ON STABILITY OF SOLUTIONS OF CERTAIN DIFFERENTIAL EQUATIONS OF THE THIRD ORDER

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1. The purpose of this paper is to obtain a set of sufficient conditions for "global asymptotic stability" of the trivial solution $x = 0$ of the differential equation

$$(1.1) \qquad \dddot{x} + af_1(x, \dot{x})\ddot{x} + f_2(x, \dot{x})\dot{x} + bf_3(x) = 0,$$

using a Lyapunov function which is substantially different from similar functions used in [2], [3] and [4], for similar differential equations. The functions f_1, f_2 and f_3 are real-valued and are smooth enough to ensure the existence of the solutions of (1.1) on $[0, \infty)$. The dot indicates differentiation with respect to t. We are taking a and b to be some positive parameters. We also assume smoothness properties for $\partial f_2(x, y)/\partial x$, $\partial^2 f_1(x, y)/\partial x^2$ and $f_3'(x)$ to ensure the existence of the integrals appearing in our work.

Our main result appears in Section 2. In Section 3 we have generalized a result of Simanov [4]. And in the same section is obtained a result for the boundedness of the solutions of the differential equation

$$(1.2) \qquad \dddot{x} + f_1(x, \dot{x})\ddot{x} + f_2(x, \dot{x})\dot{x} + bx = p(t),$$

in which $p(t)$ is an integrable function.

2. We use the following notations:

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$$G(x, y) = \int_0^y f_1(x,\eta)d\eta / y \qquad (y \neq 0),$$

$$W(x, y) = ab \int_0^x f_3(\xi)d\xi + bf_3(x)y + \int_0^y \eta g(x, \eta)d\eta$$

$$g(x, y) = f_2(x, y) - a \int_0^y \partial_x f_1(x,\eta)d\eta \quad (\partial_x = \frac{\partial}{\partial x}), \quad \text{and}$$

$$I(x, y) = y \int_0^y \eta\, \partial_x g(x, \eta)d\eta .$$

It is convenient to consider, instead of (1.1), an equivalent system:

$$(2.1) \quad \dot{x} = y, \quad \dot{y} = z - a \int_0^y f_1(x, \eta)d\eta, \quad \dot{z} = -yg(x, y) - bf_3(x)$$

We have the following:

THEOREM 2.1. Let there exist positive constants β, γ, μ, and c such that

(i) $g(x, y) \geq \beta$, and $f_3(x)/x \geq \gamma$, $x \neq 0$.

(ii) $a\beta - bc \geq \mu^2$, and $f_3'(x) < C$

(iii) $y\, \partial_x g(x, y) \leq 0$ and $1 \leq G(x, y) \leq 1 + 2\mu\sqrt{|I(x, y)|}\,/ab\,|f_3|$,

$x \neq 0$, $y \neq 0$,

then every solution $x(t)$ of (1.1) has the property that $x(t) \to 0$, as $t \to \infty$.

Proof. Consider the function

$$V(x, y, z) = \frac{1}{2} z^2 + W(x, y)$$

Differentiating V with respect to t and using the values of $\dot{x}$, $\dot{y}$ and $\dot{z}$ from (2.1), we get

$$\dot{V} = -y^2[ag(x,y)G(x,y) - bf_3'(x)]$$

$$-abf_3(x)[G(x,y)-1]y + I(x,y).$$

Obviously $\dot{V} = 0$ for $y = 0$. However, if $y \neq 0$, then the second part of (iii) implies that $f_1(x,y) \geq 1$. We write $\dot{V}$ as:

