is either $B$, some $A_i$, or the union of two parts $A_i$ and $A_j$.

https://doi.org/10.1017/S0963548324000269 Published online by Cambridge University Press

Step 3: Choose integers $a_{t-1} \geq a_t \geq 1$ such that

$$a_{t-1} + a_t = |B| \quad \text{and} \quad (a_{t-1}+1)(a_t-1) < |G[B]| \leq a_{t-1}a_t.$$

Note that this is possible by Mantel's theorem since $G[B]$ is triangle-free. Let $A_{t-1} \sqcup A_t = B$ be a partition with $|A_{t-1}| = a_{t-1}$ and $|A_t| = a_t$. If $|G[B]| = t_2(|B|)$, then $a_{t-1} = \lceil |B|/2 \rceil$ and $a_t = \lfloor |B|/2 \rfloor$ and we assume that $A_{t-1} \sqcup A_t = B$ is the original partition of $G[B]$ with the two parts labelled so that $|A_{t-1}| \geq |A_t|$.

Step 4: Let $H'$ be obtained from $K[A_1, \ldots, A_t]$ by removing a star whose centre lies in $A_t$ and $m'$ leaves lie in $A_{t-1}$, where

$$m' := \sum_{ij \in \binom{[t]}{2}} a_i a_j - e = a_{t-1}a_t - |G[B]|. \tag{9}$$

This is possible because, by Step 3,

$$0 \leq m' = a_{t-1}a_t - |G[B]| \leq a_{t-1}a_t - ((a_{t-1}+1)(a_t-1)+1) = a_{t-1} - a_t. \tag{10}$$

Notice that to obtain $H'$ we only change the structure of $G$ on $B$ while keeping $|G[B]| = |H'[B]|$. Thus, $H' \in \mathcal{H}_0(n, e)$ and, since $G[B, V(G) \setminus B]$ is complete bipartite and $G[B]$ is triangle-free, it follows from Fact 2.2 that $N(K_r, H') = N(K_r, G)$, and hence, $H' \in \mathcal{H}_0^{\min}(n, e)$.

**Lemma 3.2.** *For all $r \geq 3$, integers $n$ and $e$ with $t_{r-1}(n) < e \leq \binom{n}{2}$ and $G \in \mathcal{H}_0^{\min}(n, e)$, the graph $H'$ produced by Steps 1–4 above is isomorphic to $H^*(n, e)$.*

**Proof.** To prove that $H' \cong H^*(n, e)$, it suffices to show that $t = k$ and $(|A_1|, \ldots, |A_t|) = a^*$, where $k$ and $a^*$ are as defined in Section 1.

**Claim 3.3.** *If $m' = 0$, then $|H'[A_h \cup A_i \cup A_j]| > t_2(a_h + a_i + a_j)$ for all $\{h, i, j\} \in \binom{[t]}{3}$. If $m' > 0$, then $|H'[A_h \cup A_{t-1} \cup A_t]| > t_2(a_h + a_{t-1} + a_t)$ for all $h \in [t-2]$.*

**Proof.** Let $S := A_h \cup A_i \cup A_j$, with $\{i, j\} = \{t-1, t\}$ if $m' > 0$. Suppose to the contrary that $|H'[S]| \leq t_2(|S|)$. Then let $G_1$ be a new graph obtained from $H'$ by replacing $H'[S]$ with a bipartite graph of the same size. Note that the induced bipartite graph $H'[S, \overline{S}]$ is complete. (Indeed, this is trivially true if $m' = 0$ as then $H' = K[A_1, \ldots, A_t]$; if $m' > 0$, then the only non-complete pair is $[A_{t-1}, A_t]$, but both sets lie in $S$.) Since $H'$ is $t$-partite, the graph $G_1$ is $(t-1)$-partite (and with at most one non-complete pair of parts). By Steps 2–3, we have $t \leq 2(k-1)$. So we can represent $G_1$ as the union of a complete $(k-1)$-partite graph and a triangle-free graph, which implies that $G_1 \in \mathcal{H}_0(n, e)$. It is easy to see from Fact 2.4 that $N(K_r, G_1) \leq N(K_r, H')$, since $0 = N(K_3, G_1[S]) \leq N(K_3, H'[S])$. So it follows from the minimality of $H'$ that $N(K_3, H'[S]) = 0$. If $\{t-1, t\}$ is not a subset of $\{h, i, j\}$, then $H'[S]$ is a complete 3-partite graph and contains at least one traingle, contradicting $N(K_3, H'[S]) = 0$. Therefore, $\{t-1, t\} \subseteq \{h, i, j\}$. By symmetry, we may assume that $\{t-1, t\} = \{i, j\}$ (thus being consistent with our earlier assumption if $m' > 0$). Note that $|H[A_{t-1}, A_t]| \geq 1$, since otherwise, we would have $m' \geq a_{t-1}a_t > a_{t-1} - a_t$, contradicting (10). Note that each edge in $H[A_{t-1}, A_t]$ is in $|A_h|$ triangles in $H[S]$, contradicting $N(K_3, H'[S]) = 0$. □

