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# A CONSTRUCTION OF SURFACES WITH LARGE HIGHER CHOW GROUPS

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*On the occasion of 60th birthday of Shuji Saito*

**Abstract.** In this paper, we construct surfaces in $\mathbf{P}^3$ with large higher Chow groups defined over a Laurent power series field. Explicit elements in higher Chow group are constructed using configurations of lines contained in the surfaces. To prove the independentness, we compute the extension class in the Galois cohomologies by comparing them with the classical monodromies. It is reduced to the computation of linear algebra using monodromy weight spectral sequences.

## §1. Introduction

### 1.1 Introduction

Let $k$ be a field and $X$ be a variety over $k$. For nonnegative integers $i$ and $j$, Bloch [B] defined the $j$th higher Chow group $CH^i(X,j)$ of codimension $i$ of $X$. To study the higher Chow group, it is useful to consider the cycle map:

$$c_\star^{i,j}(X): CH^i(X,j) \to H_\star^{2i-j}(X,\mathbf{Q}(i))$$

for $\star$-cohomology theory for $\star = et(\text{ale}), B(\text{etti}), dR(\text{ham}), \mathcal{D}(\text{eligne})$, when $X$ is smooth over $k$. Rich arithmetic invariants are conjectured to appear from these cohomology theories. First natural question is how big the image of $c_\star^{i,j}(X)$ is. For a smooth surface $X$ in $\mathbf{P}^3$ over an algebraically closed field $k$, the map is trivial for $\star = et$ and $i=2$, $j=1$. We are mainly interested in the case where $k$ is not algebraically closed. Concerning this question, there are some previous works, for example, [M], [CMS]. In this paper, we give an example of a surface over $\mathbf{C}((t))$, whose image of the cycle map is of big dimension. We construct many elements in higher Chow group explicitly, which can be studied in detail.

Let $S = \mathrm{Spec}(\mathbf{C}[t])$ and $\mathcal{L}_1,\dots,\mathcal{L}_d,\mathcal{M}_1,\dots,\mathcal{M}_d$ be linear forms on $\mathbf{P}^3_{\mathbf{C}}$. Let $l_i$ and $m_i$ be the zero loci of $\mathcal{L}_i$ and $\mathcal{M}_i$, respectively. In the following,

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we assume that $\bigcup_{i=1}^{d}(l_i \cup m_i)$ is a normal crossing divisor. Let $f : X \to S$ be a family of surfaces over $S$ in $\mathbf{P}^3_S$ defined by

$$X := \{(x,t) \in \mathbf{P}^3 \times S \mid \mathcal{L}_1(x) \cdots \mathcal{L}_d(x) + t\mathcal{M}_1(x) \cdots \mathcal{M}_d(x) = 0\}.$$

Let $k = \mathbf{C}((t))$ and $\eta : \mathrm{Spec}(k) \to S$ the natural morphism. The fiber of $X$ at $\eta$ is denoted by $X_\eta$. In this paper, we consider the image of the etale cycle map

$$c^{2,1}_{et}(X_\eta) : CH^2(X_\eta, 1) \otimes \mathbf{Q}_l \to H^3_{et}(X_\eta, \mathbf{Q}_l(2)). \tag{1}$$

The main theorem of the paper is as follows.

Theorem 1.1. *Under the above notations, we have*

$$\dim_{\mathbf{Q}_l}(\mathrm{Im}(c^{2,1}_{et}(X_\eta))) \geqslant \frac{(d-1)^2(d-2)}{2}.$$

As a corollary, we have the following:

Corollary 1.2.

$$\dim_{\mathbf{Q}}(CH^2(X_\eta, 1)) \geqslant \frac{(d-1)^2(d-2)}{2}.$$

Let us explain the contents of the paper. In Section 2, we construct elements in the higher Chow group of $X_\eta$. Moreover, we give a relation between the image of these elements under the etale cycle map and the extension classes of relative homologies. In Section 3, we compare the etale extension classes and extension classes as representations of the (classical) fundamental group of a punctured disc. Thus we reduce the proof of Theorem 1.1 to the proof of the relevant statement for representations of fundamental groups. We recall how to compute the extension classes as representations of the fundamental group. In Section 4, we recall the monodromy weight spectral sequence and compute its terms for $X_{\Delta^*}$. The monodromy weight spectral sequence is used to compute the cokernel of the logarithm of monodromy action. To compute every term of the monodromy weight spectral sequence explicitly, we use the model obtained by the blowing up of the original model. In Section 5, we execute a local computation for the extension class using a nice topological model. In this model, the extension class is computed by a topological lifting whose period map can be expressed by the dilogarithmic function. By these computations, we have the local description for the extensions, which will be used in the next section. In Section 6, we sum up the previous results to compute the lower bound for the dimension of the image of the cycle map.

## §2. Construction of elements in the higher Chow group and extension

From now on, we use the same notations for $X$, $X_\eta$ in the previous section. In this section, we define an element $\Gamma_{ijk.l}$ in $CH^2(X,1)$, and study the extension class obtained by the generic fiber $\Gamma_{ijk,l,\eta}$ of $\Gamma_{ijk.l}$ at $\eta$.

