THE PACKING OF SPHERES IN THE SPACE l_p

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1. Introduction. A point $\mathbf{x}$ in the real or complex space l_p is an infinite sequence $(x_1, x_2, x_3, \ldots)$ of real or complex numbers such that $\sum_{r=1}^{\infty} |x_r|^p$ is convergent. Here $p \geqslant 1$ and we write

$$\|\mathbf{x}\| = \left\{\sum_{r=1}^{\infty} |x_r|^p\right\}^{1/p}.$$

The unit sphere S consists of all points $\mathbf{x} \in l_p$ for which $\|\mathbf{x}\| \leqslant 1$. The sphere of radius $a \geqslant 0$ and centre $\mathbf{y}$ is denoted by $S_a(\mathbf{y})$ and consists of all points $\mathbf{x} \in l_p$ such that $\|\mathbf{x}-\mathbf{y}\| \leqslant a$. The sphere $S_a(\mathbf{y})$ is contained in S if and only if $\|\mathbf{y}\| \leqslant 1-a$, and the two spheres $S_a(\mathbf{y})$ and $S_a(\mathbf{z})$ do not overlap if and only if

$$\|\mathbf{y}-\mathbf{z}\| \geqslant 2a.$$

The statement that a finite or infinite number of spheres $S_a(\mathbf{y})$ of fixed radius a can be packed in S means that each sphere $S_a(\mathbf{y})$ is contained in S and that no two such spheres overlap.

In a recent paper [3] the packing of spheres $S_a(\mathbf{y})$ in S was considered for the case $p=2$. It is the object of the present paper to extend this work to all $p \geqslant 1$. The results obtained are of a different character according as $p \leqslant 2$ or $p > 2$, and in the latter case are somewhat surprising.

We write

$$\lambda_p = \{1+2^{1-1/p}\}^{-1}, \quad \mu_p = \{1+2^{1/p}\}^{-1}.$$

Observe that $\lambda_2 = \mu_2$ and that $\lambda_p < \mu_p$ when $p > 2$. As usual, δ_{nr} is 1 or 0 according as $n=r$ or $n \neq r$.

THEOREM 1. *If $1 \leqslant p \leqslant 2$, an infinity of spheres $S_a(\mathbf{y})$ of fixed radius a can be packed in S if and only if $a \leqslant \lambda_p$. If* (i) *$a \leqslant \lambda_p$, the spheres may be centred at the points $\mathbf{y}_n = (y_{n1}, y_{n2}, y_{n3}, \ldots)$, where $y_{nr} = (1-a)\delta_{nr}$ $(n \geqslant 1,\ r \geqslant 1)$. If* (ii) *$\lambda_p < a \leqslant 1$, the maximum number of spheres $S_a(\mathbf{y})$ which can be packed in S does not exceed $L_p(a)$, where $L_1(a) = 1$, and*

$$L_p(a) = \left\{1 - \frac{1}{2}\left(\frac{1-a}{a}\right)^{p/(p-1)}\right\}^{-1} \quad (1 < p \leqslant 2).$$

THEOREM 2. *If $p > 2$, an infinity of spheres $S_a(\mathbf{y})$ of fixed radius a can be packed in S if and only if $a \leqslant \lambda_p$. If* (i) *$a \leqslant \lambda_p$, the spheres may be centred at the points $\mathbf{y}_n = (y_{n1}, y_{n2}, y_{n3}, \ldots)$, where $y_{nr} = (1-a)\delta_{nr}$ $(n \geqslant 1, r \geqslant 1)$. If* (ii) *$\lambda_p < a \leqslant \mu_p$, any finite number, no matter how large, of spheres $S_a(\mathbf{y})$ can be packed in S, but an infinite number cannot. If* (iii) *$\mu_p < a \leqslant 1$, the maximum number of spheres $S_a(\mathbf{y})$ which can be packed in S does not exceed*

$$M_p(a) = \left\{1 - \frac{1}{2}\left(\frac{1-a}{a}\right)^{p}\right\}^{-1}.$$

2. The case $1 \leqslant p \leqslant 2$. Suppose that m spheres $S_a(\mathbf{y}_j)$ $(1 \leqslant j \leqslant m)$ can be packed in S, where $m \geqslant 1$, $a \leqslant 1$. By a simple extension of Lemma 1 of [2] from n-dimensional Euclidean space to l_p, we find that