**Claim 3.4.** *If $m' > 0$, then $a_{t-2} \geq a_{t-1}$.*

**Proof.** Suppose to the contrary that $a_{t-2} \leq a_{t-1} - 1$. Then let $G_2$ be a new graph obtained from $H'$ by moving edges from $[A_{t-2}, A_t]$ to $[A_{t-1}, A_t]$ until this is no longer possible. Let $S := A_{t-2} \cup A_{t-1} \cup A_t$. If $A_{t-2} \cup A_t$ is an independent set in $G_2$ (i.e. if $m' \geq a_{t-2}a_t$), then $|H'[S]| = |G_2[S]| \leq t_2(|S|)$, contradicting Claim 3.3. Thus $G_2[S]$ can be viewed as a graph obtained from $K[A_{t-2}, A_{t-1}, A_t]$ by removing $m'$ edges from $K[A_{t-2}, A_t]$. So $G_2 \in \mathcal{H}_0(n, e)$. Note that

$$N(K_3, G_2[S]) - N(K_3, H'[S]) = m'\,(a_{t-2} - a_{t-1}) < 0,$$

https://doi.org/10.1017/S0963548324000269 Published online by Cambridge University Press

which combined with Fact 2.4 implies that $N(K_r, G_2) - N(K_r, H') < 0$, contradicting the minimality of $H'$. □

If $m' > 0$, let $C_i := A_i$ for $i \in [t]$. If $m' = 0$, let $C_1, \ldots, C_t$ be a relabelling of $A_1, \ldots, A_t$ so that the sizes of the sets are non-increasing. Regardless of the value of $m'$, the following statements clearly hold:

(i) $c_1 \geq \cdots \geq c_t$, where $c_i := |C_i|$ for $i \in [t]$,
(ii) $0 \leq m' \leq c_{t-1} - c_t$,
(iii) Claim 3.3 applies to all triples $\{C_i, C_{t-1}, C_t\}$ for $i \in [t-2]$.

The rest of the proof is written so that it works for both $m' = 0$ and $m' > 0$.

**Claim 3.5.** *We have* $c_1 \leq c_{t-1} + 1$.

**Proof.** Let $S := C_1 \cup C_{t-1} \cup C_t$. Note that

$$|K_{c_1-1,c_{t-1}+1,c_t}| - |H'[S]| = m' - c_{t-1} + c_1 - 1 =: m''.$$

Suppose to the contrary that $c_1 \geq c_{t-1} + 2$. Then $m'' \geq m' + 1$. Take a partition $C_1' \cup C_{t-1}' \cup C_t' = S$ of sizes $c_1 - 1, c_{t-1} + 1, c_t$, respectively. Let $H_S$ be the graph obtained from $K[C_1', C_{t-1}', C_t']$ by removing $m''$ edges between $C_{t-1}'$ and $C_t'$. This is possible since $m'' \leq (c_{t-1} + 1)c_t$. (Indeed, otherwise $|H'[S]| \leq (c_1 - 1)(c_{t-1} + c_t + 1) \leq t_2(|S|)$, contradicting Claim 3.3.) We have $|H_S| = |H'[S]|$. Let $H''$ be the graph obtained from $H'$ by replacing $H'[S]$ with $H_S$. Note that $H'' \in \mathcal{H}_0(n, e)$. It follows from $m' \leq c_{t-1} - c_t$ that

