STABLE CANONICAL RULES

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Abstract. We introduce stable canonical rules and prove that each normal modal multi-conclusion consequence relation is axiomatizable by stable canonical rules. We apply these results to construct finite refutation patterns for modal formulas, and prove that each normal modal logic is axiomatizable by stable canonical rules. We also define stable multi-conclusion consequence relations and stable logics and prove that these systems have the finite model property. We conclude the paper with a number of examples of stable and nonstable systems, and show how to axiomatize them.

§**1. Introduction.** It is a well-known result of Zakharyaschev [38] that each normal extension of **K4** is axiomatizable by canonical formulas. This result was generalized in two directions. In [2] it was generalized to all normal extensions of **wK4**, and in [21] Zakharyaschev's canonical formulas were generalized to multi-conclusion canonical rules and it was proved that each normal modal multi-conclusion consequence relation over **K4** is axiomatizable by canonical rules.

The key ingredients of Zakharyaschev's technique include the concepts of subreduction, closed domain condition, and selective filtration. While selective filtration is very effective in the transitive case [15], and also generalizes to the weakly transitive case [2, 6], it is less effective for **K**. This is one of the reasons why canonical formulas and rules do not work well for **K** [15, 21]. In [3] a different approach to canonical formulas for intuitionistic logic was developed that uses the technique of filtration instead of selective filtration. The new canonical formulas were called stable canonical formulas, and it was shown that each superintuitionistic logic is axiomatizable by stable canonical formulas.

In this paper we generalize the technique of [3] to the modal setting. Since the technique of filtration works well for **K**, we show that this new technique is effective in the nontransitive case as well. We give an algebraic account of filtration, introduce stable canonical rules, and prove that each normal modal multi-conclusion consequence relation is axiomatizable by stable canonical rules. This allows us to construct finite refutation patterns for modal formulas, and to show that each normal modal logic is axiomatizable by stable canonical rules. For normal extensions of **K4** we prove that stable canonical rules can be replaced by stable canonical formulas, thus providing an alternative to Zakharyaschev's axiomatization [38].

This approach also yields a new class of multi-conclusion consequence relations and logics. Following [3], we call these systems stable. We show that every stable multi-conclusion consequence relation and every stable logic has the finite model property. We also give a number of examples of stable and nonstable logics, and show how to axiomatize them. For more in-depth development of the theory of stable modal systems see [5]. The theory of stable superintuitionistic logics and stable intuitionistic multi-conclusion consequence relations is developed in [3,4].

Stable canonical rules have several applications. For example, they are utilized in [10] to obtain an alternative proof of the existence of explicit bases of admissible rules for the intuitionistic logic, **K4**, and **S4**. In [11] it is shown that stable logics have the bounded proof property, thus yielding a large class of logics with nice proof-theoretic properties.

The paper is organized as follows. In Section 2 we recall basic facts about modal logics, modal multi-conclusion consequence relations, modal algebras, and modal spaces (descriptive Kripke frames). In Section 3 we introduce stable homomorphisms, their dual stable maps, and the closed domain condition (CDC) for stable maps. Section 4 provides an algebraic approach to filtrations and connects them with CDC. The main results of the paper are proved in Section 5. We give finite refutation patterns for normal modal multi-conclusion consequence relations and normal modal logics. We introduce stable canonical rules and prove that every normal modal multi-conclusion consequence relation and every normal modal logic is axiomatizable by stable canonical rules. In Section 6 we provide an algebraic approach to transitive filtrations, introduce stable canonical formulas for **K4**, and prove that every normal extension of **K4** is axiomatizable by stable canonical formulas. This provides an alternative to Zakharyaschev's axiomatization. In Section 7 we introduce stable multi-conclusion consequence relations and stable logics, and prove that all stable systems have the finite model property. We also give a characterization of splitting multi-conclusion consequence relations and splitting logics by means of Jankov rules and Jankov formulas, thus yielding alternative proofs of the results of Jeřábek [21] and Blok [13], respectively. Finally, in Section 8 we show how to axiomatize some well-known modal multi-conclusion consequence relations and modal logics via stable canonical rules and formulas.

§**2. Preliminaries.** In this section we briefly discuss some of the basic facts that will be used throughout the paper. We use [12, 15, 22, 36] as our main references for the basic theory of normal modal logics, including their algebraic and relational semantics, and the dual equivalence between modal algebras and modal spaces (descriptive Kripke frames). We also use [14] for universal algebra, [23, 31] for modal rules, and [20, 21] for multi-conclusion modal rules.

2.1. Modal logic. We recall that a *normal modal logic* is the set of formulas of the basic modal language containing classical tautologies and $\Box(p \to q) \to (\Box p \to \Box q)$, and closed under Modus Ponens ($\varphi, \varphi \to \psi/\psi$), Necessitation ($\varphi/\Box\varphi$), and Substitution ($\varphi(p_1, \ldots, p_n)/\varphi(\psi_1, \ldots, \psi_n)$). The least normal modal logic is denoted by **K**, and NExt**K** denotes the complete lattice of normal extensions of **K**.

A *modal algebra* is a pair $\mathfrak{A} = (A, \Diamond)$, where A is a Boolean algebra and $\Diamond$ is a unary function on A that commutes with finite joins. As usual, the dual operator $\Box$ is defined as $\neg\Diamond\neg$. A *modal homomorphism* between two modal algebras is a

Boolean homomorphism h satisfying $h(\Diamond a) = \Diamond h(a)$. Let MA be the category of modal algebras and modal homomorphisms.

A *modal space* (or *descriptive Kripke frame*) is a pair $\mathfrak{X} = (X, R)$, where X is a Stone space (zero-dimensional compact Hausdorff space) and R is a binary relation on X satisfying

$$R[x] := \{y \in X : x \mathrel{R} y\}$$

is closed for each $x \in X$ and

$$R^{-1}[U] := \{x \in X : \exists y \in U \text{ with } x \mathrel{R} y\}$$

is clopen (closed and open) for each clopen U of X. A *bounded morphism* (or *p-morphism*) between two modal spaces is a continuous map f such that $f(R[x]) = R[f(x)]$. Let MS be the category of modal spaces and bounded morphisms.

It is a well-known theorem in modal logic that MA is dually equivalent to MS. The functors $(-)_* : \mathsf{MA} \to \mathsf{MS}$ and $(-)^* : \mathsf{MS} \to \mathsf{MA}$ that establish this dual equivalence are constructed as follows. For a modal algebra $\mathfrak{A} = (A, \Diamond)$, let $\mathfrak{A}_* = (A_*, R)$, where A_* is the Stone space of A (that is, the set of ultrafilters of A topologized by the basis $\{\beta(a) : a \in A\}$, where $\beta(a) = \{x \in A_* : a \in x\}$ is the Stone map) and xRy iff $(\forall a \in A)(a \in y \Rightarrow \Diamond a \in x)$. We call R the *dual* of $\Diamond$. For a modal homomorphism $h : A \to B$, its dual $h_* : B_* \to A_*$ is given by h^{-1}. For a modal space $\mathfrak{X} = (X, R)$, let $\mathfrak{X}^* = (X^*, \Diamond)$, where X^* is the Boolean algebra of clopens of X and $\Diamond(U) = R^{-1}[U]$. For a bounded morphism $f : X \to Y$, its dual $f^* : Y^* \to X^*$ is given by f^{-1}.

Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra and let $\mathfrak{X} = (X, R)$ be its dual space. Then it is well-known that R is reflexive iff $a \leq \Diamond a$, and R is transitive iff $\Diamond\Diamond a \leq \Diamond a$. A modal algebra $\mathfrak{A}$ is a **K4**-*algebra* if $\Diamond\Diamond a \leq \Diamond a$ holds in $\mathfrak{A}$, and it is an **S4**-*algebra* if in addition $a \leq \Diamond a$ holds in $\mathfrak{A}$. **S4**-algebras are also known as closure algebras, interior algebras, or topological Boolean algebras. Let K4 be the full subcategory of MA consisting of **K4**-algebras, and let S4 be the full subcategory of K4 consisting of **S4**-algebras. A modal space $\mathfrak{X} = (X, R)$ is a *transitive space* if R is transitive, and it is a *quasi-ordered space* if R is reflexive and transitive. Let TS be the full subcategory of MS consisting of transitive spaces, and let QS be the full subcategory of TS consisting of quasi-ordered spaces. Then the dual equivalence of MA and MS restricts to the dual equivalence of K4 and TS, which restricts further to the dual equivalence of S4 and QS.

For a modal algebra $\mathfrak{A} = (A, \Diamond)$ and $a \in A$, define $\Diamond^0 a = a$ and $\Diamond^{n+1} a = \Diamond\Diamond^n a$, and set $\blacklozenge_n a = \bigvee_{k \leq n} \Diamond^k a$. We define $\square^n$ and $\blacksquare_n$ similarly. By Rautenberg's criterion [30], $\mathfrak{A}$ is subdirectly irreducible (s.i. for short) iff there is $c \neq 1$ such that for each $a \neq 1$ there is $n \in \omega$ with $\blacksquare_n a \leq c$. Such a c is called an *opremum* of $\mathfrak{A}$. It is not unique.

If $\mathfrak{A}$ is a **K4**-algebra, then for each $n \geq 1$, we have $\blacklozenge_n a = a \vee \Diamond a$ and $\blacksquare_n a = a \wedge \square a$. Let $\Diamond^+ a := a \vee \Diamond a$ and $\square^+ a := a \wedge \square a$. Then $\mathfrak{A}^+ = (A, \Diamond^+)$ is an **S4**-algebra, the set H of fixed points of $\square^+$ forms a Heyting algebra, and up to isomorphism, each Heyting algebra arises this way. Therefore, a **K4**-algebra $\mathfrak{A}$ is s.i. iff the **S4**-algebra $\mathfrak{A}^+$ is s.i., which happens iff the Heyting algebra H is s.i., which in turn means that there is a greatest element c in $H - \{1\}$. If $\mathfrak{X} = (X, R)$ is the dual of $\mathfrak{A}$, then the dual of $\mathfrak{A}^+$ is $\mathfrak{X}^+ = (X, R^+)$, where $R^+ = R \cup \{(x, x) : x \in X\}$ is the reflexive closure of R.

A filter F of a modal algebra $\mathfrak{A} = (A, \Diamond)$ is called a *modal filter* if $a \in F$ implies $\Box a \in F$. It is well-known that there is a 1–1 correspondence between congruences and modal filters of $\mathfrak{A}$, hence homomorphic images of $\mathfrak{A}$ are determined by modal filters. For a modal space $\mathfrak{X} = (X, R)$, a subset Y of X is called an *up-set* if from $x \in Y$ and xRy it follows that $y \in Y$. If $\mathfrak{A}$ is a modal algebra and $\mathfrak{X}$ is its dual space, then modal filters of $\mathfrak{A}$ correspond to closed up-sets of $\mathfrak{X}$. Thus, homomorphic images of $\mathfrak{A}$ are determined by closed up-sets of $\mathfrak{X}$.

For a modal space $\mathfrak{X} = (X, R)$ and $Y \subseteq X$, let $R^0[Y] = Y$, $R^{n+1}[Y] = R[R^n[Y]]$, and $R^\omega[Y] = \bigcup_{n \in \omega} R^n[Y]$. If Y is a singleton $\{x\}$, then we write $R^n[x]$ and $R^\omega[x]$. We call $\mathfrak{X}$ *rooted* provided there is $x \in X$, called a *root* of $\mathfrak{X}$, such that $X = R^\omega[x]$. Note that if R is transitive, then $R^\omega[x] = \{x\} \cup R[x]$ and if R is reflexive and transitive, then $R^\omega[x] = R[x]$. By [32, Thm. 3.1], a finite modal algebra $\mathfrak{A}$ is s.i. iff its dual modal space $\mathfrak{X}$ is rooted. This result extends to the infinite case as follows [35]. Let $\mathfrak{X}$ be a modal space. Call $x \in X$ a *topo-root* of $\mathfrak{X}$ if X is the closure of $R^\omega[x]$. Then a modal algebra $\mathfrak{A}$ is s.i. iff in its dual modal space $\mathfrak{X}$ the set of topo-roots has a nonempty interior [35, Thm. 2]. We call such modal spaces *topo-rooted*.

2.2. Multi-conclusion modal rules. A *multi-conclusion modal rule* is an expression Γ/Δ, where Γ, Δ are finite sets of modal formulas. If $\Delta = \{\varphi\}$, then Γ/Δ is called a *single-conclusion modal rule* and is written Γ/φ. If $\Gamma = \varnothing$, then Γ/Δ is called an *assumption-free modal rule* and is written $/\Delta$. Assumption-free single-conclusion modal rules $/\varphi$ can be identified with modal formulas φ.

A *normal modal multi-conclusion consequence relation* is a set $\mathcal{S}$ of modal rules such that

(1) $\varphi/\varphi \in \mathcal{S}$.
(2) $\varphi, \varphi \to \psi/\psi \in \mathcal{S}$.
(3) $\varphi/\Box\varphi \in \mathcal{S}$.
(4) $/\varphi \in \mathcal{S}$ for each theorem φ of **K**.
(5) If $\Gamma/\Delta \in \mathcal{S}$, then $\Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S}$.
(6) If $\Gamma/\Delta, \varphi \in \mathcal{S}$ and $\Gamma, \varphi/\Delta \in \mathcal{S}$, then $\Gamma/\Delta \in \mathcal{S}$.
(7) If $\Gamma/\Delta \in \mathcal{S}$ and s is a substitution, then $s(\Gamma)/s(\Delta) \in \mathcal{S}$.

Normal modal multi-conclusion consequence relations are also called normal modal rule systems. We denote the least normal modal multi-conclusion consequence relation by $\mathbf{S_K}$, and the complete lattice of normal modal multi-conclusion consequence relations extending $\mathbf{S_K}$ by $\mathrm{NExt}\mathbf{S_K}$. For a set Ξ of multi-conclusion modal rules, let $\mathbf{S_K} + \Xi$ be the least normal modal multi-conclusion consequence relation containing Ξ. If $\mathcal{S} = \mathbf{S_K} + \Xi$, then we say that $\mathcal{S}$ is *axiomatizable* by Ξ, and if Ξ is finite, then we call $\mathcal{S}$ *finitely axiomatizable*. If $\rho \in \mathcal{S}$, then we say that the normal modal multi-conclusion consequence relation $\mathcal{S}$ *entails* or *derives* the modal rule ρ, and write $\mathcal{S} \vdash \rho$.

Given a normal modal multi-conclusion consequence relation $\mathcal{S}$, let $\Lambda(\mathcal{S}) = \{\varphi : /\varphi \in \mathcal{S}\}$ be the corresponding normal modal logic, and for a normal modal logic L, let $\Sigma(L) = \mathbf{S_K} + \{/\varphi : \varphi \in L\}$ be the corresponding normal modal multi-conclusion consequence relation. Then $\Lambda : \mathrm{NExt}\mathbf{S_K} \to \mathrm{NExt}\mathbf{K}$ and $\Sigma : \mathrm{NExt}\mathbf{K} \to \mathrm{NExt}\mathbf{S_K}$ are order-preserving maps such that $\Lambda(\Sigma(L)) = L$ for each $L \in \mathrm{NExt}\mathbf{K}$ and $\mathcal{S} \supseteq \Sigma(\Lambda(\mathcal{S}))$ for each $\mathcal{S} \in \mathrm{NExt}\mathbf{S_K}$. We say that a normal modal logic L is *axiomatized* (over **K**) by a set Ξ of multi-conclusion modal rules if $L = \Lambda(\mathbf{S_K} + \Xi)$.

A *valuation* in a modal algebra $\mathfrak{A} = (A, \Diamond)$ is a map V from the propositional variables to A, which naturally extends to all formulas. The modal algebra $\mathfrak{A}$ *validates* a multi-conclusion modal rule Γ/Δ provided for every valuation V on A, if $V(\gamma) = 1$ for all $\gamma \in \Gamma$, then $V(\delta) = 1$ for some $\delta \in \Delta$. Otherwise $\mathfrak{A}$ *refutes* Γ/Δ. If $\mathfrak{A}$ validates Γ/Δ, we write $\mathfrak{A} \models \Gamma/\Delta$, and if $\mathfrak{A}$ refutes Γ/Δ, we write $\mathfrak{A} \not\models \Gamma/\Delta$. If $\Gamma = \{\phi_1, \ldots, \phi_n\}$, $\Delta = \{\psi_1, \ldots, \psi_m\}$, and $\phi_i(\underline{x})$ and $\psi_j(\underline{x})$ are the terms in the first-order language of modal algebras corresponding to the ϕ_i and ψ_j, then $\mathfrak{A} \models \Gamma/\Delta$ iff $\mathfrak{A}$ is a model of the universal sentence $\forall \underline{x} \, (\bigwedge_{i=1}^{n} \phi_i(\underline{x}) = 1 \rightarrow \bigvee_{j=1}^{m} \psi_j(\underline{x}) = 1)$. Consequently, normal modal multi-conclusion consequence relations correspond to universal classes of modal algebras. It is well-known (see, e.g., [14, Thm. V.2.20]) that a class of modal algebras is a universal class iff it is closed under isomorphisms, subalgebras, and ultraproducts.

On the other hand, normal modal logics correspond to equationally definable classes of modal algebras; that is, models of the sentences $\forall \underline{x} \, \phi(\underline{x}) = 1$ in the first-order language of modal algebras. It is well-known (see, e.g., [14, Thm. II.11.9]) that a class of modal algebras is an equational class iff it is a variety (that is, it is closed under homomorphic images, subalgebras, and products).

We also point out that a modal algebra $\mathfrak{A}$ validates a single-conclusion modal rule Γ/ψ iff $\mathfrak{A}$ is a model of the sentence $\forall \underline{x} \, (\bigwedge_{i=1}^{n} \phi_i(\underline{x}) = 1 \rightarrow \psi(\underline{x}) = 1)$, where $\Gamma = \{\phi_1, \ldots, \phi_n\}$ and $\phi_i(\underline{x})$ and $\psi(\underline{x})$ are the terms in the first-order language of modal algebras corresponding to the ϕ_i and ψ. Consequently, normal modal consequence relations correspond to classes of modal algebras axiomatized by quasi-identities. It is well-known (see, e.g., [14, Thm. V.2.25]) that a class of modal algebras is axiomatized by quasi-identities iff it is a quasivariety (that is, it is closed under isomorphisms, subalgebras, products, and ultraproducts).

For a normal modal multi-conclusion consequence relation $\mathcal{S}$, we denote by $\mathcal{U}(\mathcal{S})$ the universal class of modal algebras corresponding to $\mathcal{S}$, and for a universal class of modal algebras $\mathcal{U}$, we denote by $\mathcal{S}(\mathcal{U})$ the normal modal multi-conclusion consequence relation corresponding to $\mathcal{U}$. Then $\mathcal{S}(\mathcal{U}(\mathcal{S})) = \mathcal{S}$ and $\mathcal{U}(\mathcal{S}(\mathcal{U})) = \mathcal{U}$. This yields an isomorphism between NExt$\mathbf{S_K}$ and the complete lattice U(MA) of universal classes of modal algebras (ordered by reverse inclusion).