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$$\sum_{j=1}^{m} \| \mathbf{y}_j \|^p \geqslant 2^{p-1} m^{2-p} (m-1)^{p-1} a^p,$$

where the right-hand side denotes zero when $m = p = 1$. Hence, for at least one sphere $S_a(\mathbf{y}_j)$,

$$\| \mathbf{y}_j \| \geqslant a \left\{ 2 \left(1 - \frac{1}{m} \right) \right\}^{1-1/p},$$

and so

$$1 \geqslant a + \| \mathbf{y}_j \| \geqslant a \left[1 + \left\{ 2 \left(1 - \frac{1}{m} \right) \right\}^{1-1/p} \right]. \quad \text{..............................(1)}$$

If infinitely many spheres $S_a(\mathbf{y})$ can be packed in S it follows that $1 \geqslant a(1 + 2^{1-1/p})$; i.e. $a \leqslant \lambda_p$.

If $\lambda_p < a \leqslant 1$ and $1 < p \leqslant 2$, we deduce from (1) that

$$\frac{1}{m} \geqslant 1 - \frac{1}{2} \left(\frac{1-a}{a} \right)^{p/(p-1)},$$

and, since the right-hand side is positive, $m \leqslant L_p(a)$. If $p = 1$, (1) shows that $m = 1 = L_1(a)$, because of the convention stated above; for otherwise we should have $a \leqslant \frac{1}{2} = \lambda_1$.

Finally, part (i) of Theorem 1 follows since, for $a \leqslant \lambda_p$ and $j \neq k$,

$$\| \mathbf{y}_j - \mathbf{y}_k \| = 2^{1/p}(1-a) \geqslant 2a,$$

and $\| \mathbf{y}_k \| = 1 - a$. Thus no two of the spheres $S_a(\mathbf{y}_k)$ overlap and they are all contained in S. This completes the proof of Theorem 1.

3. The case $p > 2$. Suppose that m spheres $S_a(\mathbf{y}_j)$ $(1 \leqslant j \leqslant m)$ can be packed in S, where $m \geqslant 1$, $a \leqslant 1$. By a simple extension of Lemma 2 of [**1**] (with $\beta = p$, $c_j = 1$), we find that

$$\sum_{j=1}^{m} \| \mathbf{y}_j \|^p \geqslant 2(m-1) a^p,$$

so that, for at least one sphere $S_a(\mathbf{y}_j)$,

$$\| \mathbf{y}_j \| \geqslant a \left\{ 2 \left(1 - \frac{1}{m} \right) \right\}^{1/p}.$$

Hence

$$1 \geqslant a + \| \mathbf{y}_j \| \geqslant a \left[1 + \left\{ 2 \left(1 - \frac{1}{m} \right) \right\}^{1/p} \right],$$

from which we deduce that

$$\frac{1}{m} \geqslant 1 - \frac{1}{2} \left(\frac{1-a}{a} \right)^p. \quad \text{..(2)}$$

The right-hand side of (2) is positive if $\mu_p < a \leqslant 1$, so that we then obtain $m \leqslant M_p(a)$, which proves part (iii) of Theorem 2.

Part (i) of Theorem 2 can be proved as in §2. To prove part (ii) we suppose that $\lambda_p < a \leqslant \mu_p$ and take any positive integer m. For $1 \leqslant j \leqslant m$ and $n \geqslant 1$ put

$$y_{jn} = \varepsilon_{jn} 2^{(1-m)/p} (1-a) = \varepsilon_{jn} b \quad (n \leqslant 2^{m-1}), \quad y_{jn} = 0 \quad (n > 2^{m-1}),$$

where ε_{jn} is 1 or -1 according as the integral part of $(n-1)2^{j-m}$ is even or odd. Then

$\mathbf{y}_j = (y_{j1}, y_{j2}, y_{j3}, \ldots) \in l_p$. For example, when $m=4$ the first 8 components of $\mathbf{y}_1$, $\mathbf{y}_2$, $\mathbf{y}_3$ and $\mathbf{y}_4$ are as follows :

$$\begin{array}{cccccccc} b & b & b & b & b & b & b & b \\ b & b & b & b & -b & -b & -b & -b \\ b & b & -b & -b & b & b & -b & -b \\ b & -b & b & -b & b & -b & b & -b \end{array}$$

Clearly $\| \mathbf{y}_j \| = 1-a$ for $1 \leqslant j \leqslant m$ and, for $1 \leqslant j < k \leqslant m$,

$$\| \mathbf{y}_j - \mathbf{y}_k \| = 2^{1-1/p}(1-a) \geqslant 2a,$$

so that the m spheres $S_a(\mathbf{y}_j)$ are packed in S. Since m can be any positive integer this proves part (ii).