$$\begin{aligned} N(K_3, H'[S]) - N(K_3, H''[S]) &= \big(c_1 c_{t-1} c_t - m' c_1\big) \\ &\quad - \big((c_1 - 1)(c_{t-1} + 1)c_t - (m' - c_{t-1} + c_1 - 1)(c_1 - 1)\big) \\ &\geq (c_1 - c_t)(c_1 - c_{t-1} - 2) + 1 \geq 1, \end{aligned}$$

which combined with Fact 2.4 implies that $N(K_r, H') - N(K_r, H'') > 0$, contradicting the minimality of $H'$. □

**Claim 3.6.** *We have* $t = k$.

**Proof.** It suffices to show that $t_{t-1}(n) < e \leq t_t(n)$. The upper bound $e \leq t_t(n)$ is trivial, since $H'$ is $t$-partite. So it remains to show that $e > t_{t-1}(n)$. Let $T := H'[C_1 \cup \cdots \cup C_{t-1}]$. It follows from Claim 3.5 that $T \cong T_{t-1}(n - c_t)$. Therefore,

$$|H'| - t_{t-1}(n - c_t) = |H' \setminus T| = c_t(n - c_t) - m'. \tag{11}$$

On the other hand, by viewing $T_{t-1}(n)$ as a graph obtained from $T_{t-1}(n - c_t)$ by adding $c_t$ new vertices into some parts, we obtain

$$t_{t-1}(n) - t_{t-1}(n - c_t) \leq c_t(n - c_{t-1} - 1).$$

By combining these two inequalities, we obtain

$$|H'| - t_{t-1}(n) \geq c_t(c_{t-1} + 1 - c_t) - m' \geq (c_t - 1)(c_{t-1} - c_t) + c_t > 0,$$

proving that $e > t_{t-1}(n)$. □

**Claim 3.7.** *The sequence* $(|C_1|, \ldots, |C_k|)$ *of part sizes is equal to* $a^* = a^*(n, e)$.

**Proof.** Recall that $t = k$ and, by (11), we have that

$$|H'| - t_{k-1}(n - c_k) = c_k(n - c_k) - m' \leq c_k(n - c_k). \tag{12}$$

Let us show that $c_k$ is the smallest nonnegative integer $a$ satisfying

$$f(a) := a(n - a) + t_{k-1}(n - a) \geq e.$$

https://doi.org/10.1017/S0963548324000269 Published online by Cambridge University Press

This inequality holds for $a = c_k$ by (12). Note that $c_k \le n/k$ as it is the smallest among $c_1 + \cdots + c_k = n$. Thus, by Lemma 2.6, it is enough to check that $a = c_k - 1$ violates this condition. Notice that

$$f(c_k - 1) - f(c_k) \le 2c_k - n - 1 + (n - c_k - c_{k-1}) = c_k - c_{k-1} - 1.$$

Therefore, it follows from $m' \le c_{k-1} - c_k$ that

$$f(c_k - 1) \le f(c_k) - (m' + 1) \le |H'| + m' - (m' + 1) < |H'|,$$

as desired.

Thus $c_k = a_k^*$ and (since $t = k$ by Claim 3.6) we have $(c_1, \ldots, c_k) = a^*$ by Claim 3.5, as desired. □

Also, it follows from the definitions that $m' = m^*$ and thus $H'$ is isomorphic to $H^*(n, e)$. This completes the proof of Lemma 3.2. □

Now we are ready to prove Proposition 3.1.

**Proof of Proposition 3.1.** Let $G \in \mathcal{H}_0^{\min}(n, e)$ be arbitrary. Let $B_1, \ldots, B_{k-1}$ be a vertex partition such that $G$ is the union of $K[B_1, \ldots, B_{k-1}]$ with a triangle-free graph $J$. Let $b_i := |B_i|$ for $i \in [k-1]$. Apply Steps 1–4 to $G$ to obtain a $k$-partite graph $H'$ with parts $A_1, \ldots, A_k$. By Lemma 3.2, we have $H' \cong H^* := H^*(n, e)$. Assume that $|A_i| = a_i^*$ for $i \in [k]$ and that all missing edges of $H'$ (if any exist) go between $A_{k-1}$ and $A_k$.