$$\dot{V} = -y^2[ag(x,y)G(x,y)-bf_3'(x)] - abf_3(x)[G(x,y)-1]y$$

$$-\frac{1}{4}a^2b^2f_3^2(x)[G(x,y)-1]^2/[a\beta-bc] + I(x,y)$$

$$+\frac{1}{4}a^2b^2f_3^2(x)[G(x,y)-1]^2/[a\beta-bc]$$

$$= -U(x,y) + E(x,y).$$

Since $ag(x,y)G(x,y)-bf_3'(x) > a\beta - bc > 0$, it is easy to check that

$$U(x,y) = y^2[ag(x,y)G(x,y)-bf_3'(x)] + abf_3(x)[G(x,y)-1]y$$

$$+\frac{1}{4}a^2b^2f_3^2(x)[G(x,y)-1]^2/[a\beta-bc] > 0, \quad y \neq 0.$$

If we could show that

$$E(x,y) = I(x,y) + \frac{1}{4}a^2b^2f_3^2(x)[G(x,y)-1]^2/[a\beta-bc] \leq 0, \quad y \neq 0$$

then $\dot{V} < 0$ for $y \neq 0$.

Now $E(x,y) \leq I(x,y) + \frac{1}{4\mu^2}a^2b^2f_3^2(x)[G(x,y)-1]^2$, $y \neq 0$ and therefore $E(x,y) \leq 0$ provided that

$$\frac{1}{4}\mu^2a^2b^2f_3^2(x)[G(x,y)-1]^2 \leq -I(x,y), \quad y \neq 0$$

i.e. if $G(x,y) \leq 1 + 2\mu\sqrt{|I(x,y)|}/ab|f_3|$, which is true by the second part of (iii).

Our next step is to show that V is positive-definite. For this it suffices to show that $W(x, y) > 0$ and $W(0, 0) = 0$.

$$2W(x, y) = 2ab\int_0^x f_3(\xi)d\xi + 2byf_3(x) + 2\int_0^y \eta g(x, \eta)d\eta$$

$$\geq 2ab\int_0^x f_3(\xi)d\xi + 2byf_3(x) + \beta y^2$$

$$= 2ab\int_0^x f_3(\xi)d\xi + \frac{1}{\beta}\{\beta y + bf_3(x)\}^2 - \frac{b^2}{\beta}f_3^2(x)$$

$$= 2\frac{b}{\beta}\int_0^x \{a\beta - bf_3'(\xi)\}f_3(\xi)\, d\xi + \frac{1}{\beta}\{\beta y + bf_3(x)\}^2$$

$$> \frac{1}{\beta}\{\beta y + bf_3(x)\}^2 + \frac{b\gamma}{\beta}(a\beta - bc)x^2$$

$$> 0 \text{ for } x \neq 0, \quad y \neq 0 .$$

Observe that $W(x, y) \to \infty$ as $|x| + |y| \to \infty$, which implies that $V \to \infty$ as $|x| + |y| + |z| \to \infty$. The remainder of the proof follows the method described by Ezeilo [2].

REMARK 1. We observe that our hypotheses reduce to the Routh-Hurwicz criteria for $\dddot{x} + a\ddot{x} + \dot{x} + bx = 0$, since (iii) is trivially satisfied.

REMARK 2.

(a) If $f_1(x, y)$ and $f_2(x, y)$ are both functions of x alone then hypothesis iii of Theorem 2.1 implies that $f_1 = 1$ and f_2 is constant. This case then reduces to one considered in ([1], [2], [3] and [4]).

(b) If f_2 is a function of y alone or of x alone then this reduces to the problem of Ezeilo.

3. The differential equation.

$$\dddot{x} + f_1(x, \dot{x})\ddot{x} + f_2(x, \dot{x})\dot{x} + bx = 0 \tag{3.1}$$

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is a special case of (1.1). However we will construct a different Lyapunov function for (3.1) in order to establish a claim made in Section 1. For (3.1) we have the following:

THEOREM 3.1. If $f_1(x, y)$ and $f_2(x, y)$ are continuously differentiable for all values of x, and

(i) $f_1(x, y) \geq b > 0$, $f_2(x, y) \geq 1$ for all x and y, (with strict inequality in at least one of the above conditions).