It remains to prove that if an infinity of spheres $S_a(\mathbf{y})$ can be packed in S, then $a \leqslant \lambda_p$. We therefore suppose that the spheres $S_a(\mathbf{y}_n)$ $(n=1, 2, 3, \ldots)$ can be packed in S, where $\mathbf{y}_n = (y_{n1}, y_{n2}, y_{n3}, \ldots)$. By considering each coordinate y_{nr} $(r=1, 2, 3, \ldots)$ in turn, picking out convergent subsequences, and renumbering the spheres, we may suppose that, for each fixed $r \geqslant 1$, $y_{nr} \to y_r$, say, as $n \to \infty$.

Since, for every positive integer N and all $n \geqslant 1$,

$$\sum_{r=1}^{N} |y_{nr}|^p \leqslant \| \mathbf{y}_n \|^p \leqslant (1-a)^p,$$

we have

$$\sum_{r=1}^{N} |y_r|^p \leqslant (1-a)^p,$$

and it follows that $\mathbf{y} = (y_1, y_2, y_3, \ldots) \in l_p$ and

$$\| \mathbf{y} \| \leqslant 1-a.$$

Now take any positive integer n and any $\varepsilon > 0$ and choose an integer N, depending on n and ε, such that

$$\sum_{r>N} |y_{nr}|^p < \varepsilon^p.$$

Then, for $m > n$, since $S_a(\mathbf{y}_m)$ and $S_a(\mathbf{y}_n)$ do not overlap,

$$(2a)^p \leqslant \sum_{r=1}^{\infty} |y_{mr} - y_{nr}|^p \qquad \text{(3)}$$

$$= \sum_{r=1}^{N} |y_{mr} - y_{nr}|^p + \sum_{r>N} |y_{mr} - y_{nr}|^p. \qquad \text{(4)}$$

Now by Minkowski's inequality,

$$\left\{ \sum_{r>N} |y_{mr} - y_{nr}|^p \right\}^{1/p} \leqslant \left\{ \sum_{r>N} |y_{mr}|^p \right\}^{1/p} + \left\{ \sum_{r>N} |y_{nr}|^p \right\}^{1/p}$$

$$\leqslant (1-a) + \varepsilon.$$

Hence, by (4),

$$(2a)^p - (1-a+\varepsilon)^p \leqslant \sum_{r=1}^{N} |y_{mr} - y_{nr}|^p.$$

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If we let $m \to \infty$ in this inequality, we obtain

$$(2a)^p - (1-a+\varepsilon)^p \leqslant \sum_{r=1}^{N} |y_r - y_{nr}|^p \leqslant \sum_{r=1}^{\infty} |y_r - y_{nr}|^p,$$

from which, since ε is arbitrary, we deduce that

$$(2a)^p - (1-a)^p \leqslant \sum_{r=1}^{\infty} |y_r - y_{nr}|^p. \qquad \text{(5)}$$

We now apply to (5) an argument similar to that applied to (3). For any $\varepsilon > 0$, choose a positive integer N, depending on ε, such that

$$\sum_{r>N} |y_r|^p < \varepsilon^p.$$

In place of (4) we have

$$(2a)^p - (1-a)^p \leqslant \sum_{r=1}^{N} |y_r - y_{nr}|^p + \sum_{r>N} |y_r - y_{nr}|^p$$
$$\leqslant \sum_{r=1}^{N} |y_r - y_{nr}|^p + (1-a+\varepsilon)^p.$$

On letting $n \to \infty$ we get, since ε is arbitrary,

$$(2a)^p \leqslant 2(1-a)^p$$

which is equivalent to $a \leqslant \lambda_p$. This completes the proof of Theorem 2.

4. The case $p = \infty$. In this case S can be interpreted as the "cube" consisting of points $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ for which $|x_r| \leqslant 1$ for $r = 1, 2, 3, \ldots$, and similarly for $S_a(\mathbf{y})$. It is clear that for $\frac{1}{2} < a \leqslant 1$, only one cube $S_a(\mathbf{y})$ can be packed in S, while, for $a \leqslant \frac{1}{2}$, infinitely many can; for we may take their centres at the points $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \ldots)$.

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