The following claim follows from the definitions of Steps 1–4.

**Claim 3.8.** *If $i \in [k-1]$ satisfies $|G[B_i]| \in \{0, t_2(b_i)\}$, then $H'[B_i] = G[B_i]$.*

Since $H'$ is $k$-partite, it follows from the definitions of Steps 1–4 that exactly one part $B_p$ of $G$ is divided into $A_q \cup A_s$ in Steps 2–3, where, say, $1 \le q < s \le k$, while the remaining parts of $G$ correspond to the remaining parts of $H'$. In particular, $b_p = a_q^* + a_s^*$.

**Claim 3.9.** *We have $|G[B_p]| > 0$.*

**Proof.** It follows from $m^* \le a_{k-1}^* - a_k^*$ that

$$|H'[B_p]| = a_q^* a_s^* - m^* \ge a_q^* a_s^* - (a_{k-1}^* - a_k^*) > 0.$$

Combined with Claim 3.8, we see that $|G[B_p]| > 0$. □

Suppose first that $m^* = 0$. Then $H' = K[A_1, \ldots, A_k]$, and $G$ can be obtained from $H'$ by replacing $H'[A_q \cup A_s]$ with $G[B_p]$. Moreover, $G[B_p]$ is a triangle-free graph with $a_q^* + a_s^*$ vertices and $a_q^* a_s^*$ edges. If $a_s^* = a_k^*$, then it follows from the definition of $\mathcal{H}_1^*(n, e)$ that $G \in \mathcal{H}_1^*(n, e)$. Otherwise, $|a_q^* - a_s^*| \le 1$ (by the definition of $a^*$), and hence, $G[B_p] \cong T_2(a_q^* + a_s^*)$. This implies that $G$ does not contain any partially full part, and hence, $G = H' \in \mathcal{H}_1^*(n, e)$.

Suppose that $m^* > 0$. Since $G[A_i, A_j]$ is complete for all $\{i, j\} \ne \{q, s\}$ and $H'[A_i, A_j]$ is complete iff $\{i, j\} \ne \{k-1, k\}$, we have $\{q, s\} = \{k-1, k\}$. Thus $G$ can be obtained from $K[A_1, \ldots, A_k]$ by replacing $K[A_{k-1} \cup A_k]$ with a triangle-free graph with $a_{k-1}^* a_k^* - m^*$ edges. This gives $G \in \mathcal{H}_1^*(n, e)$. We conclude that $\mathcal{H}_0^{\min}(n, e) \subseteq \mathcal{H}_1^*(n, e)$. Since $\mathcal{H}_1^*(n, e) \subseteq \mathcal{H}_0(n, e)$ and every graph in $\mathcal{H}_1^*(n, e)$ contains the same number of $K_r$'s, we have $\mathcal{H}_0^{\min}(n, e) = \mathcal{H}_1^*(n, e)$. □

## 4. Proof of Theorem 1.1

With Proposition 3.1 in hand, we are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Fix integers $n \ge r \ge 3$ and $e \le \binom{n}{2}$. Notice that (4) can be reduced to $\min\{N(K_r, G) : G \in \mathcal{K}(n, e)\} \ge h_r^*(n, e)$, since the other direction is trivially true. Suppose that

https://doi.org/10.1017/S0963548324000269 Published online by Cambridge University Press

$G \in \mathcal{K}^{\min}(n, e)$ is a graph obtained from a complete $\ell$-partite graph by adding a triangle-free graph to one part. We aim to show that $N(K_r, G) \geq h_r^*(n, e)$ when $r \geq 3$ and, in addition, $G \in \mathcal{H}_1^*(n, e)$ when $r \geq 4$ and $e > t_{r-1}(n)$. We prove this statement by induction on $\ell + r$. Notice that if $\ell = k - 1$ (where $k = k(n, e)$) and $r \geq 4$, then $G \in \mathcal{H}_0(n, e)$, and it follows from Proposition 3.1 that $G \in \mathcal{H}_1^*(n, e)$, as desired. If $\ell = k - 1$ and $r = 3$, then $G \in \mathcal{H}_0(n, e)$, and it follows from [9, Proposition 1.5] that $N(K_3, G) \geq h_3^*(n, e)$. So the statement is true for all pairs $(\ell, r)$ with $\ell = k - 1$ and $r \geq 3$, and this serves as our base case.