(ii) $y\, \partial_x(f_1(x, y) + \frac{1}{b} f_2(x, y)) \leq 0$, for all x and y, then every solution $x(t)$ of (3.1) satisfies

$$x(t) \to 0, \quad \dot{x}(t) \to 0, \quad \ddot{x}(t) \to 0 \quad \text{as} \quad t \to \infty \tag{3.2}$$

Proof. Reduce (3.1) to the equivalent system

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -f_1(x, y)z - f_2(x, y)y - bx. \tag{3.3}$$

Our result follows if we consider the function

$$2V(x, y, z) = (z+by)^2 + (bx+y)^2 + 2\int_0^y \eta(bf_1(x,\eta) + f_2(x, \eta))d\eta - (b^2 + 1)y^2. \tag{3.4}$$

This result is a generalisation of a result of Simanov [4].

If $p(t) \neq 0$ the result (3.2) does not, in general, hold for the solutions of (1 2), but we shall show that

THEOREM 3.2. If along a solution curve $x = x(t)$ of (1.2) we have

(i) $f_1(x, y) \geq b + |p(t)|$

(ii) $f_2(x, y) \geq 1 + |p(t)|$

(iii) $y\, \partial_x(f_1(x, y) + \frac{1}{b} f_2(x, y)) \leq 0$, for all x, y

and if further

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$$\text{(iv)} \quad \int_0^t |p(\tau)| d\tau \leq A < \infty$$

then given any finite x_0, y_0, z_0 there is a finite constant $B(x_0, y_0, z_0)$ such that the (unique) solution $x(t)$ of (1.2), which is determined by the initial conditions

$$(3.5) \qquad x(0) = x_0, \quad \dot{x}(0) = y_0, \quad \ddot{x}(0) = z_0$$

satisfies

$$(3.6) \qquad |x(t)| \leq B, \quad |\dot{x}(t)| \leq B, \quad |\ddot{x}(t)| \leq B$$

for all $t \geq 0$.

Proof. Our treatment of this theorem is again indirect. We consider the equivalent system

$$(3.7) \quad \dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = -f_1(x, y)z - f_2(x, y)y - bx + p(t)$$

and the function (3.4). Let $(x(t), y(t), z(t))$ be a solution of (3.7) satisfying the initial conditions (3.5). Since $V \to \infty$ as $x^2 + y^2 + z^2 \to \infty$, in order to prove (3.6) it suffices to show that there is a constant $C > 0$, depending only on x_0, y_0 and z_0 such that

$$(3.8) \qquad V(x(t), y(t), z(t)) \leq C, \quad t \geq 0.$$

By virtue of (3.7) we have

$$2\dot{V} = -2z^2(f_1(x, y) - b) - 2y^2(f_2(x, y) - 1)b$$

$$+ 2y \int_0^y \eta \, \partial_x(bf_1(x, \eta) + f_2(x, \eta))d\eta$$

$$+ 2p(t)z + 2bp(t)y$$

or

$$2\dot{V} \le -2z^2(f_1(x,y)-b) - 2by^2(f_2(x,y)-1) + 2|p(t)||z| + 2b|p(t)||y|$$

$$\le -2z^2(f_1(x,y)-b) - 2by^2(f_2(x,y)-1) + 2|p(t)|(1+z^2) + 2b|p(t)|(1+y^2)$$

$$= -2z^2(f_1(x,y)-b-|p(t)|) - 2by^2(f_2(x,y)-1-|p(t)|) + 2|p(t)|(1+b).$$

Thus we have

$$\dot{V} \le (1+b)|p(t)| \quad \text{for all} \quad t \ge 0$$

or

$$V(t) \le V(0) + (1+b)\int_0^t |p(\tau)|\,d\tau$$

$$\le V(0) + (1+b)A = C.$$

This proves (3.8) and Theorem 3.2 follows.

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