### 2.1 Elements in the higher Chow group of $X$

Let us briefly recall the definition of higher Chow group after Bloch [B]. Let $\Delta^j$ be the scheme defined by

$$\Delta^j = \mathrm{Spec}\left(\mathbf{C}[x_0,\dots,x_j]\middle/\left(1-\sum_{k=0}^{j}x_k\right)\right).$$

Let $X$ be a variety over $\mathbf{C}$. The simplicial faces of $X\times\Delta^j$ are defined in [B]. The Bloch cycle complex $Z^i(X,j)$ is defined by the group of $\mathbf{Z}$-linear combinations of codimension $i$ algebraic cycles in $X\times\Delta^j$, which intersect properly to the simplicial faces. We consider the boundary operator

$$Z^i(X,j)\to Z^i(X,j-1)$$

using simplicial structure defined as in [B]. Then we have a complex

$$Z^i(X,\bullet):\cdots\to Z^i(X,j)\to Z^i(X,j-1)\to\cdots\to Z^i(X,1)\to Z^i(X,0).$$

The $j$th cohomology of $Z^i(X,\bullet)$ is called the $j$th higher Chow group of codimension $i$ and denoted by $CH^i(X,j)$.

We set

$$L_i = l_i\times S,\qquad M_l = m_l\times S.$$

For integers $1\leqslant i,\ l\leqslant d$, the intersection $L_i\cap M_l$ is a family of projective lines over $S$ contained in $X$. For $i\neq j$, we set

$$p_{ij,l} = l_i\cap l_j\cap m_l,\qquad P_{ij,l} = p_{ij,l}\times S.$$

We fix $1\leqslant i<j<k\leqslant d$ and $1\leqslant l\leqslant d$. The three lines $L_i\cap M_l$, $L_j\cap M_l$, $L_k\cap M_l$ form a family of triangles in $X$. Let $(l_i\cap m_l)^0$, $(l_j\cap m_l)^0$ and $(l_k\cap m_l)^0$ be one-dimensional affine spaces contained in $(l_i\cap l_l)$, $(l_j\cap m_l)$ and $(l_k\cap m_l)$ such that $(l_i\cap m_l)^0\cup(l_j\cap m_l)^0\cup(l_k\cap m_l)^0$ contains $p_{ij,l}, p_{jk,l}$ and $p_{ki,l}$. We choose an isomorphism $\varphi_{i,jk,l}:\Delta^1\simeq(l_i\cap m_l)^0$ such that $\varphi_{i,jk,l}(0)=p_{ij,l}$, $\varphi_{i,jk,l}(1)=p_{ik,l}$. The isomorphism $\varphi_{i,jk,l}$ induces an isomorphism $\Delta^1\times S\simeq(l_i\cap m_l)^0\times S$ over $S$ which is also denoted by $\varphi_{i,jk,l}$.

DEFINITION 2.1.

(1) We set $\Gamma_{i,jk,l} = \{(x, y) \in \Delta^1 \times X \mid \varphi_{i,jk,l}(x) = y\}$. Then it is an element in $Z^2(X, 1)$.
(2) We define an element $\Gamma_{ijk,l}$ in $Z^2(X, 1)$ by

$$\Gamma_{ijk,l} = \Gamma_{i,jk,l} + \Gamma_{j,ki,l} + \Gamma_{k,ij,l}.$$

It is easy to see that $\partial\Gamma_{ijk,l} = 0$. The class of $\Gamma_{ijk,l}$ in $CH^2(X, 1)$ is also denoted by $\Gamma_{ijk,l}$.

For a point $x$ over $S$, the fiber of $f : X \to S$ is denoted by $X_x$. We set

$$T = (L_i \cap M_l) \cup (L_j \cap M_l) \cup (L_k \cap M_l).$$

Let $f_T : T \to S$ be the natural projection. The fiber of $T$ and $\Gamma_{ijk,l}$ at $x$ are written as $T_x$ and $\Gamma_{ijk,l,x}$, respectively.

Let $\Gamma_{\varphi_{i,jk,l}}$ be the graph of $\varphi_{i,jk,l}$. We define an element $\delta$ by

$$\begin{aligned}
\delta &= (P_{ij,l}, P_{jk,l}, P_{ki,l}, \Gamma_{\varphi_{i,jk,l}}, \Gamma_{\varphi_{j,ki,l}}, \Gamma_{\varphi_{k,ij,l}}) \\
&\in (Z^0(P_{ij,l} \times \Delta^0) \oplus Z^0(P_{jk,l} \times \Delta^0) \oplus Z^0(P_{ki,l} \times \Delta^0)) \\
&\quad \oplus (Z^1((L_i \cap M_l) \times \Delta^1) \oplus Z^1((L_j \cap M_l) \times \Delta^1) \oplus Z^1((L_k \cap M_l) \times \Delta^1)).
\end{aligned}$$

Then the sum of cycles $\delta$ determines an element in $CH^1(T, 1)$ by the Mayer–Vietoris theorem. We have a natural map $\tau : CH^1(T, 1) \to CH^2(X, 1)$. The cycle $\Gamma_{ijk.l} \in CH^2(X, 1)$ is equal to $\tau(\delta)$.