Assume that $\ell \geq k$ and $r \geq 3$. Let $U_1 \cup \cdots \cup U_\ell = V(G)$ be a partition such that $G$ is obtained from the complete $\ell$-partite graph $K[U_1, \ldots, U_\ell]$ by adding a triangle-free graph into $U_\ell$. We can assume that $U_\ell$ is not an independent set (otherwise consider instead the $(\ell - 1)$-partition of $V(G)$ where $U_{\ell-1}$ and $U_\ell$ are merged together).

First, we prove (4). Assume that $\ell \geq r - 1$, as otherwise $h_r^*(n, e) = 0$ and there is nothing to do. Note that $U_\ell$ is as large as any other part: if some part $U_i$ has strictly larger size then by moving all edges from $U_\ell$ to $U_i$ (by $|U_i| > |U_\ell|$ there is enough space for this) we strictly decrease the number of $r$-cliques (since $\ell \geq r - 1$), a contradiction. By relabelling parts $U_1, \ldots, U_{\ell-1}$, we may assume that $U_1$ is of smallest size among $U_1, \ldots, U_{\ell-1}$. Let $\hat{G}$ denote the induced subgraph of $G$ on $U_2 \cup \cdots \cup U_\ell$. Let $\hat{n} := n - |U_1|$ and $\hat{e} := |\hat{G}|$. Let $\hat{k} := k(\hat{n}, \hat{e})$ be as defined in (2) (while we reserve $k$ for $k(n, e)$).

**Claim 4.1.** *We have* $\hat{k} \leq k$.

**Proof.** Let $H^* = H^*(n, e)$ be the $k$-partite graph as defined in Section 1. Assume that $A_1^*, \ldots, A_k^*$ are the corresponding parts of $H^*$ of sizes $a_1^* \geq \cdots \geq a_k^*$, respectively. It is clear that $|A_1^*| \geq \frac{n}{k}$. It follows from the minimality of $U_1$ that $|U_1| \leq \frac{n - |U_\ell|}{\ell - 1} \leq \frac{n}{k} \leq |A_1^*|$. Let $W_1 \subseteq A_1^*$ be a set of size $|U_1|$ and let $H'$ be the induced subgraph of $H^*$ on $V(H) \setminus W_1$. Observe that $H'$ is still a $k$-partite graph and $|H'| \geq |\hat{G}|$. So it follows from the definition that $\hat{k} \leq k$. □

Note that $\hat{G}$ can be viewed as a graph obtained from a complete $(\ell - 1)$-partite graph by adding a triangle-free graph into one part; in particular, $\hat{G} \in \mathcal{K}(\hat{n}, \hat{e})$. Let $\hat{H}$ be $H^*(\hat{n}, \hat{e})$ and let $G'$ be the graph obtained from $G$ by replacing $\hat{G}$ with $\hat{H}$. It follows from the inductive hypothesis that

$$N(K_r, \hat{H}) = h_r^*(\hat{n}, \hat{e}) \leq N(K_r, \hat{G}) \quad \text{and} \quad N(K_{r-1}, \hat{H}) \leq N(K_{r-1}, \hat{G}).$$

Hence,

$$\begin{aligned} h_r^*(n, e) \leq N(K_r, G') &= N(K_r, \hat{H}) + |U_1| \cdot N(K_{r-1}, \hat{H}) \\ &\leq N(K_r, \hat{G}) + |U_1| \cdot N(K_{r-1}, \hat{G}) = N(K_r, G), \end{aligned}$$

finishing the inductive step for proving (4).

Now suppose that $r \geq 4$ and $e > t_{r-1}(n)$, and suppose for contradiction that $G \notin \mathcal{H}_1^*(n, e)$. Reusing the notation introduced above, let us first derive a contradiction from assuming that $\hat{G} \notin \mathcal{H}_1^*(\hat{n}, \hat{e})$.