Similarly, for a normal modal logic L, let $\mathcal{V}(L)$ denote the variety of modal algebras corresponding to L, and for a variety $\mathcal{V}$, let $L(\mathcal{V})$ denote the normal modal logic corresponding to $\mathcal{V}$. Then $L(\mathcal{V}(L)) = L$ and $\mathcal{V}(L(\mathcal{V})) = \mathcal{V}$, yielding an isomorphism between NExt$\mathbf{K}$ and the complete lattice V(MA) of varieties of modal algebras (ordered by reverse inclusion).

Under this correspondence, for a normal modal multi-conclusion consequence relation $\mathcal{S}$, the variety $\mathcal{V}(\Lambda(\mathcal{S}))$ corresponding to the modal logic $\Lambda(\mathcal{S})$ is the variety generated by the universal class $\mathcal{U}(\mathcal{S})$. We will utilize this fact later on in the paper.

§3. **Stable homomorphisms and the closed domain condition.** In this section we introduce the key concepts of stable homomorphisms and the closed domain condition, and show how the two relate to each other.

DEFINITION 3.1. Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras and let $h : A \rightarrow B$ be a Boolean homomorphism. We call h a *stable homomorphism* provided $\Diamond h(a) \leq h(\Diamond a)$ for each $a \in A$.

It is easy to see that $h : A \to B$ is stable iff $h(\Box a) \leq \Box h(a)$ for each $a \in A$. Stable homomorphisms were considered in [7] under the name of semi-homomorphisms and in [18] under the name of continuous morphisms.

DEFINITION 3.2. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be modal spaces and let $f : X \to Y$ be a continuous map. We call f *stable* if xRy implies $f(x) \, R \, f(y)$.

LEMMA 3.3. *Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras, $\mathfrak{X} = (X, R)$ be the dual of $\mathfrak{A}$, $\mathfrak{Y} = (Y, R)$ be the dual of $\mathfrak{B}$, and $h : A \to B$ be a Boolean homomorphism. Then $h : A \to B$ is stable iff $h_* : Y \to X$ is stable.*

PROOF. By Stone duality, it is sufficient to show that $\Diamond h(a) \leq h(\Diamond a)$ for each $a \in A$ iff $x \, R \, y$ implies $h_*(x) \, R \, h_*(y)$ for each $x, y \in Y$. First suppose that $\Diamond h(a) \leq h(\Diamond a)$ for each $a \in A$. Let $x, y \in Y$ with $x \, R \, y$, and let $a \in h_*(y)$. Then $h(a) \in y$. From $x \, R \, y$ it follows that $\Diamond h(a) \in x$. Since $\Diamond h(a) \leq h(\Diamond a)$ and x is a filter, $h(\Diamond a) \in x$, hence $\Diamond a \in h_*(x)$. Therefore, $h_*(x) \, R \, h_*(y)$.

Conversely, suppose xRy implies $h_*(x) \, R \, h_*(y)$ for each $x, y \in Y$. Let $a \in A$ and let $x \in R^{-1}h_*^{-1}\beta(a)$. Then there is $y \in Y$ such that $x \, R \, y$ and $h_*(y) \in \beta(a)$. Therefore, $h_*(x) \, R \, h_*(y)$ and $h_*(y) \in \beta(a)$. Thus, $h_*(x) \in R^{-1}\beta(a)$, and so $x \in h_*^{-1}R^{-1}\beta(a)$. This implies $R^{-1}h_*^{-1}\beta(a) \subseteq h_*^{-1}R^{-1}\beta(a)$. But $R^{-1}h_*^{-1}\beta(a) = \beta(\Diamond h(a))$ and $h_*^{-1}R^{-1}\beta(a) = \beta(h(\Diamond a))$. This yields $\beta(\Diamond h(a)) \subseteq \beta(h(\Diamond a))$, and since β is an isomorphism, we conclude that $\Diamond h(a) \leq h(\Diamond a)$ for each $a \in A$. $\dashv$

DEFINITION 3.4. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be modal spaces, $f : X \to Y$ be a map, and U be a clopen subset of Y. We say that f satisfies the *closed domain condition* (abbreviated as CDC) *for U* if

$$R[f(x)] \cap U \neq \varnothing \Rightarrow f(R[x]) \cap U \neq \varnothing.$$

Let $\mathfrak{D}$ be a collection of clopen subsets of Y. We say that $f : X \to Y$ satisfies the CDC *for $\mathfrak{D}$* if f satisfies CDC for each $U \in \mathfrak{D}$.

LEMMA 3.5. *Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$ be modal spaces, $f : X \to Y$ be a map, and U be a clopen subset of Y. Then the following two conditions are equivalent:*

(1) *f satisfies CDC for U.*
(2) *$f^{-1}R^{-1}U \subseteq R^{-1}f^{-1}U$.*

PROOF. (1)$\Rightarrow$(2): Suppose that f satisfies CDC for U and $x \in f^{-1}R^{-1}U$. Then $R[f(x)] \cap U \neq \varnothing$. By CDC, $f(R[x]) \cap U \neq \varnothing$. Thus, $x \in R^{-1}f^{-1}U$.

(2)$\Rightarrow$(1): Suppose that $f^{-1}R^{-1}U \subseteq R^{-1}f^{-1}U$ and $R[f(x)] \cap U \neq \varnothing$. Then $x \in f^{-1}R^{-1}U$. By (2), $x \in R^{-1}f^{-1}U$, which means that $f(R[x]) \cap U \neq \varnothing$. Thus, CDC is satisfied. $\dashv$

THEOREM 3.6. *Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras, $h : A \to B$ be a stable homomorphism, and $a \in A$. The following two conditions are equivalent:*

(1) *$h(\Diamond a) = \Diamond h(a)$.*
(2) *$h_* : B_* \to A_*$ satisfies CDC for $\beta(a)$.*

PROOF. Since $h : A \to B$ is a stable homomorphism, $\Diamond h(a) \leq h(\Diamond a)$. Therefore, $h(\Diamond a) = \Diamond h(a)$ iff $h(\Diamond a) \leq \Diamond h(a)$, which happens iff $h_*^{-1}R^{-1}\beta(a) \subseteq R^{-1}h_*^{-1}\beta(a)$. By Lemma 3.5, the last condition is equivalent to $h_* : B_* \to A_*$ satisfying CDC for $\beta(a)$. $\dashv$

Theorem 3.6 motivates the following definition.

DEFINITION 3.7. Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras and let $h : A \to B$ be a stable homomorphism.

(1) We say that h satisfies CDC *for* $a \in A$ if $h(\Diamond a) = \Diamond h(a)$.
(2) We say that h satisfies CDC *for* $D \subseteq A$ if h satisfies CDC for each $a \in D$.

§4. **Filtrations and the closed domain condition.** The filtration method is the main tool for establishing the finite model property in modal logic. The method can be developed either algebraically [28, 29] or frame-theoretically [26, 33], and the two are connected via duality [24, 25]. For a recent account of filtrations we refer to [16, 18]. In this section we give a slightly different account which is more suited for our purposes, and also discuss the connection with stable homomorphisms and the closed domain condition.

We start by recalling the frame-theoretic approach to filtrations (see, e.g., [12, Def. 2.36] or [15, Sec. 5.3]). Let $\mathfrak{M} = (X, R, V)$ be a Kripke model and let Θ be a set of formulas closed under subformulas. For our purposes, Θ will always be assumed to be finite. Define an equivalence relation $\sim_\Theta$ on X by

$$x \sim_\Theta y \text{ iff } (\forall \varphi \in \Theta)(x \models \varphi \Leftrightarrow y \models \varphi).$$

Let $X' = X/{\sim_\Theta}$ and let $V'(p) = \{[x] : x \in V(p)\}$, where $[x]$ is the equivalence class of x with respect to $\sim_\Theta$.

DEFINITION 4.1. For a binary relation R' on X', we say that the triple $\mathfrak{M}' = (X', R', V')$ is a *filtration of* $\mathfrak{M}$ *through* Θ if the following two conditions are satisfied:

(F1) $x R y \Rightarrow [x] R' [y]$.
(F2) $[x] R' [y] \Rightarrow (\forall \Diamond \varphi \in \Theta)(y \models \varphi \Rightarrow x \models \Diamond \varphi)$.

Note that if Θ is finite, then X' is finite. In fact, if Θ consists of n elements, then X' consists of no more than 2^n elements.

Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra and let $\mathfrak{X} = (X, R)$ be the dual of $\mathfrak{A}$. If V is a valuation on A, then by identifying A with the clopen subsets of X, we can view V as a valuation on X.

THEOREM 4.2. *Let* $\mathfrak{A} = (A, \Diamond)$ *be a modal algebra and let* $\mathfrak{X} = (X, R)$ *be the dual of* $\mathfrak{A}$. *For a valuation* V *on* A *and a set of formulas* Θ *closed under subformulas, let* A' *be the Boolean subalgebra of* A *generated by* $V(\Theta) \subseteq A$ *and let* $D = \{V(\varphi) : \Diamond \varphi \in \Theta\}$. *For a modal operator* $\Diamond'$ *on* A', *the following two conditions are equivalent*:

(1) *The inclusion* $(A', \Diamond') \rightarrowtail (A, \Diamond)$ *is a stable homomorphism satisfying* CDC *for* D.
(2) *Viewing* V *as a valuation on* X, *there is a filtration* $\mathfrak{M}' = (X', R', V')$ *of* $\mathfrak{M} = (X, R, V)$ *through* Θ *such that* R' *is the dual of* $\Diamond'$.

PROOF. Since A' is a Boolean subalgebra of A, it follows from Stone duality that the dual of A' can be described as the quotient of X by the equivalence relation given by $x \sim y$ iff $x \cap A' = y \cap A'$. As A' is generated by $V(\Theta)$, we have $x \sim y$ iff $x \sim_\Theta y$, so we identify the dual of A' with X'. Define V' on X' by $V'(p) = \{[x] : x \in V(p)\}$. Let $\Diamond'$ be a modal operator on A', and let $R' \subseteq X' \times X'$ be the dual of $\Diamond'$. By Lemma 3.3, $\mathfrak{M}' = (X', R', V')$ satisfies (F1) iff the inclusion

$(A', \lozenge') \rightarrowtail (A, \lozenge)$ is a stable homomorphism. Therefore, it remains to see that $\mathfrak{M}'$ satisfies (F2) iff the inclusion $(A', \lozenge') \rightarrowtail (A, \lozenge)$ satisfies CDC for D. The former means that $[x] \, R' \, [y] \Rightarrow (\forall a \in D)(a \in y \Rightarrow \lozenge a \in x)$, and the latter means that $\lozenge' a = \lozenge a$ for each $a \in D$. First suppose that the inclusion satisfies CDC for D. Let $[x] \, R' \, [y]$, $a \in D$, and $a \in y$. Since $[x] \, R' \, [y]$, we have that $(\forall b \in A')(b \in y \Rightarrow \lozenge' b \in x)$. As $a \in D \subseteq A'$, from $a \in y$ it follows that $\lozenge' a \in x$. By CDC for D we see that $\lozenge' a = \lozenge a$, so $\lozenge a \in x$, and hence $\mathfrak{M}'$ satisfies (F2). Conversely, suppose that $\mathfrak{M}'$ satisfies (F2). Let $a \in D$. Since the inclusion $(A', \lozenge') \rightarrowtail (A, \lozenge)$ is stable, we have $\lozenge a \leq \lozenge' a$. To see the reverse inequality, let $x \in \beta(\lozenge' a)$, and let β' be the Stone map for A'. Since $a \in D \subseteq A'$, from $x \in \beta(\lozenge' a)$ it follows that $[x] \in \beta'(\lozenge' a)$. Therefore, $[x] \in (R')^{-1} \beta'(a)$, so there is $y \in X$ with $[x] \, R' \, [y]$ and $[y] \in \beta'(a)$. As $a \in D \subseteq A'$, from $[y] \in \beta'(a)$ it follows that $a \in y$. By (F2), this yields $\lozenge a \in x$. So $x \in \beta(\lozenge a)$. Thus, $\beta(\lozenge' a) \subseteq \beta(\lozenge a)$, yielding $\lozenge' a \leq \lozenge a$. Consequently, $\lozenge' a = \lozenge a$ for each $a \in D$, and hence the embedding satisfies CDC for D. $\dashv$

Theorem 4.2 motivates the following definition.

DEFINITION 4.3. Let $\mathfrak{A} = (A, \lozenge)$ be a modal algebra, V be a valuation on A, and Θ be a set of formulas closed under subformulas. Let A' be the Boolean subalgebra of A generated by $V(\Theta) \subseteq A$ and let $D = \{V(\varphi) : \lozenge \varphi \in \Theta\}$. Suppose that $\lozenge'$ is a modal operator on A' such that the inclusion $(A', \lozenge') \rightarrowtail (A, \lozenge)$ is a stable homomorphism satisfying CDC for D. Then we call $\mathfrak{A}' = (A', \lozenge')$ a *filtration* of $\mathfrak{A}$ through Θ.

LEMMA 4.4. *Let $\mathfrak{A}' = (A', \lozenge')$ be a filtration of $\mathfrak{A}$ through Θ and let V' be a valuation on A' that coincides with V on the propositional variables occurring in Θ. If $\varphi \in \Theta$, then $V(\varphi) = V'(\varphi)$.*

PROOF. Easy induction on the complexity of φ. Since A' is a Boolean subalgebra of A, the proof for Boolean connectives is obvious, and since $(A', \lozenge') \rightarrowtail (A, \lozenge)$ is a stable embedding satisfying CDC for D, the proof for $\lozenge$ follows. $\dashv$

Let $D^\vee$ denote the $(\vee, 0)$-subsemilattice of A' generated by D. Then $0 \in D^\vee$, and $a \in D^\vee$ iff $a = \bigvee F$ for some finite subset F of D. Since $\lozenge$ commutes with finite joins, $a \in D^\vee$ implies $\lozenge a \in A'$.

LEMMA 4.5. *Let $\mathfrak{A} = (A, \lozenge)$, V, Θ, A', and D be as above with Θ and hence A' finite. Define $\lozenge^l$ and $\lozenge^g$ on A' by*

$$\lozenge^l a = \bigwedge \{b \in A' : \lozenge a \leq b\} \text{ and } \lozenge^g a = \bigwedge \{\lozenge b : a \leq b \ \& \ b \in D^\vee\}.$$

Then

(1) $\lozenge a \leq \lozenge^l a \leq \lozenge^g a$.
(2) *If $a \in D^\vee$, then $\lozenge a = \lozenge^l a = \lozenge^g a$.*
(3) $(A', \lozenge^l)$ *and* $(A', \lozenge^g)$ *are modal algebras.*
(4) *The inclusions of $(A', \lozenge^l)$ and $(A', \lozenge^g)$ into $\mathfrak{A}$ are stable.*
(5) *The inclusions of $(A', \lozenge^l)$ and $(A', \lozenge^g)$ into $\mathfrak{A}$ satisfy CDC for D.*
(6) $(A', \lozenge^l)$ *and* $(A', \lozenge^g)$ *are filtrations of $\mathfrak{A}$ through Θ.*
(7) *If $\mathfrak{A}' = (A', \lozenge')$ is a filtration of $\mathfrak{A}$ through Θ, then $\lozenge^l a \leq \lozenge' a \leq \lozenge^g a$ for each $a \in A'$.*

PROOF. (1). It follows from the definition that $\lozenge a \leq \lozenge^l a$. As $a \leq b \Rightarrow \lozenge a \leq \lozenge b$, we have $\{\lozenge b : a \leq b \ \& \ b \in D^\vee\} \subseteq \{b \in A' : \lozenge a \leq b\}$, so $\lozenge^l a \leq \lozenge^g a$.

(2). If $a \in D^{\vee}$, then $\Diamond^g a \leq \Diamond a$. This by (1) yields $\Diamond a = \Diamond^l a = \Diamond^g a$.

(3). Since $\Diamond 0 = 0$ and $0 \in A'$, it is clear that $\Diamond^l 0 = 0$. Moreover,

$$
\begin{aligned}
\Diamond^l a \vee \Diamond^l b &= \bigwedge\{x \in A' : \Diamond a \leq x\} \vee \bigwedge\{y \in A' : \Diamond b \leq y\} \\
&= \bigwedge\{x \vee y : x, y \in A' \,\&\, \Diamond a \leq x \,\&\, \Diamond b \leq y\} \\
&= \bigwedge\{z \in A' : \Diamond a \vee \Diamond b \leq z\} \\
&= \bigwedge\{z \in A' : \Diamond(a \vee b) \leq z\} \\
&= \Diamond^l(a \vee b).
\end{aligned}
$$

Therefore, $(A', \Diamond^l)$ is a modal algebra. As $\Diamond 0 = 0$ and $0 \in D^{\vee}$, by (2), $\Diamond^g 0 = 0$. Because $D^{\vee}$ is closed under finite joins,

$$
\begin{aligned}
\Diamond^g a \vee \Diamond^g b &= \bigwedge\{\Diamond x : a \leq x \,\&\, x \in D^{\vee}\} \vee \bigwedge\{\Diamond y : b \leq y \,\&\, y \in D^{\vee}\} \\
&= \bigwedge\{\Diamond x \vee \Diamond y : a \leq x \,\&\, b \leq y \,\&\, x, y \in D^{\vee}\} \\
&= \bigwedge\{\Diamond(x \vee y) : a \leq x \,\&\, b \leq y \,\&\, x, y \in D^{\vee}\} \\
&= \bigwedge\{\Diamond z : a \vee b \leq z \,\&\, z \in D^{\vee}\} \\
&= \Diamond^g(a \vee b).
\end{aligned}
$$

Thus, $(A', \Diamond^g)$ is a modal algebra.

In view of (3), (4) follows from (1), (5) follows from (2), and (6) follows from (4) and (5).