### 2.2 Cycle maps and connecting homomorphisms

Let $D_T = D_{T_\eta/\eta}$ be the $\mathbf{Q}_l$-dualizing complex of $T_\eta$ over $\eta$. For a subvariety $W$ in $X$ the restriction of $f$ to $W$ is denoted by $f_W$. The restriction of $f$ to $X_\eta$ is denoted by $f_\eta$. For simplicity we consider the case $(i, j, k) = (1, 2, 3)$. The complex $\mathbf{R}f_{T_\eta *}D_T(1)[2]$ is quasi-isomorphic to

$$\bigoplus_{1\leqslant i<j\leqslant 3} \mathbf{R}f_{P_{ij,l,\eta}*}, \qquad \mathbf{Q}_l \to \bigoplus_{1\leqslant i\leqslant 3} \mathbf{R}f_{(L_i\cap M_l)_\eta *}\mathbf{Q}_l(1)[2]. \tag{2}$$

The homomorphism $\bigoplus_{1\leqslant i\leqslant 3} \mathbf{R}f_{(L_i\cap M_l)_\eta,*}\mathbf{Q}_l(1)[2] \to \mathbf{R}f_{\eta *}\mathbf{Q}_l(2)[4]$ induces a homomorphism

$$\mathbf{R}f_{T_\eta *}D_T(1)[2] \to \mathbf{R}f_{\eta *}\mathbf{Q}_l(2)[4]. \tag{3}$$

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Its cone is denoted by $\mathcal{C}_\eta$. Let $S^0$ be the maximal open set of $S$ over which $f:X\to S$ is smooth. By the definition of the cone $\mathcal{C}_\eta$, we have the long exact sequence:

$$\begin{aligned}\cdots\to H^{i-1}(T_\eta,D(1))\to H^{i+1}(X_\eta,\mathbf{Q}_l(2))\to H^{i+1}(\mathcal{C}_\eta)\\ \to H^{i}(T_\eta,D(1))\to H^{i+2}(X_\eta,\mathbf{Q}_l(2))\to\cdots\end{aligned}\tag{4}$$

The same complex of (3) can be considered over $\overline{\eta}$. The corresponding complex are denoted as $\mathcal{C}_{\overline{\eta}}$. We have the following similar long exact sequence of $G=\mathrm{Gal}(\overline{k}/k)$ modules:

$$\begin{aligned}\cdots\to H^{i-1}(T_{\overline{\eta}},D(1))\to H^{i+1}(X_{\overline{\eta}},\mathbf{Q}_l(2))\to H^{i+1}(\mathcal{C}_{\overline{\eta}})\\ \to H^{i}(T_{\overline{\eta}},D(1))\to H^{i+2}(X_{\overline{\eta}},\mathbf{Q}_l(2))\to\cdots\end{aligned}\tag{5}$$

By the compatibility of cycle maps and the connecting homomorphisms, we have the following commutative diagram:

$$\begin{array}{ccc} CH^1(T_\eta,1) & \xrightarrow{\tau} & CH^2(X_\eta,1)\\ c_{et}^{1,1}\downarrow & & \downarrow c_{et}^{2,1}\\ H^1(T_\eta,D(1)) & \to & H^3(X_\eta,\mathbf{Q}_l(2)).\end{array}$$

The image of $\delta$ in $H^1(T_\eta,D(1))$ under the cycle map is denoted by $\overline{\delta}$. Then we have

$$H^1(T_\eta,D(1))=\overline{\delta}\cdot\mathbf{Q}_l.$$

### 2.3 Cycle map and extension group

We study extensions arising from $\Gamma_{ijk,l}$. Until the end of this section, we consider the fibers $X_\eta$ over $\eta\to S$. The varieties $(L_i\cap M_l)_\eta$, $T_\eta$, etc. are subvarieties of $X_\eta$. We set $\overline{\eta}=\mathrm{Spec}(\overline{k})$. Since $X_\eta$ is a hypersurface in $\mathbf{P}^3$, we have $H^0(k_{et},H^3(X_{\overline{\eta}},\mathbf{Q}_l(2)))=0$ and by Hochschild–Serre spectral sequence, we have a map

$$H^3(X_\eta,\mathbf{Q}_l(2))\to H^1(k_{et},H^2(X_{\overline{\eta}},\mathbf{Q}_l(2))).$$

PROPOSITION 2.2. *The image of $\Gamma_{ijk,l}\in CH^2(X_\eta,1)$ under the map*

$$CH^2(X_\eta,1)\xrightarrow{c_{et}^{2,1}} H^3(X_\eta,\mathbf{Q}_l(2))\to H^1(k_{et},H^2(X_{\overline{\eta}},\mathbf{Q}_l(2)))$$

*is equal to the extension class of*

$$0\to H^2(X_{\overline{\eta}},\mathbf{Q}_l(2))\to H^2(\mathcal{C}_{\overline{\eta}})\to H^1(T_{\overline{\eta}},D(1))\to 0$$

*as $G$-module. In other words, the image of $\overline{\delta}$ under the connecting homomorphism:*

$$H^1(T_\eta, D(1)) \to H^0(G, H^1(T_{\overline{\eta}}, D(1))) \to H^1(G, H^2(X_{\overline{\eta}}, \mathbf{Q}_l(2))). \tag{6}$$

*Proof.* We consider the long exact sequence (5). We obtain similar long exact sequence by replacing $k$ by $\overline{k}$. Applying the functor $H^p(G, *)$, we have a complex

$$\begin{aligned} \cdots \to H^p(G, H^{i-1}(T_{\overline{\eta}}, D(1))) &\to H^p(G, H^{i+1}(X_{\overline{\eta}}, \mathbf{Q}_l(2))) \\ &\to H^p(G, H^{i+1}(\mathcal{C}_{\overline{\eta}})) \\ \to H^p(G, H^i(T_{\overline{\eta}}, D(1))) &\to H^p(G, H^{i+2}(X_{\overline{\eta}}, \mathbf{Q}_l(2))) \to \cdots \end{aligned}$$