If $\hat{e} > t_{r-1}(\hat{n})$, then it follows from the inductive hypothesis that

$$N(K_r, \hat{H}) < N(K_r, \hat{G}) \quad \text{and} \quad N(K_{r-1}, \hat{H}) \leq N(K_{r-1}, \hat{G}).$$

Therefore,

$$\begin{aligned} N(K_r, G') &= N(K_r, \hat{H}) + |U_1| \cdot N(K_{r-1}, \hat{H}) \\ &< N(K_r, \hat{G}) + |U_1| \cdot N(K_{r-1}, \hat{G}) = N(K_r, G), \end{aligned} \tag{13}$$

contradicting the minimality of $G$.

So suppose that $\hat{e} \leq t_{r-1}(\hat{n})$. We have that $\ell \geq k \geq r$. Recall that $\hat{G}$ is a graph obtained from an $(\ell - 1)$-partite graph by adding a non-empty triangle-free graph. Thus, we have $N(K_r, \hat{H}) = 0 <$

https://doi.org/10.1017/S0963548324000269 Published online by Cambridge University Press

$N(K_r, \hat{G})$. In addition, by (4), we have $N(K_{r-1}, \hat{H}) = h^*_{r-1}(\hat{n}, \hat{e}) \le N(K_{r-1}, \hat{G})$. But then the same calculation as in (13) gives a contradiction to the minimality of $G$.

Thus we have that $\hat{G} \in \mathcal{H}^*_1(\hat{n}, \hat{e})$. Let $\hat{A}^*_1 \cup \ldots \cup \hat{A}^*_{\hat{k}} = V(\hat{G})$ be the partition of $\hat{G}$ as in the definition of $\mathcal{H}^*_1(\hat{n}, \hat{e})$. Let $B_1 := U_1 \cup \hat{A}^*_1$, $B_i := \hat{A}^*_i$ for $2 \le i \le \hat{k} - 2$, and $B_{\hat{k}-1} := \hat{A}_{\hat{k}-1} \cup \hat{A}_{\hat{k}}$. We can view $G$ as a graph obtained from $K[B_1, \ldots, B_{\hat{k}-1}]$ by adding triangle-free graphs into two parts, namely $G[B_1]$ and $G[B_{\hat{k}-1}]$. Since $\hat{k} \le k$ by Claim 4.1, it holds that $G \in \mathcal{H}_0(n, e)$. Therefore, it follows from Proposition 3.1 that $G \in \mathcal{H}^*_1(n, e)$, finishing the proof of Theorem 1.1. □

Let us remark that if we replace the family $\mathcal{K}(n, e)$ in Theorem 1.1 by the larger family $\mathcal{K}'(n, e)$ that consists of all graphs obtained from a complete partite graph by adding a triangle-free graph (that is, we allow to add edges into more than one part) then the theorem will remain true. Indeed, for $r \ge 4$, the proof of Lemma 2.5 (which in fact works for any number of parts) shows that every extremal graph $\mathcal{K}'(n, e)$ has at most one partially full part and thus belongs to $\mathcal{K}(n, e)$. For $r = 3$, the equality in (4), will also remain true (again by the proof of Lemma 2.5 except the inequality in (6) becomes equality).

## 5. Proof of Proposition 1.2

**Proof of Proposition 1.2.** First, we prove that $N(K_r, H) = h^*_r(n, e)$ for all $H \in \mathcal{H}^*_2(n, e)$. Fix $H \in \mathcal{H}^*_2(n, e)$.

First consider the case when $(|A_1|, \ldots, |A_k|) = a^*$, where the sets $A_1, \ldots, A_k$ are as in the definition of $\mathcal{H}^*_2(n, e)$. Let $K := K[A_1, \ldots, A_k]$, and $m^*_i := |\overline{H}[B_i, A_i]|$ for $i \in I := \{j \in [k-1] :\ |A_j| = |A_{k-1}|\}$. Note from the definition of $I$ that for all $i \in I$, we have that

$$N(K_{r-2}, K[A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{k-1}]) = N(K_{r-2}, K[A_1, \ldots, A_{k-2}]),$$

because we count $r$-cliques in two isomorphic graphs. Therefore,

$$\begin{aligned} N(K_r, K) - N(K_r, H) &= \sum_{i \in I} m^*_i \cdot N(K_{r-2}, K[A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_{k-1}]) \\ &= \sum_{i \in I} m^*_i \cdot N(K_{r-2}, K[A_1, \ldots, A_{k-2}]) \\ &= m^* \cdot N(K_{r-2}, K[A_1, \ldots, A_{k-2}]) = N(K_r, K) - N(K_r, H^*). \end{aligned} \tag{14}$$