(7). Suppose $\mathfrak{A}' = (A', \Diamond')$ is a filtration of $\mathfrak{A}$ through Θ. Let $a \in A'$. Since the inclusion $\mathfrak{A}' \rightarrowtail \mathfrak{A}$ is a stable homomorphism, we have $\Diamond a \leq \Diamond' a$. Therefore, $\Diamond' a \in \{b \in A' : \Diamond a \leq b\}$, which yields $\Diamond^l a \leq \Diamond' a$. Let $b \in D^{\vee}$ with $a \leq b$. Then $b = \bigvee F$ for some finite $F \subseteq D$. Since $\mathfrak{A}'$ is a modal algebra, $a \leq b$ implies $\Diamond' a \leq \Diamond' b = \Diamond' \bigvee F = \bigvee\{\Diamond' x : x \in F\}$. As the inclusion $\mathfrak{A}' \rightarrowtail \mathfrak{A}$ satisfies CDC for D, from $x \in F \subseteq D$ it follows that $\Diamond' x = \Diamond x$. Thus, $\Diamond' a \leq \bigvee\{\Diamond x : x \in F\} = \Diamond \bigvee F = \Diamond b$, yielding $\Diamond' a \leq \Diamond^g a$. $\dashv$

As a consequence, we obtain that $(A', \Diamond^l)$ is the least filtration and $(A', \Diamond^g)$ is the greatest filtration of $\mathfrak{A}$ through the finite set of formulas Θ. We next show that these correspond to the least and greatest filtrations of the dual of $\mathfrak{A}$. We recall (see, e.g., [12, Sec. 2.3] or [15, Sec. 5.3]) that the least filtration of $\mathfrak{M} = (X, R, V)$ through Θ is $\mathfrak{M}^l = (X', R^l, V')$ and the greatest filtration is $\mathfrak{M}^g = (X, R^g, V')$, where

$[x] \, R^l \, [y]$ iff $(\exists x', y' \in X)(x \sim_{\Theta} x' \,\&\, y \sim_{\Theta} y' \,\&\, x' \, R \, y')$.

$[x] \, R^g \, [y]$ iff $(\forall \Diamond \varphi \in \Theta)(y \models \varphi \Rightarrow x \models \Diamond \varphi)$.

LEMMA 4.6. *Let* $\mathfrak{A} = (A, \Diamond)$, $\mathfrak{X} = (X, R)$, A', *and* X' *be as in Theorem* 4.2, *with* A' *and* X' *finite. Then* R^l *on* X' *is the dual of* $\Diamond^l$ *on* A' *and* R^g *on* X' *is the dual of* $\Diamond^g$ *on* A'.

PROOF. Let $R_{\Diamond^l}$ be the dual of $\Diamond^l$. Then $[x] \, R_{\Diamond^l} \, [y]$ iff $(\forall a \in A')(a \in y \Rightarrow \Diamond^l a \in x)$. On the other hand, $[x] \, R^l \, [y]$ iff $(\exists x', y' \in X)(x \sim_{\Theta} x' \,\&\, y \sim_{\Theta} y' \,\&\, x' \, R \, y')$. First suppose that $[x] \, R^l \, [y]$. Let $a \in A'$ and $a \in y$. Since $a \in A'$ and $y \sim_{\Theta} y'$, from $a \in y$ it follows that $a \in y'$, so $x' \, R \, y'$ implies $\Diamond a \in x'$. By Lemma 4.5(1), $\Diamond a \leq \Diamond^l a$, so $\Diamond^l a \in x'$, and as $\Diamond^l a \in A'$, we conclude that $\Diamond^l a \in x$. Thus, $[x] \, R_{\Diamond^l} \, [y]$. Conversely, suppose that $[x] \, \cancel{R^l} \, [y]$. Let β' be the Stone map for A'. Since A' is finite, there is $a \in A'$ such that $[y] = \beta'(a)$. As $[x] \, \cancel{R^l} \, [y]$, we have $[x] \notin (R^l)^{-1}\beta'(a)$.

CLAIM 4.7. *If* $a \in A'$, *then* $(R^l)^{-1}\beta'(a) = \beta'(\Diamond^l a)$.

PROOF OF CLAIM. First suppose that $[x] \in (R^l)^{-1}\beta'(a)$. Then there is $[y] \in \beta'(a)$ with $[x] \, R^l \, [y]$. Therefore, there are $x' \sim_{\Theta} x$ and $y' \sim_{\Theta} y$ with $x' \, R \, y'$. Since $a \in A'$, from $[y] \in \beta'(a)$ it follows that $a \in y$, so as $y \sim_{\Theta} y'$, we see that $a \in y'$.

Thus, $x'\ R\ y'$ yields $\Diamond a \in x'$. Since $x \sim_\Theta x'$, this gives $\Diamond a \in x$. By Lemma 4.5(1), $\Diamond a \leq \Diamond^l a$, so we conclude that $\Diamond^l a \in x$, giving $[x] \in \beta'(\Diamond^l a)$.

Conversely, suppose that $[x] \in \beta'(\Diamond^l a)$. Then $\Diamond^l a \in x$, so for each $b \in A'$ with $\Diamond a \leq b$, we have $b \in x$. There are two cases, either $\Diamond a \in x$ or $\Diamond a \notin x$.

CASE 1. Suppose $\Diamond a \in x$. Then $x \in \beta(\Diamond a) = R^{-1}\beta(a)$, so there is $y \in \beta(a)$ with xRy. Therefore, $[y] \in \beta'(a)$ and $[x]\ R^l\ [y]$, yielding $[x] \in (R^l)^{-1}\beta'(a)$.

CASE 2. Suppose $\Diamond a \notin x$. If $\Diamond a \in A'$, then $\Diamond a = \Diamond^l a$, so $\Diamond a \in x$, a contradiction. Thus, $\Diamond a \notin A'$. We show that there is $x' \in X$ with $\Diamond a \in x'$ and $x \cap A' = x' \cap A'$. Let F be the filter generated by $\{\Diamond a\} \cup (x \cap A')$. If $0 \in F$, then there is $b \in x \cap A'$ with $\Diamond a \wedge b = 0$, so $\Diamond a \leq \neg b$, yielding $\neg b \in x \cap A'$, a contradiction. Thus, F is proper. Clearly $x \cap A' \subseteq F \cap A'$. Suppose $c \in F \cap A'$. Then there is $b \in x \cap A'$ with $c \geq \Diamond a \wedge b$, so $\Diamond a \leq b \to c$. From this we conclude that $b \to c \in x \cap A'$, so $c \in x \cap A'$, and hence $F \cap A' = x \cap A'$. Therefore, a Zorn's lemma argument produces $x' \in X$ such that $F \subseteq x'$ and $x' \cap A' = x \cap A'$. Thus, $\Diamond a \in x'$ and $x' \cap A' = x \cap A'$. Let $I = \{c \in A : \Diamond c \notin x'\}$. Then it is easy to see that I is an ideal and $\uparrow a \cap I = \varnothing$. Therefore, there is $y \in X$ with $\uparrow a \subseteq y$ and $y \cap I = \varnothing$. Thus, $a \in y$ and $x'\ R\ y$. Consequently, $[x]\ R^l\ [y]$ and $[y] \in \beta'(a)$, yielding $[x] \in (R^l)^{-1}\beta'(a)$. $\dashv$

Since $[x] \notin (R^l)^{-1}\beta'(a)$, by Claim 4.7, $[x] \notin \beta'(\Diamond^l a)$. Because $\Diamond^l a \in A'$, this yields $\Diamond^l a \notin x$. Therefore, we found $a \in A'$ such that $a \in y$ but $\Diamond^l a \notin x$. Thus, $[x]\ \cancel{R}_{\Diamond^l}\ [y]$, and so R^l is the dual of $\Diamond^l$.

Let $R_{\Diamond^g}$ be the dual of $\Diamond^g$. Then $[x]\ R_{\Diamond^g}\ [y]$ iff $(\forall a \in A')(a \in y \Rightarrow \Diamond^g a \in x)$. On the other hand, $[x]\ R^g\ [y]$ iff $(\forall a \in D)(a \in y \Rightarrow \Diamond a \in x)$. By Lemma 4.5, $\Diamond a \leq \Diamond^g a$ for each $a \in A'$ and $\Diamond a = \Diamond^g a$ for each $a \in D$. Therefore, $[x]\ R_{\Diamond^g}\ [y]$ implies $[x]\ R^g\ [y]$. Conversely, if $[x]\ \cancel{R}_{\Diamond^g}\ [y]$, then there is $a \in A'$ such that $a \in y$ and $\Diamond^g a \notin x$. Thus, there is $b \in D^\vee$ such that $a \leq b$ and $\Diamond b \notin x$. As $a \leq b$ and $a \in y$, we see that $b \in y$. Suppose that $b = \bigvee F$ for some finite $F \subseteq D$. Then $\bigvee F \in y$ implies that there is $f \in F$ with $f \in y$. Since $f \leq b$, we have $\Diamond f \leq \Diamond b$, so $\Diamond f \notin x$. Therefore, we found $f \in D$ with $f \in y$, but $\Diamond f \notin x$. Thus, $[x]\ \cancel{R}^g\ [y]$, and hence R^g is the dual of $\Diamond^g$. $\dashv$

§5. Finite refutation patterns and stable canonical rules.

In this section we show how to construct finite refutation patterns for multi-conclusion modal rules. We introduce stable canonical rules, develop their basic properties, and prove that each normal modal multi-conclusion consequence relation is axiomatizable by stable canonical rules. We apply these results to construct finite refutation patterns for modal formulas, and prove that each normal modal logic is axiomatizable by stable canonical rules.

THEOREM 5.1.

(1) *For each multi-conclusion modal rule Γ/Δ, there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying* CDC *for D_i.*

(2) *For each modal formula φ, there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \varphi$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying* CDC *for D_i.*

PROOF. (1). If $\mathbf{S_K} \vdash \Gamma/\Delta$, then we take $n = 0$. Suppose that $\mathbf{S_K} \nvdash \Gamma/\Delta$. Let Θ be the set of all subformulas of the formulas in $\Gamma \cup \Delta$. Then Θ is finite. Let m be the cardinality of Θ. Since Boolean algebras are locally finite, up to isomorphism, there are only finitely many pairs $(\mathfrak{A}, D)$ satisfying the following two conditions:

(i) $\mathfrak{A} = (A, \Diamond)$ is a finite modal algebra such that A is at most m-generated as a Boolean algebra and $\mathfrak{A} \nvDash \Gamma/\Delta$.

(ii) $D = \{V(\psi) : \Diamond\psi \in \Theta\}$, where V is a valuation on $\mathfrak{A}$ witnessing $\mathfrak{A} \nvDash \Gamma/\Delta$.

Let $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ be the enumeration of such pairs. For a modal algebra $\mathfrak{B} = (B, \Diamond)$, we prove that $\mathfrak{B} \nvDash \Gamma/\Delta$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying CDC for D_i.

($\Leftarrow$): First suppose that there is $i \leq n$ and a stable embedding $h_i : A_i \rightarrowtail B$ satisfying CDC for D_i. Define a valuation V_B on B by $V_B(p) = h_i \circ V_i(p)$ for each propositional letter p. For each $\Diamond\psi \in \Theta$, we have $V_i(\psi) \in D_i$. Therefore, $V_B(\Diamond\psi) = \Diamond V_B(\psi)$. Thus, since $V_i(\gamma) = 1_{A_i}$ for each $\gamma \in \Gamma$ and $V_i(\delta) \neq 1_{A_i}$ for each $\delta \in \Delta$, we see that $V_B(\gamma) = 1_B$ for each $\gamma \in \Gamma$ and $V_B(\delta) \neq 1_B$ for each $\delta \in \Delta$. Consequently, $\mathfrak{B} \nvDash \Gamma/\Delta$.

($\Rightarrow$): Next suppose that $\mathfrak{B} \nvDash \Gamma/\Delta$. We show that there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying CDC for D_i. Since $\mathfrak{B} \nvDash \Gamma/\Delta$, there is a valuation V_B on B such that $V_B(\gamma) = 1_B$ for each $\gamma \in \Gamma$ and $V_B(\delta) \neq 1_B$ for each $\delta \in \Delta$. Let B' be the Boolean subalgebra of B generated by $V_B(\Theta)$. As $|V_B(\Theta)| \leq |\Theta|$, we see that $|B'| \leq 2^{2^m}$. Let V' be the restriction of V_B to B' and set $D = \{V'(\psi) : \Diamond\psi \in \Theta\}$. Let $\mathfrak{B}' = (B', \Diamond')$ be a filtration of $\mathfrak{B}$ through Θ. Then the embedding $B' \rightarrowtail B$ is a stable embedding satisfying CDC for D. By Lemma 4.4, V' refutes Γ/Δ on $\mathfrak{B}'$. Since $|B'| \leq 2^{2^m}$, there is $i \leq n$ such that $\mathfrak{B}' = \mathfrak{A}_i$ and $D = D_i$. Thus, the embedding $A_i \rightarrowtail B$ is a stable embedding satisfying CDC for D_i.

(2). If $\mathbf{K} \vdash \varphi$, then we take $n = 0$. Otherwise, since for a modal algebra $\mathfrak{A}$, we have $\mathfrak{A} \vDash \varphi$ iff $\mathfrak{A} \vDash /\varphi$, we see that $\mathbf{K} \nvdash \varphi$ iff $\mathbf{S_K} \nvdash /\varphi$. Now apply (1). ⊣

DEFINITION 5.2. Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra and let D be a subset of A. For each $a \in A$ we introduce a new propositional letter p_a and define the *stable canonical rule* $\rho(\mathfrak{A}, D)$ associated with $\mathfrak{A}$ and D as Γ/Δ, where:

$$\Gamma = \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \cup$$
$$\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\} \cup$$
$$\{\Diamond p_a \rightarrow p_{\Diamond a} : a \in A\} \cup$$
$$\{p_{\Diamond a} \rightarrow \Diamond p_a : a \in D\},$$

and

$$\Delta = \{p_a : a \in A, a \neq 1\}.$$

LEMMA 5.3. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra and let $D \subseteq A$. Then $\mathfrak{A} \nvDash \rho(\mathfrak{A}, D)$.*

PROOF. Define a valuation V on A by $V(p_a) = a$ for each $a \in A$. Then $V(\gamma) = 1$ for each $\gamma \in \Gamma$ and $V(\delta) \neq 1$ for each $\delta \in \Delta$. Therefore, $\mathfrak{A} \nvDash \rho(\mathfrak{A}, D)$. ⊣

THEOREM 5.4. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra, $D \subseteq A$, and $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. Then $\mathfrak{B} \nvDash \rho(\mathfrak{A}, D)$ iff there is a stable embedding $h : A \rightarrowtail B$ satisfying CDC for D.*

Proof. First suppose that there is a stable embedding $h : A \rightarrowtail B$ satisfying CDC for D. By Lemma 5.3, the valuation $V(p_a) = a$ refutes $\rho(\mathfrak{A}, D)$ on $\mathfrak{A}$. We define a valuation V_B on B by $V_B(p_a) = h(V(p_a)) = h(a)$ for each $a \in A$. Since h is a stable homomorphism, $h(a \vee b) = h(a) \vee h(b)$, $h(\neg a) = \neg h(a)$, and $\Diamond h(a) \leq h(\Diamond a)$ for each $a, b \in A$. Therefore,

$$V_B(p_{a \vee b} \leftrightarrow p_a \vee p_b) = V_B(p_{a \vee b}) \leftrightarrow V_B(p_a) \vee V_B(p_b)$$
$$= h(a \vee b) \leftrightarrow h(a) \vee h(b) = 1,$$
$$V_B(p_{\neg a} \leftrightarrow \neg p_a) = V_B(p_{\neg a}) \leftrightarrow \neg V_B(p_a) = h(\neg a) \leftrightarrow \neg h(a) = 1,$$
$$V_B(\Diamond p_a \to p_{\Diamond a}) = V_B(\Diamond p_a) \to V_B(p_{\Diamond a}) = \Diamond h(a) \to h(\Diamond a) = 1.$$

Since h satisfies CDC for D,

$$V_B(p_{\Diamond a} \to \Diamond p_a) = V_B(p_{\Diamond a}) \to \Diamond V_B(p_a) = h(\Diamond a) \to \Diamond h(a) = 1$$

for each $a \in D$. Thus, $V_B(\gamma) = 1$ for each $\gamma \in \Gamma$. On the other hand, since h is an embedding, from $a \neq 1$ it follows that $V_B(p_a) = h(a) \neq 1$. This yields $V_B(\delta) \neq 1$ for each $\delta \in \Delta$. Consequently, $\rho(\mathfrak{A}, D)$ is refuted on $\mathfrak{B}$.

Conversely, let $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$. Then there is a valuation V on B such that $V(\gamma) = 1$ for each $\gamma \in \Gamma$ and $V(\delta) \neq 1$ for each $\delta \in \Delta$. Define a map $h : A \to B$ by $h(a) = V(p_a)$ for each $a \in A$. We show that $h : A \to B$ is a stable embedding satisfying CDC for D.

Let $a, b \in A$. Since $V(\gamma) = 1$ for each $\gamma \in \Gamma$, we have $V(p_{a \vee b}) \leftrightarrow V(p_a) \vee V(p_b) = 1$. Therefore, $V(p_{a \vee b}) = V(p_a) \vee V(p_b)$. By a similar argument,

$$V(p_{\neg a}) = \neg V(p_a),$$
$$\Diamond V(p_a) \leq V(p_{\Diamond a}), \text{ and}$$
$$V(p_{\Diamond a}) = \Diamond V(p_a) \text{ for } a \in D.$$

Since $h(a) = V(p_a)$ for each $a \in A$, we have:

$$h(a \vee b) = h(a) \vee h(b),$$
$$h(\neg a) = \neg h(a),$$
$$\Diamond h(a) \leq h(\Diamond a), \text{ and}$$
$$h(\Diamond a) = \Diamond h(a) \text{ for } a \in D.$$

Thus, h is a stable homomorphism satisfying CDC for D. To see that h is an embedding, let $a \in A$ with $a \neq 1$. Since $V(\delta) \neq 1$ for each $\delta \in \Delta$, we have $V(p_a) \neq 1$, so $h(a) \neq 1$, yielding that h is an embedding.   ⊣

As a consequence of Theorems 5.1 and 5.4, we obtain:

Theorem 5.5.

(1) *For a multi-conclusion modal rule Γ/Δ, there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have:*

$$\mathfrak{B} \models \Gamma/\Delta \text{ iff } \mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n).$$

(2) *For a modal formula φ, there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have*:

$$\mathfrak{B} \models \varphi \text{ iff } \mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n).$$

PROOF. (1). By Theorem 5.1(1), there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite modal algebra, $D_i \subseteq A_i$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \Gamma/\Delta$ iff there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying CDC for D_i. By Theorem 5.4, this is equivalent to the existence of $i \leq n$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A}_i, D_i)$. Thus, $\mathfrak{B} \models \Gamma/\Delta$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n)$.

(2). This is proved similarly but uses Theorem 5.1(2).   ⊣

We are ready to prove the main result of the paper.

THEOREM 5.6.