Since the cohomological dimension of $G$ is one, we have the following exact sequences by Hochschild–Serre spectral sequence.

$$\begin{aligned} &0 \to H^1(G, H^{i-2}(T_{\overline{\eta}}, D(1))) \to H^{i-1}(T_\eta, D(1)) \\ &\quad \to H^0(G, H^{i-1}(T_{\overline{\eta}}, D(1))) \to 0, \\ &0 \to H^1(G, H^i(X_{\overline{\eta}}, \mathbf{Q}_l(2))) \to H^{i+1}(X_\eta, \mathbf{Q}_l(2)) \\ &\quad \to H^0(G, H^{i+1}(X_{\overline{\eta}}, \mathbf{Q}_l(2))) \to 0, \\ &0 \to H^1(G, H^i(\mathcal{C}_{\overline{\eta}})) \to H^{i+1}(\mathcal{C}_\eta) \to H^0(G, H^{i+1}(\mathcal{C}_{\overline{\eta}})) \to 0. \end{aligned}$$

Since the sequence (5) is exact, the homology of

$$H^0(G, H^{i+1}(\mathcal{C}_{\overline{\eta}})) \to H^0(G, H^i(T_{\overline{\eta}}, D(1))) \xrightarrow{\alpha} H^0(G, H^{i+2}(X_{\overline{\eta}}, \mathbf{Q}_l(2))) \tag{7}$$

and that of

$$H^1(G, H^{i-1}(T_{\overline{\eta}}, D(1))) \xrightarrow{\beta} H^1(G, H^{i+1}(X_{\overline{\eta}}, \mathbf{Q}_l(2))) \to H^1(G, H^{i+1}(\mathcal{C}_{\overline{\eta}})) \tag{8}$$

are isomorphic. The following lemma is straight forward.

LEMMA 2.3. *Assume that*

$$0 \to H^{i+1}(X_{\overline{\eta}}, \mathbf{Q}_l(2)) \to H^{i+1}(\mathcal{C}_{\overline{\eta}}) \to H^i(T_{\overline{\eta}}, D(1)) \to 0 \tag{9}$$

*is exact. Then $\alpha$ and $\beta$ are zero maps.*

The homomorphism

$$H^0(G, H^i(T_{\overline{\eta}}, D(1))) \to H^1(G, H^{i+1}(X_{\overline{\eta}}, \mathbf{Q}_l(2))) \tag{10}$$

induced by the isomorphism between cohomologies of (7) and (8) is equal to the connecting homomorphism arising from the exact sequence (9). The connecting homomorphism (6) is induced by the following zigzag.

$$\begin{array}{ccc} & & H^1(G, H^2(X_{\overline{\eta}}, \mathbf{Q}_l(2))) \\ & & \downarrow \\ H^1(T_\eta, D_{T^0}(1)) & \to & H^3(X_\eta, \mathbf{Q}_l(2)) \\ \downarrow & & \\ H^0(G, H^1(T_{\overline{\eta}}, D(1))). & & \end{array}$$

Therefore the image $c_{et}^{2,1}(\Gamma_{ijk,l}) = c_{et}^{2,1}(\tau(\delta))$ of $\Gamma_{ijk.l}$ under the cycle map is equal to the image of $\overline{\delta} \in H^0(G, H^1(T_{\overline{k}}, D))$ under the connecting homomorphism. ∎

## §3. Comparison to classical cohomology theory

In this section, we compare extensions for etale cohomologies with those for classical cohomology theory.

### 3.1 Comparison to classical theory

Let $S^0$ be the maximal open set of $S$ over which $f : X \to S$ is smooth. Let $\Delta = \{t \in \mathbf{C} \mid |t| < \epsilon\}$ be a sufficiently small neighborhood of 0 in $S(\mathbf{C})$ such that $\Delta^*(= \Delta - \{0\}) \subset S^0(\mathbf{C})$. We fix an element $t_0$ in $\Delta^*$.

The restrictions of $f$ and $f_T$ to $f^{-1}(S^0)$ and $f^{-1}(S^0) \cap T$ are denoted by $f_{S^0}$ and $f_{T,S^0}$, respectively. We have the following short exact sequence of etale $l$-adic local systems on $S^0$.

$$0 \to \mathbf{R}^2 f_{S^0 *}\mathbf{Q}_l(2) \to \mathbf{R}^2 f_{S^0 *}\mathcal{C} \to \mathbf{R}^1 f_{T,S^0 *} D_T(1) \to 0. \tag{11}$$

We set $\overline{\eta} = \mathrm{Spec}(\overline{k})$. Let $S^{st}$ be the strict Henselization under $\overline{\eta}$ of $S$ at $t_0$ over $\eta$. Then the diagram $t_0 \leftarrow S^{st} \to \overline{\eta}$ defines an isomorphism $\pi_1^{et}(S^0, t_0) \xrightarrow{\simeq} \pi_1^{et}(S^0, \overline{\eta})$ and isomorphism between the fiber of the exact sequence (11) at $t_0$ and that at $\overline{\eta}$. We have the following diagram

$$\begin{array}{ccccc} \pi_1^{cl}(S^0(\mathbf{C}), t_0) & \to & \pi_1^{et}(S^0, t_0) & \xrightarrow{\simeq} & \pi_1(S^0, \overline{\eta}) \\ \cup & & & & \cup \\ \pi_1^{cl}(\Delta^*, t_0) & & & & \pi_1^{et}(\eta, \overline{\eta}). \end{array}$$