It follows that $N(K_r, H) = N(K_r, H^*) = h^*(n, e)$, as desired.

Now suppose that $(|A_1|, \ldots, |A_k|) \ne a^*$. Recall that then $m^* = 0$, $(|A_1|, \ldots, |A_k|) = (a^*_2, \ldots, a^*_{k-1}, a^*_1 - 1, a^*_k + 1)$, $m = a^*_1 - a^*_k + 1$, and $H$ is a graph obtained from $K[A_1, \ldots, A_k]$ by removing some $m$ edges. We may assume that these $m$ edges were removed from parts $[A_{k-1}, A_k]$, since this does not affect the value of $N(K_r, H)$ by the calculation in (14). Now, by viewing $H$ as a graph obtained from $K[A_1, \ldots, A_k]$ by replacing $K[A_{k-1}, A_k]$ with a triangle-free graph, we see that $H \in \mathcal{H}^*_1(n, e)$, and hence, $N(K_r, H) = h^*(n, e)$.

Next, we show that there are infinitely many pairs $(n, e) \in \mathbb{N}^2$ with $t_{r-1}(n) < e \le \binom{n}{2}$ such that $\mathcal{H}^*_2(n, e) \setminus \mathcal{H}^*_1(n, e) \ne \emptyset$. It is enough to chose $(n, e)$ so that $a^*_{k-2} = a^*_{k-1}$ and $m^*, a^*_k \ge 2$; the choice that we use (in (15) below) is rather arbitrary.

Take any integers $p \ge r - 1$, $q \ge 100$, and $2 \le m \le q$. Let $n := 2pq + q$ and $e := \binom{p}{2}(2q)^2 + 2pq^2 - m$. Note that $e + m$ is the number of edges in the complete $(p+1)$-partite graph $K_{2q,\ldots,2q,q}$ with $p$ parts of size $2q$ and one part of size $q$. The choice of $(p, q, m)$ ensures that

$$e = \binom{p}{2}(2q)^2 + 2pq^2 - m > \binom{p}{2}\left(\frac{2pq + q}{p}\right)^2 \ge t_p(n).$$

By $e < e + m \le t_{p+1}(n)$, we have that $k(n, e) = p$.

https://doi.org/10.1017/S0963548324000269 Published online by Cambridge University Press

Let us show that $a_p^* = q$. By Lemma 2.6, it is enough to show that $(q-1)(n-q-1) + t_{k-1}(n-q-1) < e$. The left-hand side here is the size of the graph obtained from the complete partite graph $K_{2q,\ldots,2q,q}$ by moving a vertex from the part of size $q$ into one of size $2q$. This results in losing $q+1 > m$ edges, giving the required. Thus,

$$a_1^* = \cdots = a_{p-1}^* = 2q, \quad a_p^* = q, \quad \text{and} \quad m^* = m. \tag{15}$$

Let $V_1 \cup \cdots \cup V_{p+1} = [n]$ be a partition such that $|V_1| = \cdots = |V_p| = 2q$ and $|V_{p+1}| = q$. Fix $m$ distinct vertices $v_1, \ldots, v_m \in V_{p+1}$, and choose a vertex $u_i \in V_i$ for every $i \in [m]$. Let $G$ be the graph obtained from $K[V_1, \ldots, V_{p-1}]$ by removing pairs in $\{\{v_i, u_i\} : i \in [m]\}$. It is easy to see that $G \in \mathcal{H}_2^*(n, e) \setminus \mathcal{H}_1^*(n, e)$, proving Proposition 1.2. □

## Acknowledgements

The authors would like to thank the anonymous referee for helpful comments.