(1) *Each normal modal multi-conclusion consequence relation $\mathcal{S}$ is axiomatizable by stable canonical rules. Moreover, if $\mathcal{S}$ is finitely axiomatizable, then $\mathcal{S}$ is axiomatizable by finitely many stable canonical rules.*

(2) *Each normal modal logic L is axiomatizable by stable canonical rules. Moreover, if L is finitely axiomatizable, then L is axiomatizable by finitely many stable canonical rules.*

PROOF. (1). Let $\mathcal{S}$ be a normal modal multi-conclusion consequence relation. Then there is a family $\{\rho_i : i \in I\}$ of modal rules such that $\mathcal{S} = \mathbf{S_K} + \{\rho_i : i \in I\}$. By Theorem 5.5(1), for each $i \in I$, there exist $(\mathfrak{A}_{i1}, D_{i1}), \ldots, (\mathfrak{A}_{in_i}, D_{in_i})$ such that $\mathfrak{A}_{ij} = (A_{ij}, \Diamond_{ij})$ is a finite modal algebra, $D_{ij} \subseteq A_{ij}$, and for each modal algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \models \rho_i$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_{i1}, D_{i1}), \ldots, \rho(\mathfrak{A}_{in_i}, D_{in_i})$. Thus, $\mathfrak{B} \models \mathcal{S}$ iff $\mathfrak{B} \models \{\rho_i : i \in I\}$, which happens iff $\mathfrak{B} \models \rho(\mathfrak{A}_{i1}, D_{i1}), \ldots, \rho(\mathfrak{A}_{in_i}, D_{in_i})$ for each $i \in I$. Consequently, $\mathcal{S} = \mathbf{S_K} + \bigcup_{i \in I} \{\rho(\mathfrak{A}_{i1}, D_{i1}), \ldots, \rho(\mathfrak{A}_{in_i}, D_{in_i})\}$, and so $\mathcal{S}$ is axiomatizable by stable canonical rules. In particular, if $\mathcal{S}$ is finitely axiomatizable, then $\mathcal{S}$ is axiomatizable by finitely many stable canonical rules.

(2). Let L be a normal modal logic. Then $\Sigma(L) = \mathbf{S_K} + \{/\varphi : \varphi \in L\}$ is a normal modal multi-conclusion consequence relation. Therefore, by (1), $\Sigma(L) = \mathbf{S_K} + \{\rho(\mathfrak{A}_i, D_i) : i \in I\}$. Thus, $L = \Lambda(\Sigma(L)) = \Lambda(\mathbf{S_K} + \{\rho(\mathfrak{A}_i, D_i) : i \in I\})$. In particular, if L is finitely axiomatizable, then L is axiomatizable by finitely many stable canonical rules.   ⊣

REMARK 5.7. Since the axiomatization in Theorem 5.6(2) of normal extensions of $\mathbf{K}$ is by means of stable canonical rules, we are required to work with all finite modal algebras, it is not sufficient to work with only finite s.i. modal algebras. As we will see in the next section, the situation improves for normal extensions of $\mathbf{K4}$, where stable canonical rules can be replaced by stable canonical formulas, and it is sufficient to work with only finite s.i. $\mathbf{K4}$-algebras.

REMARK 5.8. Using duality between modal algebras and modal spaces, one can rephrase all the results in this and forthcoming sections in dual terms. In fact, stable canonical rules can be defined directly for finite modal spaces (finite Kripke frames) without using modal algebras.

Let $\mathfrak{X} = (X, R)$ be a finite modal space and $\mathfrak{D} \subseteq \mathcal{P}(X)$. For each $x \in X$ we introduce a new propositional letter p_x and define the *stable canonical rule* $\sigma(\mathfrak{X}, \mathfrak{D})$

as the rule Γ/Δ, where

$$
\begin{aligned}
\Gamma = \Big\{ &\bigvee \{ p_x : x \in X \} \Big\} \cup \\
&\{ p_x \to \neg p_y : x, y \in X, x \neq y \} \cup \\
&\{ p_x \to \neg \Diamond p_y : x, y \in X, x \not\mathrel{R} y \} \cup \\
&\Big\{ p_x \to \bigvee \{ \Diamond p_y : y \in U \} : x \in X, U \in \mathfrak{D}, x \in R^{-1}[U] \Big\},
\end{aligned}
$$

and

$$
\Delta = \{ \neg p_x : x \in X \}.
$$

Then a modal space $\mathfrak{Y} = (Y, R)$ refutes $\sigma(\mathfrak{X}, \mathfrak{D})$ iff there is an onto stable map $f : Y \to X$ satisfying CDC for $\mathfrak{D}$. This provides an alternative way of defining stable canonical rules by avoiding algebraic terminology. Indeed, let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra, $\mathfrak{X} = (X, R)$ be its dual modal space, $D \subseteq A$, and $\mathfrak{D} = \{ \beta(a) : a \in D \}$. Then for each modal algebra $\mathfrak{B} = (B, \Diamond)$ with the dual space $\mathfrak{Y} = (Y, R)$, we have $\mathfrak{B} \models \rho(\mathfrak{A}, D)$ iff $\mathfrak{Y} \models \sigma(\mathfrak{X}, \mathfrak{D})$.

§6. Stable canonical formulas for K4. As we have seen, all normal modal logics are axiomatizable by stable canonical rules. In general, these rules are not equivalent to formulas. In this section we show that for transitive normal modal logics we can replace stable canonical rules by stable canonical formulas. This provides an axiomatization of transitive normal modal logics, which is an alternative to Zakharyaschev's axiomatization [38], and is a modal counterpart of [3].

6.1. Transitive filtrations. We start by developing the transitive analogues of the least and greatest filtrations.

DEFINITION 6.1. Let $\mathfrak{A} = (A, \Diamond)$ be a **K4**-algebra, V be a valuation on A, Θ be a set of formulas closed under subformulas, and $\mathfrak{A}' = (A', \Diamond')$ be a filtration of $\mathfrak{A}$ through Θ. We call $\mathfrak{A}'$ a *transitive filtration* if $\mathfrak{A}'$ is also a **K4**-algebra.

For a **K4**-algebra $\mathfrak{A} = (A, \Diamond)$ and $a \in A$, we recall that $\Diamond^+ a = a \vee \Diamond a$.

LEMMA 6.2. *Let $\mathfrak{A} = (A, \Diamond)$ be a **K4**-algebra, and let V, Θ, A', D, and $D^\vee$ be as in Lemma 4.5. Define $\Diamond^t$ and $\Diamond^L$ on A' by*

$$
\Diamond^t a = \bigwedge \{ \Diamond b : \Diamond a \leq \Diamond b \ \& \ b, \Diamond b \in A' \} \ and
$$

$$
\Diamond^L a = \bigwedge \{ \Diamond b : \Diamond a \leq \Diamond b \ \& \ \Diamond^+ a \leq \Diamond^+ b \ \& \ b \in D^\vee \}.
$$

Then both $(A', \Diamond^t)$ and $(A', \Diamond^L)$ are transitive filtrations of $\mathfrak{A}$ through Θ.

PROOF. Since $\Diamond 0 = 0$ and $0 \in A'$, it is obvious that $\Diamond^t 0 = 0$. As A' is closed under finite joins,

$$
\begin{aligned}
\Diamond^t a \vee \Diamond^t b &= \bigwedge \{ \Diamond x : \Diamond a \leq \Diamond x \ \& \ x, \Diamond x \in A' \} \vee \bigwedge \{ \Diamond y : \Diamond b \leq \Diamond y \ \& \ y, \Diamond y \in A' \} \\
&= \bigwedge \{ \Diamond x \vee \Diamond y : \Diamond a \leq \Diamond x \ \& \ \Diamond b \leq \Diamond y \ \& \ x, \Diamond x, y, \Diamond y \in A' \} \\
&= \bigwedge \{ \Diamond (x \vee y) : \Diamond a \leq \Diamond x \ \& \ \Diamond b \leq \Diamond y \ \& \ x, \Diamond x, y, \Diamond y \in A' \} \\
&= \bigwedge \{ \Diamond z : \Diamond (a \vee b) \leq \Diamond z \ \& \ z, \Diamond z \in A' \} \\
&= \Diamond^t (a \vee b).
\end{aligned}
$$

Since $\Diamond 0 = 0$ and $0 \in D^\vee$, it is obvious that $\Diamond^L 0 = 0$. As $D^\vee$ is closed under finite joins,

$$
\begin{aligned}
\Diamond^L a \vee \Diamond^L b &= \bigwedge\{\Diamond x : \Diamond a \leq \Diamond x \ \& \ \Diamond^+ a \leq \Diamond^+ x \ \& \ x \in D^\vee\} \vee \\
&\qquad \bigwedge\{\Diamond y : \Diamond b \leq \Diamond y \ \& \ \Diamond^+ b \leq \Diamond^+ y \ \& \ y \in D^\vee\} \\
&= \bigwedge\{\Diamond x \vee \Diamond y : \Diamond a \leq \Diamond x \ \& \ \Diamond^+ a \leq \Diamond^+ x \ \& \\
&\qquad \Diamond b \leq \Diamond y \ \& \ \Diamond^+ b \leq \Diamond^+ y \ \& \ x, y \in D^\vee\} \\
&= \bigwedge\{\Diamond(x \vee y) : \Diamond a \leq \Diamond x \ \& \ \Diamond^+ a \leq \Diamond^+ x \ \& \ \Diamond b \leq \Diamond y \ \& \\
&\qquad \Diamond^+ b \leq \Diamond^+ y \ \& \ x, y \in D^\vee\} \\
&= \bigwedge\{\Diamond z : \Diamond(a \vee b) \leq \Diamond z \ \& \ \Diamond^+(a \vee b) \leq \Diamond^+ z \ \& \ z \in D^\vee\} \\
&= \Diamond^L(a \vee b).
\end{aligned}
$$

Therefore, both $(A', \Diamond^t)$ and $(A', \Diamond^L)$ are modal algebras. It is obvious that $\Diamond^l a \leq \Diamond^t a \leq \Diamond^L a \leq \Diamond^g a$ for each $a \in A'$. Thus, both $(A', \Diamond^t)$ and $(A', \Diamond^L)$ are filtrations of $\mathfrak{A}$ through Θ. It remains to show that both $(A', \Diamond^t)$ and $(A', \Diamond^L)$ are **K4**-algebras. We have

$$\Diamond^t a = \bigwedge\{\Diamond x : \Diamond a \leq \Diamond x \ \& \ x, \Diamond x \in A'\}$$

and

$$\Diamond^t \Diamond^t a = \bigwedge\{\Diamond y : \Diamond\Diamond^t a \leq \Diamond y \ \& \ y, \Diamond y \in A'\}.$$

Let $x, \Diamond x \in A'$ and $\Diamond a \leq \Diamond x$. Then

$$
\begin{aligned}
\Diamond\Diamond^t a &= \Diamond \bigwedge\{\Diamond y : \Diamond a \leq \Diamond y \ \& \ y, \Diamond y \in A'\} \\
&\leq \bigwedge\{\Diamond\Diamond y : \Diamond a \leq \Diamond y \ \& \ y, \Diamond y \in A'\} \\
&\leq \bigwedge\{\Diamond y : \Diamond a \leq \Diamond y \ \& \ y, \Diamond y \in A'\} \ \leq \ \Diamond x,
\end{aligned}
$$

so $\Diamond^t \Diamond^t a \leq \Diamond^t a$. Also,

$$\Diamond^L a = \bigwedge\{\Diamond x : \Diamond a \leq \Diamond x \ \& \ \Diamond^+ a \leq \Diamond^+ x \ \& \ x \in D^\vee\}$$

and

$$\Diamond^L \Diamond^L a = \bigwedge\{\Diamond y : \Diamond\Diamond^L a \leq \Diamond y \ \& \ \Diamond^+ \Diamond^L a \leq \Diamond^+ y \ \& \ y \in D^\vee\}.$$

Let $x \in D^\vee$, $\Diamond a \leq \Diamond x$, and $\Diamond^+ a \leq \Diamond^+ x$. Then

$$
\begin{aligned}
\Diamond\Diamond^L a &= \Diamond \bigwedge\{\Diamond y : \Diamond a \leq \Diamond y \ \& \ \Diamond^+ a \leq \Diamond^+ y \ \& \ y \in D^\vee\} \\
&\leq \bigwedge\{\Diamond\Diamond y : \Diamond a \leq \Diamond y \ \& \ \Diamond^+ a \leq \Diamond^+ y \ \& \ y \in D^\vee\} \\
&\leq \bigwedge\{\Diamond y : \Diamond a \leq \Diamond y \ \& \ \Diamond^+ a \leq \Diamond^+ y \ \& \ y \in D^\vee\} \ \leq \ \Diamond x
\end{aligned}
$$

and

$$\lozenge^+ \lozenge^L a = \lozenge^+ \bigwedge \{\lozenge y : \lozenge a \le \lozenge y \ \& \ \lozenge^+ a \le \lozenge^+ y \ \& \ y \in D^\vee\}$$
$$\le \bigwedge \{\lozenge^+ \lozenge y : \lozenge a \le \lozenge y \ \& \ \lozenge^+ a \le \lozenge^+ y \ \& \ y \in D^\vee\}$$
$$= \bigwedge \{\lozenge y : \lozenge a \le \lozenge y \ \& \ \lozenge^+ a \le \lozenge^+ y \ \& \ y \in D^\vee\}$$
$$\le \bigwedge \{\lozenge^+ y : \lozenge a \le \lozenge y \ \& \ \lozenge^+ a \le \lozenge^+ y \ \& \ y \in D^\vee\} \ \le \ \lozenge^+ x.$$

This implies $\lozenge^L \lozenge^L a \le \lozenge^L a$. Thus, both $(A', \lozenge^t)$ and $(A', \lozenge^L)$ are **K4**-algebras. ⊣

We recall (see, e.g., [15, Sec. 5.3] or [12, Sec. 2.3]) that the *Lemmon filtration* of $\mathfrak{M} = (X, R, V)$ through Θ is given by $[x] \, R^L \, [y]$ iff $(\forall \lozenge \varphi \in \Theta)(y \models \lozenge^+ \varphi \Rightarrow x \models \lozenge \varphi)$.

LEMMA 6.3. *Suppose that $\mathfrak{A} = (A, \lozenge)$ is a **K4**-algebra and $\mathfrak{X} = (X, R)$ is its dual. Let A' and X' be as in Theorem* 4.2*, with A' and X' finite, R^l be as in Lemma* 4.5*, and $\lozenge^t$ and $\lozenge^L$ be as in Lemma* 6.2*. The dual of $\lozenge^t$ is the transitive closure of R^l and the dual of $\lozenge^L$ is the Lemmon filtration.*

PROOF. Let R^t denote the transitive closure of R^l. Then $[x] \, R^t \, [y]$ iff there exist $z_1, \ldots, z_n \in X$ such that $[x] = [z_1] \, R^l \cdots R^l \, [z_n] = [y]$. Also, $[x] \, R_{\lozenge^t} \, [y]$ iff $(\forall a \in A')$ $(a \in y \Rightarrow \lozenge^t a \in x)$. We show $R^t = R_{\lozenge^t}$. Since $\lozenge^l a \le \lozenge^t a$ for each $a \in A'$, we have $R^l \subseteq R_{\lozenge^t}$. Also, since $(A', \lozenge^t)$ is a **K4**-algebra, $R_{\lozenge^t}$ is transitive. Thus, $R^t \subseteq R_{\lozenge^t}$. Conversely, suppose that $[x] \, \cancel{R^t} \, [y]$. To see that $[x] \, \cancel{R_{\lozenge^t}} \, [y]$, it is sufficient to find $a \in A'$ such that $a \in y$ and $\lozenge^t a \notin x$. Let $a \in A'$ be such that $\beta(a) = [y]$. Then $a \in y$. Since $[x] \, \cancel{R^t} \, [y]$, we have $[x] \cap R^{-1}[y] = \varnothing$. If $R^{-1}[y]$ is saturated (that is, $R^{-1}[y]$ is a union of equivalence classes), then $\beta(\lozenge a) = R^{-1}[y]$ is saturated, so $\lozenge a \in A'$. This yields $\lozenge^t a = \lozenge a$. As $x \notin R^{-1}[y]$, we have $x \notin \beta(\lozenge a)$, so $\lozenge a \notin x$. Thus, $a \in y$ and $\lozenge^t a \notin x$. If $R^{-1}[y]$ is not saturated, then we consider the saturation $[R^{-1}[y]]$ of $R^{-1}[y]$. Since $[x] \, \cancel{R^t} \, [y]$, we have $[x] \cap (R^{-1}[R^{-1}[y]] \cup R^{-1}[y]) = \varnothing$. If $R^{-1}[R^{-1}[y]]$ is saturated, then let $b \in A'$ be such that $\beta(b) = [R^{-1}[y]] \cup [y]$. So $\beta(\lozenge b) = R^{-1}[R^{-1}[y]] \cup R^{-1}[y] \supseteq \beta(\lozenge a)$ is saturated. Therefore, $\lozenge b \in A'$, $\lozenge a \le \lozenge b$, and $x \notin \beta(\lozenge b)$. Thus, $a \in y$ and $\lozenge^t a \le \lozenge b \notin x$. If $[R^{-1}[y]]$ is not saturated, then we continue the process by taking its saturation. Since there are only finitely many saturated subsets of X, the process will end after finitely many steps, which will produce $b \in A'$ such that $\lozenge a \le \lozenge b$, $\lozenge b \in A'$, and $x \notin \beta(\lozenge b)$. Thus, $a \in y$ and $\lozenge^t a \le \lozenge b \notin x$, and hence $[x] \, \cancel{R_{\lozenge^t}} \, [y]$.

Let R^L be the Lemmon filtration. Then $[x] \, R^L \, [y]$ iff $(\forall \lozenge \varphi \in \Theta)(y \models \lozenge^+ \varphi \Rightarrow x \models \lozenge \varphi)$, which is equivalent to $(\forall a \in D)(\lozenge^+ a \in y \Rightarrow \lozenge a \in x)$. Also, $[x] \, R_{\lozenge^L} \, [y]$ iff $(\forall a \in A')(a \in y \Rightarrow \lozenge^L a \in x)$. We show $R^L = R_{\lozenge^L}$. First suppose that $[x] \, \cancel{R^L} \, [y]$. Then there exists $a \in D$ such that $\lozenge^+ a \in y$ but $\lozenge a \notin x$. From $\lozenge^+ a \in y$ it follows that $a \in y$ or $\lozenge a \in y$. As $a \in D$, we have $\lozenge^L a = \lozenge a$. So if $a \in y$, then $\lozenge^L a \notin x$. On the other hand, if $\lozenge a \in y$, then letting $b = \lozenge a$, we have $b \in A'$, $b \in y$, and $\lozenge^L b = \lozenge^L \lozenge a \le \lozenge^L \lozenge^L a \le \lozenge^L a = \lozenge a \notin x$. Therefore, in both cases we have $[x] \, \cancel{R_{\lozenge^L}} \, [y]$. Next suppose that $[x] \, \cancel{R_{\lozenge^L}} \, [y]$. Then there exists $a \in A'$ such that $a \in y$ but $\lozenge^L a \notin x$. The latter implies that there exists $b \in D^\vee$ such that $\lozenge a \le \lozenge b$, $\lozenge^+ a \le \lozenge^+ b$, and $\lozenge b \notin x$. As $a \le \lozenge^+ a \le \lozenge^+ b$, the former implies that $\lozenge^+ b \in y$. Since b is a finite join of elements of D, we conclude that $[x] \, \cancel{R^L} \, [y]$. ⊣

6.2. Refutation patterns and stable canonical formulas for K4. Next we apply the results of Section 5 to obtain refutation patterns for **K4**. We will utilize the following corollary of Venema's characterization [35] of s.i. modal algebras.