We can easily see that the above homomorphism induces a homomorphism

$$c_\pi : \pi_1^{cl}(\Delta^*, t_0) \to \pi_1^{et}(\eta, \overline{\eta}).$$

By composing the comparison map the fiber of the exact sequence (11) at $\overline{\eta}$

$$0 \to H^2_{et}(X_{\overline{\eta}}, \mathbf{Q}_l) \to H^2_{et}(X_{\overline{\eta}}, \mathcal{C}_\eta) \to H^1_{et}(T_{\overline{\eta}}, D_S) \to 0 \tag{12}$$

is isomorphic to the exact sequence

$$0 \to H^2_B(X_{t_0}, \mathbf{Q}_l) \to H^2_B(X_{t_0}, \mathcal{C}_\eta) \to H^1_B(T, D_S) \to 0. \tag{13}$$

This isomorphism is equivariant under the fundamental groups via the map $c_\pi$. As a consequence, we have the following proposition.

PROPOSITION 3.1. *The extension class $\epsilon_{et}$ of (12) in $H^1(\pi_1^{et}(\eta, \overline{\eta}), H^2(X_{\overline{\eta}}, \mathbf{Q}_l(2)))$ goes to the extension class $\epsilon_{cl}$ of (13) in $H^1(\pi_1^{cl}(\Delta^*, \overline{t_0}), H^2(X_{t_0}, \mathbf{Q}_l(2)))$ under the map $c_\pi$.*

The short exact sequence (13) is isomorphic to the following short exact sequence

$$0 \to H_2(X_{t_0}, \mathbf{Q}) \to H_2(X_{t_0}, T_{t_0}, \mathbf{Q}) \to H_1(T_{t_0}, \mathbf{Q}) \to 0 \tag{14}$$

as a module over fundamental group $\pi_1^{cl}(\Delta^*, t_0)$ after tensoring with $\mathbf{Q}_l$. Therefore, we have the following proposition.

PROPOSITION 3.2. *The extension class $\epsilon_{cl}$ is equal to the extension class of (14) in $H^1(\pi_1^{cl}(\Delta^*, \overline{t_0}), H_2(X_{t_0}, \mathbf{Q}))$ via the isomorphism $H^2(X_{t_0}, \mathbf{Q}_l(2)) \simeq H_2(X_{t_0}, \mathbf{Q}) \otimes \mathbf{Q}_l$.*

### 3.2 Extensions from the topological side

The surface $X_{t_0}$ contains three affine lines $(l_i \cap m_l)^0, (l_j \cap m_l)^0, (l_k \cap m_l)^0$. We choose a topological path $\gamma_{jk,l}$ (resp. $\gamma_{ki,l}, \gamma_{ij,l}$) connecting $p_{ij,l}$ and $p_{ik,l}$ (resp. $p_{jk,l}$ and $p_{ji,l}$, $p_{ki,l}$ and $p_{kj,l}$) in $(l_i \cap m_l)^0$ (resp. $(l_j \cap m_l)^0$ and $(l_k \cap m_l)^0$). Then we have a topological cycle $\tau = \gamma_{ij,l} + \gamma_{jk,l} + \gamma_{ki,l}$. Since $X_{t_0}$ is simply connected, there exists a 2-chain $\sigma_0$ in $X_{t_0}$ such that $\tau = \partial(\sigma_0)$. Then the relative cycle $\tau$ defines a relative homology class in $H_2(X_{t_0}, T_{t_0})$. Let

$$\psi : [0, 1] \to \Delta^*,$$

be a path in $\Delta^*$ beginning from $t_0$ ending at $t_0$ turning around the origin, whose homotopy class is a positive generator $\gamma$ of $\pi_1^{cl}(\Delta^*, t_0)$. We extend the relative two-cycle $\sigma_0$ to a continuous family on of relative two-cycle $\sigma(s)$ in $X_{\psi(s)}$. $(0 \leqslant s \leqslant 1)$ such that:

(1) $\partial(\sigma(s)) = \partial(\sigma(0))$ $(\subset T_{\psi(s)})$ for all $s \in [0, 1]$;
(2) $\sigma(0) = \sigma$.

We set $\sigma(1) = \sigma'$. Then the chain $\sigma - \sigma'$ becomes a closed two-chain in $X_{t_0}$ and we have the homology class $[\sigma - \sigma']$ in $H_2(X_{t_0}, \mathbf{Q})$. The element $[\sigma - \sigma']$ constructed as above defines an element in

$$\begin{aligned} H^1(\pi_1(\Delta^*, t_0), H_2(X_{t_0}, \mathbf{Q})) &\simeq H_2(X_{t_0}, \mathbf{Q})/(\gamma - 1)H_2(X_{t_0}, \mathbf{Q}) \\ &\simeq \mathrm{Coker}(H_2(X_{t_0}, \mathbf{Q}) \xrightarrow{N} H_2(X_{t_0}, \mathbf{Q})). \end{aligned}$$

Here $N$ is the logarithm of $\gamma$. It is equal to $\epsilon_{cl}$ introduced in the last subsection. Here $\gamma$ denotes a positive generator of $\pi_1(\Delta^*, t_0)$. As a consequence, we have the following proposition.

LEMMA 3.3. *We have* $c_B^{2,1}(\Gamma_{ijk,l}) = \epsilon_{cl}$.