## Funding

Research was supported by ERC Advanced Grant 101020255 and Leverhulme Research Project Grant RPG-2018-424.

## References

[1] Balogh, J. and Clemen, F. C. (2023) On stability of the Erdős-Rademacher problem. *Illinois J. Math.* **67** 1–11.
[2] Bollobás, B. (1976) On complete subgraphs of different orders. *Math. Proc. Cambridge Philos. Soc.* **79** 19–24.
[3] Erdős, P. (1955) Some theorems on graphs. *Riveon Lematematika* **9** 13–17.
[4] Erdős, P. (1962) On a theorem of Rademacher-Turán. *Illinois J. Math.* **6** 122–127.
[5] Fisher, D. C. (1989) Lower bounds on the number of triangles in a graph. *J. Graph Theory* **13** 505–512.
[6] Goldwurm, M. and Santini, M. (2000) Clique polynomials have a unique root of smallest modulus. *Inform. Process. Lett.* **75** 127–132.
[7] Goodman, A. W. (1959) On sets of acquaintances and strangers at any party. *Am. Math. Monthly* **66** 778–783.
[8] Kim, J., Liu, H., Pikhurko, O. and Sharifzadeh, M. (2020) Asymptotic structure for the clique density theorem. *Discrete Anal.* **19** 26.
[9] Liu, H., Pikhurko, O. and Staden, K. (2020) The exact minimum number of triangles in graphs with given order and size. *Forum Math. Pi* **8** 144.
[10] Liu, X. and Mubayi, D. (2022) On a generalized Erdős-Rademacher problem. *J. Graph Theory* **100** 101–126.
[11] Lovász, L. and Simonovits, M. (1976) On the number of complete subgraphs of a graph. In *Proceedings of the Fifth British Combinatorial Conference* (Univ. Aberdeen, Aberdeen, 1975), volume No. XV of Congress. Numer., Utilitas Math, pp. 431–441.
[12] Lovász, L. and Simonovits, M. (1983) On the number of complete subgraphs of a graph II. In*Studies in pure mathematics*, Birkhäuser, pp. 459–495.
[13] Mantel, W. (1907) Problem 28. *Wiskundige Opgaven* **10** 60–61.
[14] Moon, J. W. and Moser, L. (1962) On a problem of Turán. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **7** 283–286.
[15] Mubayi, D. (2010) Counting substructures I: color critical graphs. *Adv. Math.* **225** 2731–2740.
[16] Mubayi, D. (2013) Counting substructures II: Hypergraphs. *Combinatorica* **33** 591–612.
[17] Nikiforov, V. (2011) The number of cliques in graphs of given order and size. *Trans. Amer. Math. Soc.* **363** 1599–1618.
[18] Nikiforov, V. S. (1976/77) On a problem of P. *Erdős. Annuaire Univ. Sofia Fac. Math. Méc.* **71** 157–160.
[19] Nikiforov, V. S. and Khadzhiivanov, N. G. (1981) Solution of the problem of P. Erdős on the number of triangles in graphs with $n$ vertices and $[n^2/4] + l$ edges. *C. R. Acad. Bulgare Sci.* **34** 969–970.
[20] Nordhaus, E. A. and Stewart, B. M. (1963) Triangles in an ordinary graph. *Canadian J. Math.* **15** 33–41.
[21] Pikhurko, O. and Razborov, A. (2017) Asymptotic structure of graphs with the minimum number of triangles. *Combin. Probab. Comput.* **26** 138–160.
[22] Razborov, A. A. (2008) On the minimal density of triangles in graphs. *Combin. Probab. Comput.* **17** 603–618.
[23] Reiher, C. (2016) The clique density theorem. *Ann. Math.* **184** 683–707.
[24] Turán, P. (1941) On an external problem in graph theory. *Mat. Fiz. Lapok* **48** 436–452.
[25] Xiao, C. and Katona, G. O. H. (2021) The number of triangles is more when they have no common vertex. *Discrete Math.* **344** 112330.

**Cite this article:** Liu X and Pikhurko O (2024). A note on extremal constructions for the Erdős–Rademacher problem. *Combinatorics, Probability and Computing*. https://doi.org/10.1017/S0963548324000269