PROPOSITION 6.4. *Let* $\mathfrak{A} = (A, \Diamond)$ *be a finite modal algebra and let* $\mathfrak{B} = (B, \Diamond)$ *be a s.i. modal algebra. If there is a stable embedding* $h : A \rightarrowtail B$, *then* $\mathfrak{A}$ *is also s.i.*

PROOF. Let $\mathfrak{X} = (X, R)$ be the dual of $\mathfrak{A}$, $\mathfrak{Y} = (Y, R)$ be the dual of $\mathfrak{B}$, and $f : Y \to X$ be the dual of h. Since h is 1-1, f is onto. As $\mathfrak{B}$ is s.i., by [35, Thm. 2], the set of topo-roots of $\mathfrak{Y}$ has nonempty interior. Let t belong to this interior. We show that $f(t)$ is a root of $\mathfrak{X}$. Because $\mathfrak{A}$ is finite, this will imply that $\mathfrak{A}$ is s.i. For $Y \subseteq X$, we denote by $\overline{Y}$ the topological closure of Y. Since t is a topo-root of $\mathfrak{Y}$, we have $\overline{R^\omega[t]} = Y$. Therefore, $f\left(\overline{R^\omega[t]}\right) = X$. As f is continuous and X is finite, $f\left(\overline{R^\omega[t]}\right) \subseteq \overline{f(R^\omega[t])} = f(R^\omega[t])$. So $f(R^\omega[t]) = X$. Since h is stable, by Lemma 3.3, f is stable. So for each $y \in Y$, we have $f(R[y]) \subseteq R[f(y)]$, and hence $f(R^\omega[y]) \subseteq R^\omega[f(y)]$. Thus, $f(R^\omega[t]) \subseteq R^\omega[f(t)]$, which yields that $R^\omega[f(t)] = X$. Consequently, $f(t)$ is a root of $\mathfrak{X}$, so $\mathfrak{X}$ is rooted, an hence $\mathfrak{A}$ is s.i. ⊣

We next prove the following version of Theorem 5.1(2) for **K4**.

THEOREM 6.5. *For a modal formula* φ, *there exist* $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ *such that each* $\mathfrak{A}_i = (A_i, \Diamond_i)$ *is a finite s.i.* **K4**-*algebra,* $D_i \subseteq A_i$, *and for each s.i. modal algebra* $\mathfrak{B} = (B, \Diamond)$, *the following conditions are equivalent*:

(1) $\mathfrak{B} \not\models \varphi$.
(2) *There is* $i \leq n$ *and a stable embedding* $h : A_i \rightarrowtail B$ *satisfying* CDC *for* D_i.
(3) *There is a s.i. homomorphic image* $\mathfrak{C} = (C, \Diamond)$ *of* $\mathfrak{B}$, $i \leq n$, *and a stable embedding* $h : A_i \rightarrowtail C$ *satisfying* CDC *for* D_i.

PROOF. If **K4** $\vdash \varphi$, then we take $n = 0$. Suppose that **K4** $\not\vdash \varphi$. Let Θ be the set of all subformulas of φ. Then Θ is finite. Let m be the cardinality of Θ. Since Boolean algebras are locally finite, up to isomorphism, there are only finitely many pairs $(\mathfrak{A}, D)$ satisfying the following two conditions:

(i) $\mathfrak{A} = (A, \Diamond)$ is a finite s.i. **K4**-algebra such that A is at most m-generated as a Boolean algebra and $\mathfrak{A} \not\models \varphi$.
(ii) $D = \{V(\psi) : \Diamond\psi \in \Theta\}$, where V is a valuation on $\mathfrak{A}$ witnessing $\mathfrak{A} \not\models \varphi$.

Let $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ be the enumeration of such pairs. Let $\mathfrak{B} = (B, \Diamond)$ be a s.i. **K4**-algebra.

(1)⇒(2): Suppose that $\mathfrak{B} \not\models \varphi$. As in the proof of Theorem 5.1, but using a transitive filtration instead of an arbitrary filtration, we construct a finite **K4**-algebra $\mathfrak{B}' = (B', \Diamond')$ of size $\leq 2^{2^m}$, a valuation V' on B' refuting φ, and a stable embedding $B' \rightarrowtail B$ satisfying CDC for $D = \{V'(\psi) : \Diamond\psi \in \Theta\}$. Since $\mathfrak{B}$ is s.i., by Proposition 6.4, so is $\mathfrak{B}'$. Therefore, there is $i \leq n$ such that $\mathfrak{B}' = \mathfrak{A}_i$ and $D = D_i$. Thus, there is $i \leq n$ and a stable embedding $h : A_i \rightarrowtail B$ satisfying CDC for D_i.

(2)⇒(3): This is obvious.

(3)⇒(1): Suppose that there is a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$, $i \leq n$, and a stable embedding $h : A_i \rightarrowtail C$ satisfying CDC for D_i. The same argument as in the

proof of Theorem 5.1 yields that $\mathfrak{C} \not\models \varphi$. Since $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}$, we conclude that $\mathfrak{B} \not\models \varphi$.   ⊣

REMARK 6.6.   While Theorem 6.5 also holds for **K**, unlike **K4**, it does not yield any substantial gains because the next definition, producing stable canonical formulas for **K4**, does not work for **K**.

DEFINITION 6.7.   Let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. **K4**-algebra and $D \subseteq A$. For each $a \in A$ we introduce a new propositional letter p_a and define the *stable canonical formula* $\gamma(\mathfrak{A}, D)$ associated with $\mathfrak{A}$ and D as follows:

$$\gamma(\mathfrak{A}, D) = \bigwedge \{\Box^+ \gamma : \gamma \in \Gamma\} \rightarrow \bigvee \{\Box^+ \delta : \delta \in \Delta\}$$
$$= \Box^+ \bigwedge \Gamma \rightarrow \bigvee \{\Box^+ \delta : \delta \in \Delta\},$$

where Γ and Δ are as in Definition 5.2.

THEOREM 6.8.   *Let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. **K4**-algebra, $D \subseteq A$, and $\mathfrak{B} = (B, \Diamond)$ be a **K4**-algebra. Then $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$ iff there is a s.i. homomorphic image $\mathfrak{C} = (C, \Diamond)$ of $\mathfrak{B}$ and a stable embedding $h : A \rightarrowtail C$ satisfying* CDC *for D.*

PROOF.   First suppose that there is a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a stable embedding $h : A \rightarrowtail C$ satisfying CDC for D. Define a valuation V_A on A by $V_A(p_a) = a$ for each $a \in A$. Then $V_A(\gamma) = 1_A$ for each $\gamma \in \Gamma$ and $V_A(\delta) \neq 1_A$ for each $\delta \in \Delta$. Therefore, $V_A(\Box^+ \bigwedge \Gamma) = 1_A$ and $\Box^+ \delta \neq 1_A$ for each $\delta \in \Delta$. Since $\mathfrak{A}$ is a s.i. **K4**-algebra, its opremum c is the second largest element of the Heyting algebra H of the fixed points of $\Box^+$. Thus, $\bigvee \{\Box^+ \delta : \delta \in \Delta\} \leq c$, and hence $\mathfrak{A} \not\models \gamma(\mathfrak{A}, D)$. Next define a valuation V_C on C by $V_C(p_a) = h(V_A(p_a)) = h(a)$ for each $a \in A$. The same argument as in the proof of Theorem 5.4 shows that $V_C(\gamma) = 1_C$ for each $\gamma \in \Gamma$ and $V_C(\delta) \neq 1_C$ for each $\delta \in \Delta$. Therefore, $V_C(\Box^+ \bigwedge \Gamma) = 1_C$ and $\Box^+ \delta \neq 1_C$ for each $\delta \in \Delta$. Because $\mathfrak{C}$ is s.i., it has an opremum, hence $\bigvee \{\Box^+ \delta : \delta \in \Delta\}$ is underneath the opremum, so $\mathfrak{C} \not\models \gamma(\mathfrak{A}, D)$. Since $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}$, we conclude that $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$.

Conversely, suppose that $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$. Since $\mathfrak{B}$ is a **K4**-algebra, by [1, Lem. 4.1] (which is a modal analogue of [37, Lem. 1]), there is a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a valuation V_C on C such that $V_C(\Box^+ \bigwedge \Gamma) = 1_C$ and $V_C(\bigvee \{\Box^+ \delta : \delta \in \Delta\}) \neq 1_C$. Next we define a map $h : A \rightarrow C$ by $h(a) = V_C(p_a)$ for each $a \in A$. The proof of Theorem 5.4 then shows that h is a stable embedding satisfying CDC for D.   ⊣

Combining Theorems 6.5 and 6.8 yields.

COROLLARY 6.9.   *For a modal formula φ, there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite s.i. **K4**-algebra, $D_i \subseteq A_i$, and for each s.i. **K4**-algebra $\mathfrak{B} = (B, \Diamond)$, we have*:

$$\mathfrak{B} \models \varphi \text{ iff } \mathfrak{B} \models \bigwedge_{i=1}^{n} \gamma(\mathfrak{A}_i, D_i).$$

PROOF.   By Theorem 6.5, there exist $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite s.i. **K4**-algebra, $D_i \subseteq A_i$, and for each s.i. **K4**-algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \not\models \varphi$ iff there is a s.i. homomorphic image $\mathfrak{C} = (C, \Diamond)$ of $\mathfrak{B}$, $i \leq n$, and a stable embedding $h : A_i \rightarrowtail C$ satisfying CDC for D_i. By Theorem 6.8, this

is equivalent to the existence of $i \leq n$ such that $\mathfrak{B} \not\models \gamma(\mathfrak{A}_i, D_i)$. Thus, $\mathfrak{B} \models \varphi$ iff $\mathfrak{B} \models \bigwedge_{i=1}^{n} \gamma(\mathfrak{A}_i, D_i)$.   ⊣

Consequently, we arrive at a new axiomatization of modal logics above **K4**, which is an alternative to Zakharyaschev's axiomatization.

THEOREM 6.10. *Each normal transitive logic L is axiomatizable over* **K4** *by stable canonical formulas. Moreover, if L is finitely axiomatizable, then L is axiomatizable by finitely many stable canonical formulas.*

PROOF. Let L be a normal transitive logic. Then L is obtained by adding $\{\varphi_i : i \in I\}$ to **K4** as new axioms. By Corollary 6.9, for each $i \in I$, there exist $(\mathfrak{A}_{i1}, D_{i1}), \ldots, (\mathfrak{A}_{in_i}, D_{in_i})$ such that $\mathfrak{A}_{ij} = (A_{ij}, \Diamond_{ij})$ is a finite s.i. **K4**-algebra, $D_{ij} \subseteq A_{ij}$, and for each s.i. **K4**-algebra $\mathfrak{B} = (B, \Diamond)$, we have $\mathfrak{B} \models \varphi_i$ iff $\mathfrak{B} \models \bigwedge_{j=1}^{n_i} \gamma(\mathfrak{A}_{ij}, D_{ij})$. Since every modal logic is determined by the class of its s.i. modal algebras, $L = \mathbf{K4} + \{\bigwedge_{j=1}^{m_i} \gamma(\mathfrak{A}_{ij}, D_{ij}) : i \in I\}$. In particular, if L is finitely axiomatizable, then L is axiomatizable by finitely many stable canonical formulas.   ⊣

REMARK 6.11. Let $\mathbf{S_{K4}} := \Sigma(\mathbf{K4})$ be the least normal modal multi-conclusion consequence relation containing $/\varphi$ for each $\varphi \in \mathbf{K4}$. By Theorem 5.6(1), all normal multi-conclusion consequence relations extending $\mathbf{S_{K4}}$ are axiomatizable by stable canonical rules. If in the proof of Theorem 5.6(1) we use a transitive filtration, then we obtain that all multi-conclusion consequence relations extending $\mathbf{S_{K4}}$ are axiomatizable over $\mathbf{S_{K4}}$ by stable canonical rules of finite **K4**-algebras. In other words, multi-conclusion consequence relations extending $\mathbf{S_{K4}}$ are axiomatizable over $\mathbf{S_{K4}}$ by stable canonical rules of not just finite modal algebras, but by stable canonical rules of finite **K4**-algebras.

REMARK 6.12. In [10] the technique of stable canonical rules is utilized to give an alternative proof of the existence of explicit bases of admissible rules for the intuitionistic logic, **S4**, and **K4**.

REMARK 6.13. Let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. **K4**-algebra and let $D \subseteq A$. In general, $\mathbf{K4} + \gamma(\mathfrak{A}, D)$ is not equal to $\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$. We do have that $\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D)) \subseteq \mathbf{K4} + \gamma(\mathfrak{A}, D)$. Indeed, for a s.i. modal algebra $\mathfrak{B} = (B, \Diamond)$, if $\mathfrak{B} \not\models \Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$, then $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$. Therefore, by Theorems 5.4 and 6.8, $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$. This yields $\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D)) \subseteq \mathbf{K4} + \gamma(\mathfrak{A}, D)$. The other inclusion, in general, may not be true. However, if $\mathcal{U}(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$ is a variety, then $\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D)) = \mathbf{K4} + \gamma(\mathfrak{A}, D)$. To see this, let $\mathfrak{B} \not\models \mathbf{K4} + \gamma(\mathfrak{A}, D)$. Then $\mathfrak{B} \not\models \gamma(\mathfrak{A}, D)$. Therefore, by Theorem 6.8, there is a s.i. homomorphic image $\mathfrak{C} = (C, \Diamond)$ of $\mathfrak{B}$ and a stable embedding $h : A \rightarrowtail C$ satisfying CDC for D. By Theorem 5.4, $\mathfrak{C} \not\models \rho(\mathfrak{A}, D)$. If $\mathfrak{B} \models \rho(\mathfrak{A}, D)$, then $\mathfrak{B} \in \mathcal{U}(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$, and since this class is a variety, it is closed under homomorphic images, so $\mathfrak{C} \in \mathcal{U}(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D))$. But then $\mathfrak{C} \models \rho(\mathfrak{A}, D)$, a contradiction. Thus, $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$, and hence $\Lambda(\mathbf{S_{K4}} + \rho(\mathfrak{A}, D)) = \mathbf{K4} + \gamma(\mathfrak{A}, D)$. We leave it as an interesting open question to determine when $\mathcal{U}(\mathbf{S_K} + \rho(\mathfrak{A}, D))$ is a variety.

REMARK 6.14. As noted in Remark 5.8, our results can be phrased in dual terms. As with stable canonical rules, stable canonical formulas can also be defined directly for finite rooted transitive spaces (finite rooted transitive Kripke frames).

Let $\mathfrak{X} = (X, R)$ be a finite rooted transitive space and let $\mathfrak{D} \subseteq \mathcal{P}(X)$. For each $x \in X$ we introduce a new propositional letter p_x and define the *stable canonical formula* $\tau(\mathfrak{X}, \mathfrak{D})$ as follows:

$$\tau(\mathfrak{X}, \mathfrak{D}) = \bigwedge \{\square^+ \gamma : \gamma \in \Gamma\} \rightarrow \bigvee \{\square^+ \delta : \delta \in \Delta\}$$
$$= \square^+ \bigwedge \Gamma \rightarrow \bigvee \{\square^+ \delta : \delta \in \Delta\},$$

where Γ and Δ are as in Remark 5.8. Then a transitive space $\mathfrak{Y} = (Y, R)$ refutes $\tau(\mathfrak{X}, \mathfrak{D})$ iff there is a closed topo-rooted up-set Z of Y and an onto stable map $f : Z \rightarrow X$ satisfying CDC for $\mathfrak{D}$. This provides an alternative way of defining stable canonical formulas for **K4** by avoiding algebraic terminology. Indeed, let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. **K4**-algebra, $\mathfrak{X} = (X, R)$ be its dual, $D \subseteq A$, and $\mathfrak{D} = \{\beta(a) : a \in D\}$. Then for each **K4**-algebra $\mathfrak{B} = (B, \Diamond)$ with its dual $\mathfrak{Y} = (Y, R)$, we have $\mathfrak{B} \models \gamma(\mathfrak{A}, D)$ iff $\mathfrak{Y} \models \tau(\mathfrak{X}, \mathfrak{D})$.

§7. Stable rules and Jankov rules. As we saw in Section 5, stable canonical rules $\rho(\mathfrak{A}, D)$ axiomatize all normal modal multi-conclusion consequence relations and all normal modal logics. In this section we consider two extreme cases, when $D = \varnothing$ and when $D = A$. In the first case we call the stable canonical rule $\rho(\mathfrak{A}, \varnothing)$ simply a *stable rule* and denote it by $\rho(\mathfrak{A})$. In the second case we denote the stable canonical rule $\rho(\mathfrak{A}, A)$ by $\chi(\mathfrak{A})$ and call it a *Jankov rule*. We characterize normal modal multi-conclusion consequence relations and normal modal logics axiomatized by stable rules and prove that they all have the finite model property. On the other hand, as follows from [21] and [13], Jankov rules axiomatize splittings and join splittings in the lattices of normal modal multi-conclusion consequence relations and normal modal logics, respectively. We give alternate proofs of these results.

We start by an immediate consequence of Theorem 5.4.

PROPOSITION 7.1. *Let $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$ be modal algebras with A finite.*

(1) $\mathfrak{B} \not\models \rho(\mathfrak{A})$ *iff there is a stable embedding $h : A \rightarrowtail B$.*
(2) $\mathfrak{B} \not\models \chi(\mathfrak{A})$ *iff there is a 1-1 modal homomorphism $h : A \rightarrowtail B$.*

DEFINITION 7.2.

(1) We call a class $\mathcal{K}$ of modal algebras *stable* provided for modal algebras $\mathfrak{A} = (A, \Diamond)$ and $\mathfrak{B} = (B, \Diamond)$, if $\mathfrak{B} \in \mathcal{K}$ and there is a stable embedding $h : A \rightarrowtail B$, then $\mathfrak{A} \in \mathcal{K}$.
(2) We call a normal modal multi-conclusion consequence relation $\mathcal{S}$ *stable* provided the corresponding universal class $\mathcal{U}(\mathcal{S})$ is stable.