## §4. Monodromy weight spectral sequence

### 4.1 Blowing up and strata

From now on, we consider varieties in the category of complex analytic spaces. We use the same notations for the complex analytic space associated to $X$ and the morphism $f : X \to \mathbf{A}^1$. Let $\Delta$ be a sufficiently small disc around $t = 0$ as in the previous section and $X_\Delta$ the pull back $f^{-1}(\Delta)$.

Then the total space $X_D$ has nodes at $p_{ij,l} = X_0 \cap P_{ij,l} = \{\mathcal{L}_i = \mathcal{L}_j = \mathcal{M}_l = t = 0\}$ for $1 \leqslant i < j \leqslant d$, $1 \leqslant l \leqslant d$. For example, we can choose a local coordinate $x, y, z, t$ at $p_{ij,l}$ such that $x = \mathcal{L}_i$, $y = \mathcal{L}_j$, $z = \mathcal{M}_l$. Under this coordinates, $X_\Delta$ is defined by $\{xy + tz = 0\}$ around $p_{ij,l}$. The blowing up along $\bigcup_{i<j,l}\{(p_{ij,l}, 0)\}$ is denoted by $\widetilde{X}$ and the induced morphism $\widetilde{X} \to \Delta$ is denoted by $\widetilde{f}$.

Let $h_{ij,k}$ be the exceptional divisor over $(p_{ij,l}, 0)$ $(1 \leqslant i < j \leqslant d, 1 \leqslant l \leqslant d)$. Then the singular fiber $f^{-1}(0)$ consists of the following $d + (d^2(d-1))/2$ components.

(1) Proper transforms $g_1, \ldots, g_d$ of $l_i \times 0$, where $g_i$ is isomorphic to the blowing up of $\mathbf{P}^2$ along the points $p_{ij,l}$ $(j \neq i, 1 \leqslant l \leqslant d)$.
(2) $(d^2(d-1))/2$ exceptional components $h_{ij,k}$ $(1 \leqslant i < j \leqslant d, 1 \leqslant k \leqslant d)$. Each component $h_{ij,k}$ is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$.

One-dimensional stratum of the singular fiber consists of the followings.

(1) $(d(d-1))/2$ intersections $l_{ij} = g_i \cap g_j$ $(1 \leqslant i < j \leqslant d)$.
(2) $d^2(d-1)$ intersections of $m_{ij,k} = h_{ij,k} \cap g_i$.

The zero-dimensional stratum consists of the following points:

(1) $(d(d-1)(d-2))/6$ intersection points $p_{ijk}=g_i\cap g_j\cap g_k$;
(2) $(d^2(d-1))/2$ intersection points $q_{ij,k}=h_{ij,k}\cap g_i\cap g_j=m_{ij,k}\cap m_{ji,k}$.

We have $h_{ij,k}\supset m_{ij,k}\cup m_{ji,k}$. The disjoint union of the $k$-dimensional stratum is denoted by $T^{(k)}$. Then we have

$$\begin{aligned}
T^{(0)} &= \{p_{ijk}\}_{1\leqslant i<j<k\leqslant d}\cup\{q_{ij,k}\}_{1\leqslant i<j\leqslant d,1\leqslant k\leqslant d},\\
T^{(1)} &= \{l_{ij}\}_{1\leqslant i<j\leqslant d}\cup\{m_{ij,k}\}_{1\leqslant i\neq j\leqslant d,1\leqslant k\leqslant d},\\
T^{(2)} &= \{g_i\}_{1\leqslant i\leqslant d}\cup\{h_{ij,k}\}_{1\leqslant i<j\leqslant d,1\leqslant k\leqslant d}.
\end{aligned}$$

For example, for $d=4$, we have

$$\#T^{(0)}=28,\qquad \#T^{(1)}=54,\qquad \#T^{(0)}=28.$$

### 4.2 The $E_1$-term of the monodromy weight spectral sequence

In this section, we recall the monodromy weight spectral sequence [PSt, Section 11.2, p. 259]. We set $\Delta^*=\{t\in\Delta\mid t\neq 0\}$. Let $\overline{\eta}:H\to\Delta^*$ be the universal covering of $\Delta^*$. We consider the following fiber products:

$$\begin{array}{ccccc}
X_{\overline{\eta}} & \xrightarrow{k} & X^* & \xrightarrow{j} & \widetilde{X}\\
\downarrow & & \downarrow & & \downarrow\\
H & \xrightarrow{\eta} & \Delta^* & \to & \Delta.
\end{array}$$

We set $\overline{j}=j\circ k$. We consider the following complex which is quasi-isomorphic to the near by cycle sheaf $\mathbf{R}\psi\mathbf{Q}=i^*\overline{j}_*\ \overline{j}^*\mathbf{Q}$.

$$0\to i^*\mathbf{R}j_*\mathbf{Q}(1)^{1\leqslant}[1]\to i^*\mathbf{R}j_*\mathbf{Q}(2)^{2\leqslant}[2]\to i^*\mathbf{R}j_*\mathbf{Q}(3)^{3\leqslant}[3]\to 0. \tag{15}$$

We have $H^i(T^{(j)})(-1)=0$ for odd $i$ and $0\leqslant j\leqslant 2$. Therefore the $E_1$-terms of the associate monodromy weight spectral sequence are given as follows:

$$\begin{array}{ccccc}
H^0(T^{(0)})(-2) & H^2(T^{(1)})(-1) & H^4(T^{(2)})(0) & 0 & 0\\
0 & 0 & 0 & 0 & 0\\
0 & H^0(T^{(1)})(-1) & H^2(T^{(2)})(0)\oplus H^0(T^{(0)})(-1) & H^2(T^{(1)})(0) & 0\\
0 & 0 & 0 & 0 & 0\\
0 & 0 & H^0(T^{(2)})(0) & H^0(T^{(1)})(0) & H^0(T^{(0)})(0).
\end{array}$$