REMARK 7.3. By Lemma 3.3, it is clear that dually a class $\mathcal{K}$ of modal spaces is stable provided for modal spaces $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, R)$, if $\mathfrak{X} \in \mathcal{K}$ and there is an onto stable map $f : X \twoheadrightarrow Y$, then $\mathfrak{Y} \in \mathcal{K}$.

THEOREM 7.4. *A normal modal multi-conclusion consequence relation $\mathcal{S}$ is stable iff $\mathcal{S}$ is axiomatizable by stable rules.*

PROOF. First suppose that $\mathcal{S}$ is stable. Let $\mathcal{A}_\mathcal{S}$ be the set of all nonisomorphic finite modal algebras refuting $\mathcal{S}$. We show that $\mathcal{S} = \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}_\mathcal{S}\}$. Let $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. If $\mathfrak{B} \not\models \mathcal{S}$, then there is $\rho \in \mathcal{S}$ such that $\mathfrak{B} \not\models \rho$.

The construction in the proof of Theorem 5.1 yields a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \not\models \rho$ and the inclusion $A \rightarrowtail B$ is a stable embedding. Therefore, $\mathfrak{A} \in \mathcal{A}_{\mathcal{S}}$. By Proposition 7.1(1), $\mathfrak{B} \not\models \rho(\mathfrak{A})$. Thus, $\mathfrak{B} \not\models \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}_{\mathcal{S}}\}$. Conversely, if $\mathfrak{B} \not\models \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}_{\mathcal{S}}\}$, then there is $\mathfrak{A} \in \mathcal{A}_{\mathcal{S}}$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A})$. By Proposition 7.1(1), there is a stable embedding $A \rightarrowtail B$. If $\mathfrak{B} \models \mathcal{S}$, then since $\mathcal{S}$ is stable, $\mathfrak{A} \models \mathcal{S}$, a contradiction. Therefore, $\mathfrak{B} \not\models \mathcal{S}$. Thus, $\mathcal{S} = \mathbf{S_K} + \{\rho(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}_{\mathcal{S}}\}$, and so $\mathcal{S}$ is axiomatizable by stable rules.

Next let $\mathcal{S}$ be axiomatizable by stable rules. Then $\mathcal{S} = \mathbf{S_K} + \{\rho(\mathfrak{C}_i) : i \in I\}$. Suppose that $\mathfrak{B} \models \mathcal{S}$ and $h : A \rightarrowtail B$ is a stable embedding. If $\mathfrak{A} \not\models \mathcal{S}$, then there is $i \in I$ such that $\mathfrak{A} \not\models \rho(\mathfrak{C}_i)$. By Proposition 7.1(1), there is a stable embedding $C_i \rightarrowtail A$. Therefore, there is a stable embedding $C_i \rightarrowtail B$. Applying Proposition 7.1(1) again yields $\mathfrak{B} \not\models \rho(\mathfrak{C}_i)$. The obtained contradiction proves that $\mathfrak{A} \models \mathcal{S}$. Thus, $\mathcal{S}$ is stable. $\dashv$

DEFINITION 7.5.   We call a normal modal logic L *stable* provided there is a stable normal modal multi-conclusion consequence relation $\mathcal{S}$ such that $L = \Lambda(\mathcal{S})$.

As an immediate consequence of Theorem 7.4, we obtain:

PROPOSITION 7.6.   *For every* $L \in \mathrm{Next}\mathbf{K}$, *the following are equivalent*:

(1) L *is stable.*
(2) L *is axiomatizable by stable rules.*
(3) L *is the logic of a stable universal class.*

PROOF. (1)$\Rightarrow$(2): If L is stable, then $L = \Lambda(\mathcal{S})$ for some stable $\mathcal{S} \in \mathrm{NExt}\mathbf{S_K}$. By Theorem 7.4, there is a family $\{\mathfrak{A}_i : i \in I\}$ of finite modal algebras such that $\mathcal{S} = \mathbf{S_K} + \{\rho(\mathfrak{A}_i) : i \in I\}$. Therefore, L is axiomatizable by stable rules.

(2)$\Rightarrow$(3): If L is axiomatizable by stable rules, then $L = \Lambda(\mathbf{S_K} + \{\rho(\mathfrak{A}_i) : i \in I\})$ for some family $\{\mathfrak{A}_i : i \in I\}$ of finite modal algebras. By Theorem 7.4, $\mathcal{S} := \mathbf{S_K} + \{\rho(\mathfrak{A}_i) : i \in I\}$ is a stable multi-conclusion consequence relation. Therefore, $\mathcal{U}(\mathcal{S})$ is a stable universal class. Since L is the logic of $\mathcal{U}(\mathcal{S})$, we conclude that L is the logic of a stable universal class.

(3)$\Rightarrow$(1): Suppose L is the logic of a stable universal class $\mathcal{U}$. Then $\mathcal{S}(\mathcal{U})$ is a stable multi-conclusion consequence relation and $L = \Lambda(\mathcal{S}(\mathcal{U}))$. Thus, L is a stable logic. $\dashv$

DEFINITION 7.7.

(1) A normal modal multi-conclusion consequence relation $\mathcal{S}$ has the *finite model property (fmp)* if for each rule ρ with $\mathcal{S} \not\vdash \rho$, there exists a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \models \mathcal{S}$ and $\mathfrak{A} \not\models \rho$.
(2) A normal modal logic L has the *finite model property (fmp)* if for each formula φ with $L \not\vdash \varphi$, there exists a finite modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \models L$ and $\mathfrak{A} \not\models \varphi$.

THEOREM 7.8.

(1) *Every stable normal modal multi-conclusion consequence relation has the finite model property.*
(2) *Every stable normal modal logic has the finite model property.*

PROOF. (1). Let $\mathcal{S}$ be a stable normal modal multi-conclusion consequence relation and let ρ be a multi-conclusion modal rule such that $\mathcal{S} \not\vdash \rho$. Then there is a

modal algebra $\mathfrak{A} = (A, \Diamond)$ such that $\mathfrak{A} \models \mathcal{S}$ and $\mathfrak{A} \not\models \rho$. The proof of Theorem 5.1 yields a finite modal algebra $\mathfrak{A}' = (A', \Diamond')$ such that $\mathfrak{A}' \not\models \rho$ and the embedding $A' \rightarrowtail A$ is stable. Since $\mathcal{S}$ is stable, $\mathfrak{A}' \models \mathcal{S}$. Thus, $\mathcal{S}$ has the fmp.

(2). This is an immediate consequence of (1). $\dashv$

REMARK 7.9. We will give numerous examples of stable multi-conclusion consequence relations and stable logics in Section 8. In fact, there are continuum many such systems. Their theory is developed in [5]. For the theory of stable superintuitionistic logics and stable intuitionistic multi-conclusion consequence relations consult [3, 4]. Proof-theoretic properties of stable logics are studied in [11], where it is shown that every stable modal logic has the bounded proof property.

We next turn to Jankov rules. We call a normal modal multi-conclusion consequence relation $\mathcal{S}$ *splitting* if there is a normal modal multi-conclusion consequence relation $\mathcal{T}$ such that $\mathcal{S} \not\subseteq \mathcal{T}$ and for each normal modal multi-conclusion consequence relation $\mathcal{U}$, we have $\mathcal{S} \subseteq \mathcal{U}$ or $\mathcal{U} \subseteq \mathcal{T}$. The pair $(\mathcal{S}, \mathcal{T})$ is called a *splitting pair*. We call a normal modal multi-conclusion consequence relation *join splitting* if it is a join (in the lattice $\mathrm{NExt}\mathbf{S_K}$) of splitting normal modal multi-conclusion consequence relations. Splitting and join splitting normal modal logics are defined similarly.

For a modal algebra $\mathfrak{A} = (A, \Diamond)$, let $\mathcal{S}(\mathfrak{A}) = \{\rho : \mathfrak{A} \models \rho\}$ and $L(\mathfrak{A}) = \{\varphi : \mathfrak{A} \models \varphi\}$. Then it is straightforward to verify that $\mathcal{S}(\mathfrak{A})$ is a normal modal multi-conclusion consequence relation and $L(\mathfrak{A})$ is a normal modal logic. The following theorem was first proved by Jeřábek [21, Thm. 6.5] using model-theoretic technique.

THEOREM 7.10. *Let $\mathcal{S}$ be a normal modal multi-conclusion consequence relation.*

(1) *S is splitting iff S is axiomatizable by a Jankov rule.*
(2) *S is join splitting iff S is axiomatizable by Jankov rules.*

PROOF. (1). First suppose that $\mathcal{S}$ is axiomatizable by a Jankov rule $\chi(\mathfrak{A})$. It is sufficient to show that $(\mathcal{S}, \mathcal{S}(\mathfrak{A}))$ is a splitting pair in $\mathrm{NExt}\mathbf{S_K}$. Since $\mathfrak{A} \not\models \chi(\mathfrak{A})$, we have $\mathcal{S} \not\subseteq \mathcal{S}(\mathfrak{A})$. Let $\mathcal{T} \in \mathrm{NExt}\mathbf{S_K}$ with $\mathcal{S} \not\subseteq \mathcal{T}$. Then there is a modal algebra $\mathfrak{B} = (B, \Diamond)$ such that $\mathfrak{B} \models \mathcal{T}$ and $\mathfrak{B} \not\models \mathcal{S}$. Therefore, $\mathfrak{B} \not\models \chi(\mathfrak{A})$. By Proposition 7.1(2), there is a 1-1 modal homomorphism $A \rightarrowtail B$. Thus, $\mathcal{T} \subseteq \mathcal{S}(\mathfrak{B}) \subseteq \mathcal{S}(\mathfrak{A})$, and hence $(\mathcal{S}, \mathcal{S}(\mathfrak{A}))$ is a splitting pair in $\mathrm{NExt}\mathbf{S_K}$.

Next suppose that $\mathcal{S}$ is splitting in $\mathrm{NExt}\mathbf{S_K}$. Then there is $\mathcal{T} \in \mathrm{NExt}\mathbf{S_K}$ such that $(\mathcal{S}, \mathcal{T})$ is a splitting pair. Therefore, $\mathcal{T}$ is a completely meet-prime element of $\mathrm{NExt}\mathbf{S_K}$. Thus, since $\mathbf{S_K}$ has the fmp, there is a finite modal algebra $\mathfrak{B} = (B, \Diamond)$ such that $\mathcal{S}(\mathfrak{B}) \subseteq \mathcal{T}$ (see, e.g., [27, Sec. 4]). As $\mathfrak{B}$ is finite, we see that $\mathcal{T} = \mathcal{S}(\mathfrak{A})$ for some subalgebra $\mathfrak{A}$ of $\mathfrak{B}$. This yields that $(\mathcal{S}, \mathcal{S}(\mathfrak{A}))$ is a splitting pair. By the above argument, $(\mathbf{S_K} + \chi(\mathfrak{A}), \mathcal{S}(\mathfrak{A}))$ is also a splitting pair. Thus, $\mathcal{S} = \mathbf{S_K} + \chi(\mathfrak{A})$.

(2). This follows from (1). $\dashv$

REMARK 7.11. In [2, 3, 8, 9] the theory of algebra-based (or equivalently frame-based) formulas is developed and a general criterion when a logic is axiomatized by these formulas is established. Such well-known classes of formulas as Jankov formulas, stable formulas, subframe formulas and others are particular instances of algebra-based formulas. This theory has a natural generalization to the theory of algebra-based (or equivalently frame-based) rules. We will not pursue it here, and only note that stable rules and Jankov rules are particular instances of these algebra-based rules.

We call a modal algebra $\mathfrak{A} = (A, \Diamond)$ of *height* $\leq n$ if $\square^{n+1}0 = 1$ (equivalently $\Diamond^{n+1}1 = 0$).

LEMMA 7.12. *Let $\mathfrak{A} = (A, \Diamond)$ be of height $\leq n$ and $a, b \in A$ with $\blacksquare_n a \nleq b$. Then there is a s.i. modal algebra $\mathfrak{B} = (B, \Diamond)$ and an onto modal homomorphism $h : A \twoheadrightarrow B$ such that $h(\blacksquare_n a) = 1$ and $h(b) \neq 1$.*

PROOF. The proof is similar to that of [1, Lem. 4.1] and we only sketch it. Let F be the filter generated by $\blacksquare_n a$. Then $\blacksquare_n a \in F$ and $b \notin F$. If $x \in F$, then $\blacksquare_n a \leq x$, so $\square \blacksquare_n a \leq \square x$. Since $\mathfrak{A}$ is of height $\leq n$, we have $\blacksquare_n a \leq \square \blacksquare_n a$. Therefore, $\blacksquare_n a \leq \square x$, and hence F is a modal filter. By Zorn's lemma, there is a maximal modal filter G such that $\blacksquare_n a \in G$ and $b \notin G$. Since G is maximal with this property, the quotient algebra $\mathfrak{B} = \mathfrak{A}/G$ is s.i. Let $h : A \twoheadrightarrow B$ be the quotient map. Then $h(\blacksquare_n a) = 1$ and $h(b) \neq 1$.   ⊣

DEFINITION 7.13. Let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. modal algebra of height $\leq n$, and let $D \subseteq A$. For each $a \in A$ we introduce a new propositional letter p_a and define the *stable canonical formula* $\varepsilon(\mathfrak{A}, D)$ associated with A and D as follows:

$$\varepsilon(\mathfrak{A}, D) = \left(\square^{n+1}\bot \wedge \bigwedge \{\blacksquare_n \gamma : \gamma \in \Gamma\} \right) \to \bigvee \{\blacksquare_n \delta : \delta \in \Delta\}$$
$$= \left(\square^{n+1}\bot \wedge \blacksquare_n \bigwedge \Gamma \right) \to \bigvee \{\blacksquare_n \delta : \delta \in \Delta\},$$

where Γ and Δ are as in Definition 5.2.

THEOREM 7.14. *Let $\mathfrak{A} = (A, \Diamond)$ be a finite s.i. modal algebra of height $\leq n$, $D \subseteq A$, and $\mathfrak{B} = (B, \Diamond)$ be a modal algebra. Then $\mathfrak{B} \not\models \varepsilon(\mathfrak{A}, D)$ iff there is a s.i. homomorphic image $\mathfrak{C} = (C, \Diamond)$ of $\mathfrak{B}$ and a stable embedding $h : A \rightarrowtail C$ satisfying* (CDC) *for D.*

PROOF. The proof follows the same pattern as the proof of Theorem 6.8. First suppose that there is a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a stable embedding $h : A \rightarrowtail C$ satisfying CDC for D. Define a valuation V_A on A by $V_A(p_a) = a$ for each $a \in A$. Then $V_A(\gamma) = 1_A$ for each $\gamma \in \Gamma$ and $V_A(\delta) \neq 1_A$ for each $\delta \in \Delta$. Therefore, $V_A(\blacksquare_n \bigwedge \Gamma) = 1_A$ and $V_A(\blacksquare_n \delta) \neq 1_A$ for each $\delta \in \Delta$. Moreover, since $\mathfrak{A}$ has height $\leq n$, we have $V_A(\square^{n+1}\bot) = \square^{n+1}0_A = 1_A$. As $\mathfrak{A}$ is s.i., it has an opremum c. Let $a \neq 1$. Then there is $m \in \omega$ with $\blacksquare_m a \leq c$. Because $\mathfrak{A}$ has height $\leq n$, we see that $\blacksquare_n a \leq \blacksquare_m a$, yielding $\blacksquare_n a \leq c$. Thus, $\bigvee \{V_A(\blacksquare_n \delta) : \delta \in \Delta\} \leq c$, and hence $\mathfrak{A} \not\models \varepsilon(\mathfrak{A}, D)$. Next define a valuation V_C on C by $V_C(p_a) = h(V_A(p_a)) = h(a)$ for each $a \in A$. The same argument as in the proof of Theorem 5.4 shows that $V_C(\gamma) = 1_C$ for each $\gamma \in \Gamma$ and $V_C(\delta) \neq 1_C$ for each $\delta \in \Delta$. Therefore, $V_C(\blacksquare_n \bigwedge \Gamma) = 1_C$ and $V_C(\blacksquare_n \delta) \neq 1_C$ for each $\delta \in \Delta$. Moreover, $V_C(\square^{n+1}\bot) = \square^{n+1}V_C(\bot) = \square^{n+1}0_C = \square^{n+1}h(0_A) \geq h(\square^{n+1}0_A) = h(1_A) = 1_C$. Since $\mathfrak{C}$ is s.i., it has an opremum, and the same argument as above yields that $\bigvee \{V_C(\blacksquare_n \delta) : \delta \in \Delta\}$ is underneath the opremum, hence $\mathfrak{C} \not\models \varepsilon(\mathfrak{A}, D)$. As $\mathfrak{C}$ is a homomorphic image of $\mathfrak{B}$, we conclude that $\mathfrak{B} \not\models \varepsilon(\mathfrak{A}, D)$.

Conversely, suppose that $\mathfrak{B} \not\models \varepsilon(\mathfrak{A}, D)$. Let $\mathfrak{X} = (X, R)$ be the dual of $\mathfrak{B}$. Then there exist a valuation V on $\mathcal{B}$ and $x \in X$ such that $x \in \beta(V(\square^{n+1}\bot \wedge \blacksquare_n \bigwedge \Gamma))$ but $x \notin \beta(V(\bigvee \{\blacksquare_n \delta : \delta \in \Delta\}))$. Since $x \in \beta(V(\square^{n+1}\bot))$, we have $R^\omega[x] = R^n[x]$. Therefore, $R^\omega[x]$ is a closed up-set of $\mathfrak{X}$, hence its dual modal algebra $\mathfrak{B}'$ is a homomorphic image of $\mathfrak{B}$. From $R^\omega[x] = R^n[x]$ it follows that $\mathfrak{B}'$ is of height $\leq n$. It is also clear that $V(\blacksquare_n \bigwedge \Gamma) = 1_{B'}$, $V(\bigvee \{\blacksquare_n \delta : \delta \in \Delta\}) \neq 1_{B'}$. By Lemma 7.12,

there is a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}'$, and hence of $\mathfrak{B}$, and a valuation V_C on C such that $V_C(\blacksquare_n \bigwedge \Gamma) = 1_C$ and $V_C(\bigvee\{\blacksquare_n\delta : \delta \in \Delta\}) \neq 1_C$. Next we define a map $h : A \to C$ by $h(a) = V_C(p_a)$ for each $a \in A$. The proof of Theorem 5.4 then shows that h is a stable embedding satisfying CDC for D. $\dashv$

We call a modal algebra $\mathfrak{A} = (A, \Diamond)$ of *finite height* if $\mathfrak{A}$ is of height $\leq n$ for some n. If $\mathfrak{A}$ is a finite s.i. modal algebra of finite height, then we denote $\varepsilon(\mathfrak{A}, A)$ by $\varepsilon(\mathfrak{A})$ and call it the *Jankov formula* of $\mathfrak{A}$. We next give an alternate proof of Blok's theorem [13].

THEOREM 7.15. *Let L be a normal modal logic.*

(1) *L is a splitting logic iff L is axiomatizable by the Jankov formula of a finite s.i. modal algebra of finite height.*

(2) *L is a join splitting logic iff L is axiomatizable by Jankov formulas of finite s.i. modal algebras of finite height.*

PROOF. (1). First suppose that $L = \mathbf{K} + \varepsilon(\mathfrak{A})$ for some finite s.i. modal algebra $\mathfrak{A} = (A, \Diamond)$ of finite height. Then $\varepsilon(\mathfrak{A}) \in L$. On the other hand, by Theorem 7.14, $\mathfrak{A} \not\models \varepsilon(\mathfrak{A})$. Therefore, $\varepsilon(\mathfrak{A}) \notin L(\mathfrak{A})$, and hence $L \not\subseteq L(\mathfrak{A})$. Let M be a normal modal logic such that $L \not\subseteq M$. Then there is a modal algebra $\mathfrak{B} = (B, \Diamond)$ such that $\mathfrak{B} \models M$ and $\mathfrak{B} \not\models L$. This gives $\mathfrak{B} \not\models \varepsilon(\mathfrak{A})$. By Theorem 7.14, $\mathfrak{A}$ is isomorphic to a subalgebra of a s.i. homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$. Thus, $M \subseteq L(\mathfrak{B}) \subseteq L(\mathfrak{C}) \subseteq L(\mathfrak{A})$. Consequently, $(L, L(\mathfrak{A}))$ is a splitting pair.

Conversely, suppose that L is a splitting logic. Then there is a normal modal logic M such that (L, M) is a splitting pair. It is well-known (see, e.g., [15, Cor. 3.29]) that $\mathbf{K}$ is the modal logic of all finite irreflexive trees. Since (L, M) is a splitting pair, M is a completely meet-prime element of NExt$\mathbf{K}$. Therefore, there is the dual modal algebra $\mathfrak{E}$ of some finite irreflexive tree such that $L(\mathfrak{E}) \subseteq M$. This, by Jónsson's lemma, means that $M = L(\mathfrak{A})$, where $\mathfrak{A}$ is a s.i. homomorphic image of a subalgebra of $\mathfrak{E}$. Thus, $(L, L(\mathfrak{A}))$ is a splitting pair. Because $\mathfrak{E}$ is of finite height, so is $\mathfrak{A}$. By the above argument, $(\mathbf{K} + \varepsilon(\mathfrak{A}), L(\mathfrak{A}))$ is also a splitting pair. Consequently, $L = \mathbf{K} + \varepsilon(\mathfrak{A})$.

(2). This follows from (1). $\dashv$

REMARK 7.16. As follows from Remarks 5.8 and 6.14, stable canonical rules and stable canonical formulas for $\mathbf{K4}$ can be defined directly for finite modal spaces (finite Kripke frames) without using algebraic terminology. The same is true for Jankov formulas, see [15, Sec. 10.5].

§8. **Examples.** In this final section we show how to axiomatize some well-known modal logics and multi-conclusion consequence relations via stable canonical rules and formulas. More examples can be found in [5].

We will be mostly working with modal spaces rather than modal algebras since our proofs will rely on a geometric intuition of modal spaces. Suppose $\mathfrak{A}$ is a finite modal algebra, $D \subseteq A$, $\mathfrak{X}$ is the dual of $\mathfrak{A}$, and $\mathfrak{D} = \{\beta(a) : a \in D\}$. To simplify notation, we write $\rho(\mathfrak{X}, \mathfrak{D})$ instead of $\rho(\mathfrak{A}, D)$. If $\mathfrak{D} = \varnothing$, then we simply write $\rho(\mathfrak{X})$. Also, if $\mathfrak{A}$ is a finite s.i. $\mathbf{K4}$-algebra, then we write $\gamma(\mathfrak{X}, \mathfrak{D})$ and $\gamma(\mathfrak{X})$ instead of $\gamma(\mathfrak{A}, D)$ and $\gamma(\mathfrak{A})$, respectively. When drawing modal spaces, we use the standard convention that $\bullet$ depicts an irreflexive point, while $\circ$ depicts a reflexive point.

Let $\mathfrak{X}$ be a modal space and $x, y \in X$. We say that there is an *R-path* between x and y if there is a finite sequence $z_0, \ldots, z_n$ such that $x = z_0$, $y = z_n$, and $z_i \mathrel{R} z_{i+1}$ or $z_{i+1} \mathrel{R} z_i$ for $i < n$. We call $\mathfrak{X}$ *connected* provided it is nonempty and there is an R-path between any two $x, y \in X$. Let **Con** be the class of finite modal algebras whose dual spaces are connected, and let $\mathcal{S}(\textbf{Con})$ be the stable multi-conclusion consequence relation corresponding to the universal class generated by **Con**. Similarly, let **Rooted** be the class of finite modal algebras whose dual spaces are rooted, and let $\mathcal{S}(\textbf{Rooted})$ be the stable multi-conclusion consequence relation corresponding to the universal class generated by **Rooted**.

We let **Rules** be the set of all multi-conclusion modal rules. Clearly **Rules** corresponds to the empty class of modal algebras. We denote the stable canonical rule of the empty modal space by $\rho(\ )$, and note that it corresponds to the stable canonical rule of the trivial modal algebra. We also let **Form** be the set of all modal formulas, and $\textbf{S}_{\textbf{Form}} := \Sigma(\textbf{Form})$ be the least normal modal multi-conclusion consequence relation containing $/\varphi$ for each $\varphi \in \textbf{Form}$. Then $\textbf{S}_{\textbf{Form}}$ corresponds to the universal class consisting of the trivial algebra and its isomorphic copies.

THEOREM 8.1.

(1) $\textbf{S}_{\textbf{Form}} = \textbf{S}_{\textbf{K}} + \rho(\circ)$.
(2) $\textbf{Rules} = \textbf{S}_{\textbf{K}} + \rho(\ ) + \rho(\circ)$.
(3) $\mathcal{S}(\textbf{Con}) = \textbf{S}_{\textbf{K}} + \rho(\ ) + \rho(\circ\ \circ)$.
(4) $\mathcal{S}(\textbf{Rooted}) = \textbf{S}_{\textbf{K}} + \rho(\ ) + \rho(\circ\ \circ) + \rho\left(\vcenter{\hbox{$\circ\overset{\circ}{\underset{\circ}{\vphantom{x}}}$}} \right)$.

PROOF. (1). It is easy to see that a modal space $\mathfrak{X}$ can be mapped via a stable map onto $\circ$ iff $\mathfrak{X}$ is nonempty. Therefore, for a modal algebra $\mathfrak{A}$, we have $\mathfrak{A} \not\models \rho(\circ)$ iff $\mathfrak{A}$ is nontrivial. Thus, the universal class corresponding to $\textbf{S}_{\textbf{K}} + \rho(\circ)$ consists of the trivial algebra and its isomorphic copies. Consequently, $\textbf{S}_{\textbf{Form}} = \textbf{S}_{\textbf{K}} + \rho(\circ)$.

(2). For a modal space $\mathfrak{X}$, if $\mathfrak{X}$ is empty, then $\mathfrak{X}$ can be mapped via a stable map onto the empty modal space, and if $\mathfrak{X}$ is nonempty, then $\mathfrak{X}$ can be mapped via a stable map onto $\circ$. Therefore, for a modal algebra $\mathfrak{A}$, if $\mathfrak{A}$ is trivial, then $\mathfrak{A} \not\models \rho(\ )$, and if $\mathfrak{A}$ is nontrivial, then $\mathfrak{A} \not\models \rho(\circ)$. Thus, the universal class corresponding to $\textbf{S}_{\textbf{K}} + \rho(\ ) + \rho(\circ)$ is empty. Consequently, $\textbf{Rules} = \textbf{S}_{\textbf{K}} + \rho(\ ) + \rho(\circ)$.

(3). Let $\mathfrak{A}$ be a finite modal algebra and let $\mathfrak{X}$ be the dual of $\mathfrak{A}$. Clearly $\mathfrak{A}$ is trivial iff $\mathfrak{A} \not\models \rho(\ )$. For nontrivial $\mathfrak{A}$, we show that $\mathfrak{A} \in \textbf{Con}$ iff $\mathfrak{A} \models \rho(\circ\ \circ)$. First suppose that $\mathfrak{A} \notin \textbf{Con}$. Then $\mathfrak{X}$ is not connected. Therefore, there are distinct $x, y \in X$ such that there is no R-path between them. Thus, X can be partitioned into two up-sets U, V such that $x \in U$ and $y \in V$. Define $f : X \to \circ\ \circ$ by sending U to one reflexive point and V to another. It is easy to see that f is a stable map, so $\circ\ \circ$ is a stable image of $\mathfrak{X}$, yielding that $\mathfrak{A} \not\models \rho(\circ\ \circ)$. Next suppose that $\mathfrak{A} \not\models \rho(\circ\ \circ)$. Then there is an onto stable map $f : X \to \circ\ \circ$. Let U be the inverse image of one point and V the inverse image of the other point. Pick $x \in U$ and $y \in V$. Since U and V are disjoint, x and y are distinct. Moreover, since U and V are up-sets, there is no R-path between x and y. Therefore, $\mathfrak{X}$ is not connected, and so $\mathfrak{A} \notin \textbf{Con}$. Thus, **Con** coincides with the class of finite modal algebras validating $\textbf{S}_{\textbf{K}} + \rho(\ ) + \rho(\circ\ \circ)$. By Theorem 7.4, $\textbf{S}_{\textbf{K}} + \rho(\ ) + \rho(\circ\ \circ)$ is stable, so by Theorem 7.8, it has the finite model property. From this we conclude that $\mathcal{S}(\textbf{Con}) = \textbf{S}_{\textbf{K}} + \rho(\ ) + \rho(\circ\ \circ)$.

(4). The proof is similar to that of (3). Let $\mathfrak{A}$ be a finite modal algebra and let $\mathfrak{X}$ be its dual. Clearly $\mathfrak{A}$ is trivial iff $\mathfrak{A} \not\models \rho(\ )$. For nontrivial $\mathfrak{A}$, we show that $\mathfrak{A} \in \mathbf{Rooted}$ iff $\mathfrak{A} \models \rho(\circ\ \circ)$ and $\mathfrak{A} \models \rho\left(\overset{\wedge}{\underset{\circ\ \ \circ}{}}\right)$. First suppose that $\mathfrak{A} \notin \mathbf{Rooted}$. If $\mathfrak{X}$ is not connected, then by (3), $\mathfrak{X}$ is mapped via a stable map onto $\circ\ \circ$, so $\mathfrak{A} \not\models \rho(\circ\ \circ)$. Suppose that $\mathfrak{X}$ is connected. Since $\mathfrak{X}$ is not rooted, there are $x, y \in X$ such that there is an R-path between x and y, but $(R^\omega)^{-1}[x] \cap (R^\omega)^{-1}[y] = \varnothing$. Define $f : X \to \overset{\wedge}{\underset{\circ\ \ \circ}{}}$ by sending $(R^\omega)^{-1}[x]$ to one minimal point, $(R^\omega)^{-1}[y]$ to another minimal point, and the rest to the top. It is easy to check that f is an onto stable map, so $\mathfrak{A} \not\models \rho\left(\overset{\wedge}{\underset{\circ\ \ \circ}{}}\right)$. Conversely, suppose that $\mathfrak{A} \not\models \rho(\circ\ \circ)$ or $\mathfrak{A} \not\models \rho\left(\overset{\wedge}{\underset{\circ\ \ \circ}{}}\right)$. If $\mathfrak{A} \not\models \rho(\circ\ \circ)$, then by (3), $\mathfrak{X}$ is not connected, hence not rooted, yielding that $\mathfrak{A} \notin \mathbf{Rooted}$. Suppose that $\mathfrak{A} \not\models \rho\left(\overset{\wedge}{\underset{\circ\ \ \circ}{}}\right)$. Then there is an onto stable map $f : X \to \overset{\wedge}{\underset{\circ\ \ \circ}{}}$. If $\mathfrak{X}$ were rooted, with x a root of $\mathfrak{X}$, then for each $y \in X$, we would have $f(x) R^\omega f(y)$. This is a contradiction since $f(x)$ is not R^ω-related to at least one of the minimal points of $\overset{\wedge}{\underset{\circ\ \ \circ}{}}$. Therefore, $\mathfrak{A} \notin \mathbf{Rooted}$. Now the same argument as in (3) yields that $\mathcal{S}(\mathbf{Rooted}) = \mathbf{S_K} + \rho(\ ) + \rho(\circ\ \circ) + \rho\left(\overset{\wedge}{\underset{\circ\ \ \circ}{}}\right)$. $\dashv$

REMARK 8.2. By [19, Sec. 3.2], we can translate multi-conclusion modal rules into formulas of the modal language $\mathcal{L}_U$ enriched with the universal modality $[\mathbf{u}]$ by reading Γ/Δ as $\bigwedge\{[\mathbf{u}]\gamma : \gamma \in \Gamma\} \to \bigvee\{[\mathbf{u}]\delta : \delta \in \Delta\}$. As follows from [34], connectedness is modally definable in $\mathcal{L}_U$ by the formula $[\mathbf{u}](\Diamond p \to \Box p) \to ([\mathbf{u}]p \vee [\mathbf{u}]\neg p)$. Consequently, $\mathcal{S}(\mathbf{Con})$ can alternatively be axiomatized by the rule $\Diamond p \to \Box p / p, \neg p$.

Next we turn to some examples of normal modal logics. Let $\mathbf{KD} = \mathbf{K} + (\Box p \to \Diamond p)$ be the logic of serial frames ($\forall x \exists y : x R y$) and let $\mathbf{KT} = \mathbf{K} + (p \to \Diamond p)$ be the logic of reflexive frames.

THEOREM 8.3.

(1) $\mathbf{Form} = \Lambda\left(\mathbf{S_K} + \rho(\circ)\right)$.
(2) $\mathbf{KD} = \Lambda\left(\mathbf{S_K} + \rho(\bullet) + \rho(\circ\!\!-\!\!\twoheadrightarrow\!\bullet)\right)$.
(3) $\mathbf{KT} = \Lambda\left(\mathbf{S_K} + \rho(\bullet) + \rho(\bullet\!\twoheadleftarrow\!\!-\!\!\circ)\right)$.

PROOF. (1). By Theorem 8.1(1), $\mathcal{U}\left(\mathbf{S_K} + \rho(\circ)\right)$ consists of the trivial algebra and its isomorphic copies, so $\mathcal{U}\left(\mathbf{S_K} + \rho(\circ)\right)$ is a variety. On the other hand, it is well-known that this is exactly the variety corresponding to the inconsistent logic $\mathbf{Form}$. The result follows.

(2). Since both $\mathbf{KD}$ and $\Lambda\left(\mathbf{S_K} + \rho(\bullet) + \rho(\circ\!\!-\!\!\twoheadrightarrow\!\bullet)\right)$ have the finite model property, it is sufficient to show that a finite modal algebra $\mathfrak{A}$ is a $\mathbf{KD}$-algebra iff $\mathfrak{A} \models \rho(\bullet), \rho(\circ\!\!-\!\!\twoheadrightarrow\!\bullet)$. Let $\mathfrak{X}$ be the dual of $\mathfrak{A}$. First suppose that $\mathfrak{A}$ is a $\mathbf{KD}$-algebra. Then $\mathfrak{X}$ is serial, and it is easy to see that it cannot be mapped via a stable map onto $\bullet$ or $\circ\!\!-\!\!\twoheadrightarrow\!\bullet$. Therefore, $\mathfrak{A} \models \rho(\bullet), \rho(\circ\!\!-\!\!\twoheadrightarrow\!\bullet)$. Next suppose that $\mathfrak{A}$ is not a $\mathbf{KD}$-algebra. Then there is $x \in X$ such that $R[x] = \varnothing$. If $X = \{x\}$, then $\mathfrak{X}$ is isomorphic to $\bullet$, so

$\mathfrak{A} \not\models \rho(\bullet)$. Otherwise define $f : X \to \circ\!\longrightarrow\!\bullet$ by sending x to the irreflexive point and the rest of X to the reflexive point of $\circ\!\longrightarrow\!\bullet$. It is easy to see that this is a stable map. Therefore, $\mathfrak{A} \not\models \rho(\circ\!\longrightarrow\!\bullet)$. The result follows.

(3). The proof follows the same pattern as that of (2). Since both **KT** and $\Lambda\left(\mathbf{S_K} + \rho(\bullet) + \rho(\bullet\!\longleftarrow\!\!\!\circ)\right)$ have the finite model property, it is sufficient to show that a finite modal algebra $\mathfrak{A}$ is a **KT**-algebra iff $\mathfrak{A} \models \rho(\bullet), \rho(\bullet\!\longleftarrow\!\!\!\circ)$. Let $\mathfrak{X}$ be the dual of $\mathfrak{A}$. First suppose that $\mathfrak{A}$ is a **KT**-algebra. Then $\mathfrak{X}$ is reflexive, and it is easy to see that it cannot be mapped via a stable map onto $\bullet$ or $\bullet\!\longleftarrow\!\!\!\circ$. Therefore, $\mathfrak{A} \models \rho(\bullet), \rho(\bullet\!\longleftarrow\!\!\!\circ)$. Next suppose that $\mathfrak{A}$ is not a **KT**-algebra. Then $\mathfrak{X}$ contains an irreflexive point x. If $X = \{x\}$, then $\mathfrak{X}$ is isomorphic to $\bullet$, so $\mathfrak{A} \not\models \rho(\bullet)$. Otherwise define $f : X \to \bullet\!\longleftarrow\!\!\!\circ$ by sending x to the irreflexive point and the rest to the reflexive point of $\bullet\!\longleftarrow\!\!\!\circ$. It is easy to see that this is a stable map. Therefore, $\mathfrak{A} \not\models \rho(\bullet\!\longleftarrow\!\!\!\circ)$, and the result follows. $\dashv$

REMARK 8.4. Let $\mathbf{S_{KD}} := \Sigma(\mathbf{KD})$ be the least normal modal multi-conclusion consequence relation containing $/\varphi$ for each $\varphi \in \mathbf{KD}$, and similarly let $\mathbf{S_{KT}} := \Sigma(\mathbf{KT})$. The proof of Theorem 8.3 shows that $\mathbf{S_{KD}} = \mathbf{S_K} + \rho(\bullet) + \rho(\circ\!\longrightarrow\!\bullet)$ and $\mathbf{S_{KT}} = \mathbf{S_K} + \rho(\bullet) + \rho(\bullet\!\longleftarrow\!\!\!\circ)$.