Therefore $Gr_0^W(H^2(X_{t_0},\mathbf{Q}))$ is isomorphic to the cohomology of

$$H^0(T^{(1)})(-1)\to H^2(T^{(2)})(0)\oplus H^0(T^{(0)})(-1)\to H^2(T^{(1)})(0).$$

In this section, we consider the $E_1$-differential for $H^2(X)$. We have

$$H^0(T^{(1)})(-1) = \bigoplus_{i,j} \mathbf{Q}1_{l_{ij}} \oplus \bigoplus_{i,j,k} \mathbf{Q}1_{m_{ij,k}},$$
$$H^2(T^{(1)})(0) = \bigoplus_{i,j} \mathbf{Q}[l_{ij}] \oplus \bigoplus_{i,j,k} \mathbf{Q}[m_{ij,k}],$$
$$H^0(T^{(0)})(-1) = \bigoplus_{i,j,k} \mathbf{Q}1_{p_{ijk}} \oplus \bigoplus_{i,j,k} \mathbf{Q}1_{q_{ij,k}}.$$

The cohomology class of $[l_{ij}]$ in $H^2(g_i)$ is denoted by $[l_{ij}]_{g_i}$, etc. We identify:

(1) $H^*(l_{12})$ and $H^*(l_{21})$ by $[x]_{l_{12}} = -[x]_{l_{21}}$;
(2) $H^0(q_{12,k})(-1)$ and $H^0(q_{21,k})(-1)$ by $1_{q_{12,k}} = -1_{q_{21,k}}$;
(3) $H^0(p_{123})(-1)$ and $H^0(p_{213})(-1)$ by $1_{p_{123}} = -1_{p_{213}} = -1_{p_{132}}$, etc.;
(4) $H^2(h_{12,k})(0)$ and $H^2(h_{21,k})(0)$ by $[x]_{h_{12,k}} = -[x]_{h_{21,k}}$.

We describe the differentials.

**The map** $d : H^0(T^{(1)})(-1) \to H^2(T^{(2)})(0) \oplus H^0(T^{(0)})(-1)$.
The differential is given by

$$d(1_{l_{ij}}) = [l_{ij}]_{g_i} - [l_{ij}]_{g_j} + \sum_{k\neq i,j} 1_{p_{ijk}} + \sum_{1\leqslant l\leqslant d} 1_{q_{ij,l}}, \tag{16}$$

$$d(1_{m_{ij,l}}) = [m_{ij,l}]_{g_i} + [m_{ij,l}]_{h_{ij,l}} - 1_{q_{ij,l}}. \tag{17}$$

**The map** $d : H^2(T^{(2)})(0) \oplus H^0(T^{(0)})(-1) \to H^2(T^{(1)})(0)$.
The differential is given by

$$d([x]_{g_i}) = \sum_{j\neq i}[l_{ij}](x, l_{ij})_{g_i} + \sum_{j\neq i}\sum_{k=1}^{d}[m_{ij,k}](x, m_{ij,k})_{g_i},$$
$$d([x]_{h_{ij,l}}) = -[m_{ij,l}](x, m_{ij,l})_{h_{ij,l}} + [m_{ji,l}](x, m_{ji,l})_{h_{ij,l}},$$

and

$$d(1_{p_{ijk}}) = [l_{ij}] + [l_{jk}] + [l_{ki}],$$
$$d(1_{q_{ij,l}}) = -[m_{ij,l}] + [m_{ji,l}] + [l_{ij}].$$

Since $d[x]_{h_{21,k}} = -d[x]_{h_{12,k}}$, this map is consistent with the rule of suffix. We can check that $d^2 = 0$.

### 4.3 Description of the 1-cocycle associated to $\Gamma_{ijk,l}$

We define a closed element $\gamma_{ijk,l}$ in $H^2(T^{(2)})(0) \oplus H^0(T^{(0)})(-1)$ by

$$\begin{aligned}\gamma_{ijk,l} &= [m_{ij,l} + m_{ji,l}]_{h_{ij,l}} + [m_{jk,l} + m_{kj,l}]_{h_{jk,l}} + [m_{ki,l} + m_{ik,l}]_{h_{ki,l}} \\ &\quad + 1_{p_{ijk}} - 1_{q_{ij,l}} - 1_{q_{jk,l}} - 1_{q_{ki,l}}.\end{aligned} \tag{18}$$

We prove the following proposition in the next section.

PROPOSITION 4.1. *The extension class associated to $c_B^{1,2}(\Gamma_{ijk,l})$ is equal to the image of $\gamma_{ijk,l}$ in $Gr^M H^2(X_{t_0}, \mathbf{Q})$.*

## §5. Extension class and a topological model

In this section, we compute the monodromy on the cohomologies of $W_{t_0}$. This computation will be used for the computation of $c_B^{1,2}(\Gamma_{ijk,l})$.