It follows from Theorems 8.1 and 8.3 that $\mathcal{S}(\mathbf{Con})$ and $\mathcal{S}(\mathbf{Rooted})$ are stable multi-conclusion consequence relations and **KD** and **KT** are stable modal logics. There are infinitely many stable multi-conclusion consequence relations and stable modal logics. For example, for a finite modal algebra $\mathfrak{A}$, let $\mathbf{Stable}(\mathfrak{A})$ be the class of modal algebras that are isomorphic to stable subalgebras of $\mathfrak{A}$. Obviously $\mathbf{Stable}(\mathfrak{A})$ is stable and is closed under isomorphisms and subalgebras. Since each member of $\mathbf{Stable}(\mathfrak{A})$ is finite and there are only finitely many nonisomorphic members of $\mathbf{Stable}(\mathfrak{A})$, it is also clear that $\mathbf{Stable}(\mathfrak{A})$ is closed under ultraproducts. Therefore, $\mathbf{Stable}(\mathfrak{A})$ is a stable universal class. Thus, the multi-conclusion consequence relation corresponding to $\mathbf{Stable}(\mathfrak{A})$ and the logic corresponding to the variety generated by $\mathbf{Stable}(\mathfrak{A})$ are stable. In fact, there are continuum many stable multi-conclusion consequence relations and stable modal logics [5]. On the other hand, many well-known systems are not stable. For example, as it is shown in [5], the following well-known systems **K4**, **S4**, **GL**, **S4.Grz**, and **S4.1** are not stable. This suggests the following modification of the notion of stability.

DEFINITION 8.5. Let L be a normal modal logic, $\mathbf{S}_L$ be the corresponding normal modal multi-conclusion consequence relation, and $\mathcal{V}(L)$ be the variety corresponding to L.

(1) We call a class $\mathcal{K} \subseteq \mathcal{V}(L)$ of modal algebras *stable within* $\mathcal{V}(L)$ provided for modal algebras $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}(L)$, if $\mathfrak{B} \in \mathcal{K}$ and there is a stable embedding $A \rightarrowtail B$, then $\mathfrak{A} \in \mathcal{K}$.

(2) We call a normal extension $\mathcal{S}$ of $\mathbf{S}_L$ *stable over* $\mathbf{S}_L$ provided the universal class $\mathcal{U}(\mathcal{S})$ is stable within $\mathcal{V}(L)$.

(3) We call a normal extension L' of L *stable over* L provided $\mathcal{V}(L')$ is generated by a universal class which is stable within $\mathcal{V}(L)$.

This modified notion of stability is studied in [5], where it is shown that many well-known modal logics are stable over **K4** and **S4**. Below we give a table of some of stable logics over **K4** and **S4**.

Form	=	$\mathbf{K4} + \gamma(\circ)$	**Form**	=	$\mathbf{S4} + \gamma(\circ)$
K4.2	=	$\mathbf{K4} + \gamma(\text{⟨⟩})$	**S4.2**	=	$\mathbf{S4} + \gamma(\text{⟨⟩})$
K4.3	=	$\mathbf{K4} + \gamma(\text{⟨⟩}) + \gamma(\text{◇})$	**S4.3**	=	$\mathbf{S4} + \gamma(\text{⟨⟩}) + \gamma(\text{◇})$
K4.BW$_n$	=	$\mathbf{K4} + \gamma(\text{∨}) + \gamma(\text{◇})$	**S4.BW$_n$**	=	$\mathbf{S4} + \gamma(\text{∨}) + \gamma(\text{◇})$
K4.BTW$_n$	=	$\mathbf{K4} + \gamma(\text{∨})$	**S4.BTW$_n$**	=	$\mathbf{S4} + \gamma(\text{∨})$
K4B	=	$\mathbf{K4} + \gamma(\text{⦿})$	**S5**	=	$\mathbf{S4} + \gamma(\text{⦿})$
S4	=	$\mathbf{K4} + \gamma(\bullet) + \gamma(\text{⦿})$			
D4	=	$\mathbf{K4} + \gamma(\bullet)$			

TABLE 1. Axiomatization of some stable logics over **K4** and **S4**.

To this we add that the disjunction rule $\Box^+ p \vee \Box^+ q / p, q$ is stable over $\mathbf{S_{K4}}$. Let $\mathbf{Disj} = \mathbf{S_{K4}} + (\Box^+ p \vee \Box^+ q / p, q)$.

THEOREM 8.6. $\mathbf{Disj} = \mathbf{S_{K4}} + \rho(\circ\ \circ) + \rho\left(\text{◇}\right)$.

PROOF. Let $\mathfrak{A}$ be a **K4**-algebra and let $\mathfrak{X}$ be the dual of $\mathfrak{A}$. It is easy to see that $\mathfrak{A} \models \mathbf{Disj}$ iff $\Diamond^+ a \wedge \Diamond^+ b = 0$ implies $a = 0$ or $b = 0$. We show that this happens iff $\mathfrak{X}$ is rooted. First suppose that $\mathfrak{X}$ is rooted and that x is a root of $\mathfrak{X}$. If $a, b \neq 0$, then $\beta(a), \beta(b) \neq \varnothing$. Since $\mathfrak{X}$ is transitive, this implies $x \in (R^+)^{-1}\beta(a), (R^+)^{-1}\beta(b)$. Therefore, $x \in \beta(\Diamond^+ a \wedge \Diamond^+ b)$. Thus, $\Diamond^+ a \wedge \Diamond^+ b \neq 0$. Conversely, suppose $\mathfrak{X}$ is not rooted. Then there exist $x, y \in X$ such that $(R^+)^{-1}[x] \cap (R^+)^{-1}[y] = \varnothing$. Since $\mathfrak{X}$ is a transitive modal space, there exist nonempty clopen sets U, V such that $(R^+)^{-1}[U] \cap (R^+)^{-1}[V] = \varnothing$. Therefore, there exist $a, b \neq 0$ with $\Diamond^+ a \wedge \Diamond^+ b = 0$. Thus, $\mathfrak{A} \models \mathbf{Disj}$ iff $\mathfrak{X}$ is rooted. Since a stable image of a rooted space is rooted, we obtain that $\mathbf{Disj}$ is stable over $\mathbf{S_{K4}}$. By **K4**-analogues of Theorems 7.4 and 7.8 (see [5] for details), both $\mathbf{Disj}$ and $\mathbf{S_{K4}} + \rho(\circ\ \circ) + \rho\left(\text{◇}\right)$ have the finite model property. Now apply Theorem 8.1(4) to complete the proof. ⊣

On the other hand, the Löb rule $\Box p \rightarrow p / p$ is not stable over $\mathbf{S_{K4}}$, and neither is the Grzegorczyk rule $\Box(p \rightarrow \Box p) \rightarrow p / p$ over $\mathbf{S_{S4}}$ (see [5]). Since the Löb rule is equivalent to the Löb formula $\Box(\Box p \rightarrow p) \rightarrow \Box p$ and the Grzegorczyk rule is equivalent to the Grzegorczyk formula $\Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p$, the Gödel-Löb logic **GL** is not stable over **K4** and the Gzegorczyk logic **S4.Grz** is not stable over **S4**. As we already pointed out, **S4.1** is also not stable over **S4**. We conclude the paper by axiomatizing **GL**, **S4.Grz**, and **S4.1** via stable canonical formulas.

THEOREM 8.7.

(1) $\mathbf{GL} = \mathbf{K4} + \gamma\left(\begin{matrix}\circ \\ d\end{matrix}\right) + \gamma\left(\begin{matrix} d \\ \circ\!\!\longrightarrow\!\!\circ \end{matrix}\right)$.

(2) $\mathbf{S4.Grz} = \mathbf{S4} + \gamma\left(\begin{matrix}\circ\!\!\longleftarrow\!\!\circ \\ d_1 \quad d_2\end{matrix}\right) + \gamma\left(d_1 \text{ ◇ } d_2\right)$.

(3) $\mathbf{S4.1} = \mathbf{S4} + \gamma\left(\begin{matrix}\circ\!\!\longleftarrow\!\!\circ \\ d_1 \quad d_2\end{matrix}\right)$.

Proof. (1). Let $\mathfrak{A}$ be a s.i. **K4**-algebra. It is sufficient to prove that $\mathfrak{A} \not\models$ **GL** iff $\mathfrak{A} \not\models$ **K4** $+ \gamma \left(\begin{smallmatrix} \circ \\ d \end{smallmatrix} \right) + \gamma \left(\begin{smallmatrix} d \\ \circ\!\!-\!\!\circ \end{smallmatrix} \right)$. First suppose that $\mathfrak{A} \not\models$ **GL**. Let $\mathfrak{X}$ be the dual of $\mathfrak{A}$. Then there is a bounded morphism from a clopen subset U of X onto $\circ$ (see, e.g., ([15, Sec. 9.4])). There are two cases, either $(R^+)^{-1}[U] = X$ or $(R^+)^{-1}[U] \subsetneq X$. In the first case we define f from X onto $\circ$ by sending the entire X to $\circ$, and in the second case we define g from X onto $\circ\!\!-\!\!\circ$ by sending $(R^+)^{-1}[U]$ to the root of $\circ\!\!-\!\!\circ$ and the rest to the top node. It is easy to see that f and g are stable maps. To see that f satisfies CDC for $\{\{d\}\}$, it is sufficient to show that for each $x \in X$, there is $y \in X$ such that $x\,R\,y$. For $x \in U$ such a y exists because there is a bounded morphism from U onto $\circ$. If $x \notin U$, then as $(R^+)^{-1}[U] = X$, there is $y \in U$ such that $x\,R\,y$. Thus, f satisfies CDC for $\{\{d\}\}$. That g satisfies CDC for $\{\{d\}\}$ is proved similarly. Therefore, applying Theorem 6.8, in the first case we obtain $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} \circ \\ d \end{smallmatrix} \right)$, and in the second case we obtain $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} d \\ \circ\!\!-\!\!\circ \end{smallmatrix} \right)$. Consequently, $\mathfrak{A} \not\models$ **K4** $+ \gamma \left(\begin{smallmatrix} \circ \\ d \end{smallmatrix} \right) + \gamma \left(\begin{smallmatrix} d \\ \circ\!\!-\!\!\circ \end{smallmatrix} \right)$.

Conversely, suppose that $\mathfrak{A} \not\models$ **K4** $+ \gamma \left(\begin{smallmatrix} \circ \\ d \end{smallmatrix} \right) + \gamma \left(\begin{smallmatrix} d \\ \circ\!\!-\!\!\circ \end{smallmatrix} \right)$. Then $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} \circ \\ d \end{smallmatrix} \right)$ or $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} d \\ \circ\!\!-\!\!\circ \end{smallmatrix} \right)$. If $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} d \\ \circ\!\!-\!\!\circ \end{smallmatrix} \right)$, then by Theorem 6.8, there is a s.i. homomorphic image $\mathfrak{B}$ of $\mathfrak{A}$ and a stable map f from the dual space $\mathfrak{Y}$ of $\mathfrak{B}$ onto $\circ\!\!-\!\!\circ$ satisfying CDC for $\{\{d\}\}$. Therefore, $U = f^{-1}(d)$ is a clopen subset of Y and the restriction of f to U is a bounded morphism from U onto $\circ$. Thus, $\mathfrak{B} \not\models$ **GL** (see, e.g., ([15, Sec. 9.4])). Since $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, we see that $\mathfrak{A} \not\models$ **GL**. The case $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} \circ \\ d \end{smallmatrix} \right)$ is proved similarly.

(2). The proof is similar to that of (1). Let $\mathfrak{A}$ be a s.i. **S4**-algebra. It is sufficient to prove that $\mathfrak{A} \not\models$ **S4.Grz** iff $\mathfrak{A} \not\models$ **S4** $+ \gamma \left(\begin{smallmatrix} \circ\!\!-\!\!\circ \\ d_1 \quad d_2 \end{smallmatrix} \right) + \gamma \left(d_1 \overset{\circ}{\circ\!\!-\!\!\circ} d_2 \right)$. First suppose that $\mathfrak{A} \not\models$ **S4.Grz**. Let $\mathfrak{X}$ be the dual of $\mathfrak{A}$. Then there is a bounded morphism f from a clopen subset U of X onto $\circ\!\!-\!\!\circ$ (see, e.g., [15, Sec. 9.4]). As in the proof of (1), there are two cases: $R^{-1}[U] = X$ or $R^{-1}[U] \subsetneq X$. In the first case we define g_1 from X onto $\circ\!\!-\!\!\circ$ by sending U to $\circ\!\!-\!\!\circ$ via f and $R^{-1}[U] - U$ to a point of $\circ\!\!-\!\!\circ$. In the second case we define g_2 from X onto $\overset{\circ}{\circ\!\!-\!\!\circ}$ by sending U to the minimal cluster of $\overset{\circ}{\circ\!\!-\!\!\circ}$ via f, $R^{-1}[U] - U$ to a point of the minimal cluster, and the rest to the top node of $\overset{\circ}{\circ\!\!-\!\!\circ}$. It is easy to see that g_1 and g_2 are stable maps. To see that g_1 satisfies CDC for $\{\{d_1\}, \{d_2\}\}$, let $x \in X$, and suppose $g_1(x)\,R\,y$. If $x \in U$, then as f is a bounded morphism, there is $z \in U$ such that $x\,R\,z$ and $f(z) = g_1(z) = y$. If $x \in R^{-1}[U] - U$, then there is $z \in U$ such that $x\,R\,z$. If $g_1(z) = y$, then we are done. Otherwise, $g_1(z)\,R\,y$ and $z \in U$. So as f is a bounded morphism, there is $u \in U$ such that $z\,R\,u$ and $f(u) = g_1(u) = y$.

By transitivity, $x \, R \, u$. Thus, g_1 satisfies CDC for $\{\{d_1\}, \{d_2\}\}$. That g_2 satisfies CDC for $\{\{d_1\}, \{d_2\}\}$ is proved similarly. Therefore, applying Theorem 6.8, in the first case we obtain $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} \circ\!\!\longleftrightarrow\!\!\circ \\ d_1 \quad d_2 \end{smallmatrix} \right)$, and in the second case we obtain

$\mathfrak{A} \not\models \gamma \left(d_1 \, \substack{\triangle \\ \circ\!\!-\!\!\circ} \, d_2 \right)$. Consequently, $\mathfrak{A} \not\models \mathbf{S4} + \gamma \left(\begin{smallmatrix} \circ\!\!\longleftrightarrow\!\!\circ \\ d_1 \quad d_2 \end{smallmatrix} \right) + \gamma \left(d_1 \, \substack{\triangle \\ \circ\!\!-\!\!\circ} \, d_2 \right)$.

Conversely, suppose that $\mathfrak{A} \not\models \mathbf{S4} + \gamma \left(\begin{smallmatrix} \circ\!\!\longleftrightarrow\!\!\circ \\ d_1 \quad d_2 \end{smallmatrix} \right) + \gamma \left(d_1 \, \substack{\triangle \\ \circ\!\!-\!\!\circ} \, d_2 \right)$. Then

$\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} \circ\!\!\longleftrightarrow\!\!\circ \\ d_1 \quad d_2 \end{smallmatrix} \right)$ or $\mathfrak{A} \not\models \gamma \left(d_1 \, \substack{\triangle \\ \circ\!\!-\!\!\circ} \, d_2 \right)$. If $\mathfrak{A} \not\models \gamma \left(d_1 \, \substack{\triangle \\ \circ\!\!-\!\!\circ} \, d_2 \right)$, then by Theorem 6.8, there is a s.i. homomorphic image $\mathfrak{B}$ of $\mathfrak{A}$ and a stable map f from the dual space $\mathfrak{Y}$ of $\mathfrak{B}$ onto $\substack{\triangle \\ \circ\!\!-\!\!\circ}$ satisfying CDC for $\{\{d_1\}, \{d_2\}\}$. Therefore, $U = f^{-1}(\{d_1, d_2\})$ is a clopen subset of Y and the restriction of f to U is a bounded morphism from U onto $\circ\!\!\longleftrightarrow\!\!\circ$. Thus, $\mathfrak{B} \not\models \mathbf{S4.Grz}$ (see, e.g., ([15, Sec. 9.4]). Since $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, we see that $\mathfrak{A} \not\models \mathbf{S4.Grz}$. The case $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} \circ\!\!\longleftrightarrow\!\!\circ \\ d_1 \quad d_2 \end{smallmatrix} \right)$ is proved similarly.

(3). Let $\mathfrak{A}$ be an $\mathbf{S4}$-algebra and $\mathfrak{X}$ be its dual. Then $\mathfrak{X}$ is reflexive and transitive. It is sufficient to show that $\mathfrak{A} \models \Box\Diamond p \to \Diamond\Box p$ iff $\mathfrak{A} \models \gamma \left(\begin{smallmatrix} \circ\!\!\longleftrightarrow\!\!\circ \\ d_1 \quad d_2 \end{smallmatrix} \right)$. First suppose that $\mathfrak{A} \not\models \Box\Diamond p \to \Diamond\Box p$. By [17, Sec. 4], $\mathfrak{A} \models \Box\Diamond p \to \Diamond\Box p$ iff $\mathfrak{X}$ has no nonsingleton maximal clusters. Therefore, $\mathfrak{X}$ has a nonsingleton maximal cluster C. Since C is a maximal cluster, it is a closed topo-rooted up-set of $\mathfrak{X}$. Thus, it is a modal space whose dual is a s.i. homomorphic image of $\mathfrak{A}$. Since C is nonsingleton, there is a clopen partition U, V of C into two clopens. Define $f : C \to \circ\!\!\longleftrightarrow\!\!\circ$ by sending U to one reflexive point and V to the other reflexive point. It is easy to see that f is an onto bounded morphism. Therefore, f is an onto stable map satisfying CDC for $\{\{d_1\}, \{d_2\}\}$. Thus, by Theorem 6.8, $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} \circ\!\!\longleftrightarrow\!\!\circ \\ d_1 \quad d_2 \end{smallmatrix} \right)$.

Conversely, suppose that $\mathfrak{A} \not\models \gamma \left(\begin{smallmatrix} \circ\!\!\longleftrightarrow\!\!\circ \\ d_1 \quad d_2 \end{smallmatrix} \right)$. Applying Theorem 6.8 again yields a closed topo-rooted up-set $\mathfrak{Y}$ of $\mathfrak{X}$, which is mapped onto $\circ\!\!\longleftrightarrow\!\!\circ$ via a stable map satisfying CDC for $\{\{d_1\}, \{d_2\}\}$. This gives that each maximal cluster of $\mathfrak{Y}$ is a nonsingleton cluster. Therefore, $\mathfrak{Y}$ and hence $\mathfrak{X}$ refutes $\Box\Diamond p \to \Diamond\Box p$, completing the proof. ⊣

The axiomatization of $\mathbf{K4}$ and $\mathbf{S4}$ by stable canonical rules is more involved and will be discussed elsewhere.

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