### 5.1 Computation of monodromy for the homotopical model $W_{t_0}$

We define a family of affine varieties $W$ by

$$W : xyz + t(1 - x - y - z) = 0 \tag{19}$$

in $\mathbf{A}^3 = \{(x, y, z) \mid x, y, z \in \mathbf{C}\}$. If $t \neq 0, 1/4$, then it is smooth. Let $t_0$ be a sufficiently small complex number. The fiber at $t_0$ is denoted by $W_{t_0}$. We consider the cohomology of $W_{t_0}$ in this subsection. Let $f : W_{t_0} \to \mathbf{A}^2$ be a map defined by $(x, y, z) \to (y, z)$. We set

$$\Sigma = \{(y, z) \in \mathbf{A}^2 \mid yz = t_0, 1 - y - z = 0\} = \{p_1, p_2\}.$$

Let $\widehat{\mathbf{A}^2}$ be the blowing up of $\mathbf{C}^2$ at two points $p_1, p_2$. The exceptional divisor at $p_1$ and $p_2$ are denoted by $E_1$ and $E_2$. Since the defining equation is $x(yz - t) + t(1 - y - z) = 0$, the fiber of $f$ at $p_1$ is isomorphic to $\mathbf{A}^1$ and we have a map $W_{t_0} \to \widehat{\mathbf{A}^2}$. Let $D$ be a curve in $\mathbf{A}^2$ defined by $yz = t_0$ and $\widehat{D}$ be the proper transform of $D$. Then we have

$$W_{t_0} = \widehat{\mathbf{A}^2} - \widehat{D}$$

and a long exact sequence

$$\begin{array}{ccccccccc}
H^2_{\widehat{D}}(\widehat{\mathbf{A}^2}) & \xrightarrow{\alpha} & H^2(\widehat{\mathbf{A}^2}) & \to & H^2(W_{t_0}) & \to & H^3_{\widehat{D}}(\widehat{\mathbf{A}^2}) & \to & 0. \\
\| & & \| & & & & \| & & \\
\mathbf{Q}[\widehat{D}] & & \mathbf{Q}[E_1] \oplus \mathbf{Q}[E_2] & & & & H^1(\widehat{D})(-1) & & \\
\| & & \| & & & & \| & & \\
\mathbf{Q}(-1) & & \mathbf{Q}(-1) \oplus \mathbf{Q}(-1) & & & & \mathbf{Q}(-2) & &
\end{array}$$

The map $\alpha$ is defined by $\alpha(D) = E_1 + E_2$. As a consequence, we have the following proposition.

PROPOSITION 5.1.

(1) *There is a sub Hodge structure $V_2$ in $H^2(W_{t_0})$ such that*

$$V_2 \simeq \mathbf{Q}(-1), \qquad H^2(W_{t_0})/V_2 \simeq \mathbf{Q}(-2).$$

(2) *The de Rham part $H^3_{\widehat{D},dR}(\widehat{\mathbf{A}^2})$ is generated by the image of*

$$\omega = \frac{dy\,dz}{yz - t} \in H^2_{dR}(W_{t_0}).$$

We set $V_4 = H^2(W_{t_0})/V_2$.

*Proof.* (2) Let

$$\Omega = \frac{dx\,dy\,dz}{xyz + t(1 - x - y - z)} = \frac{(yz - t)\,dx}{x(yz - t) + t(1 - y - z)} \wedge \frac{dy\,dz}{yz - t}$$

be a differential form on $\mathbf{A}^2 - W_{t_0}$. Then the residue $\mathrm{res}_{W_{t_0}}(\Omega)$ of $\Omega$ along $W_{t_0}$ is equal to $\omega$ on $\mathbf{A}^2 - D \subset \widehat{A^2} - \widehat{D}$. Therefore $\omega$ defines a holomorphic two form on $W_{t_0}$. Since the residue of

$$\omega = \frac{z\,dy}{yz - t} \wedge \frac{dz}{z}$$

along $D$ is equal to $dz/z$, the image of $\omega$ under the map $H^2_{dR}(W_{t_0}) \to H^1_{dR}(\widehat{D})(-1)$ is a generator of $H^1_{dR}(\widehat{D})(-1)$. ∎

### 5.2 Relative cycles and extension

We define four affine planes $L_1, L_2, L_3$ and $M$ in $\mathbf{A}^3$ by $L_1 = \{x = 0\}$, $L_2 = \{y = 0\}$, $L_3 = \{z = 0\}$ and $M = \{1 - x - y - z = 0\}$. Then $(L_i \cap M)_{t_0} \subset W_{t_0}$. We set $T = (L_1 \cap M) \cup (L_2 \cup M) \cup (L_3 \cap M)$. Then we have the following dual exact sequences.

$$0 \to H_2(W_{t_0}, \mathbf{Q}) \to H_2(W_{t_0}, T_{t_0}, \mathbf{Q}) \xrightarrow{\alpha} H_1(T_{t_0}, \mathbf{Q}) \to 0,$$
$$0 \to H^1(T_{t_0}, \mathbf{Q}) \to H^2(W_{t_0}, j_!\mathbf{Q}) \to H^2(W_{t_0}, \mathbf{Q}) \to 0,$$

where $j : W_{t_0} - T_{t_0} \to W_{t_0}$ is the open immersion. We set $\overline{\gamma} = \overline{\gamma}_1 + \overline{\gamma}_2 + \overline{\gamma}_3$, where

$$\overline{\gamma}_1 = \{(x, y, z) = (0, t, 1 - t) \mid t \in [0, 1]\},$$

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$$\overline{\gamma}_2 = \{(x, y, z) = (1-t, 0, t) \mid t \in [0, 1]\},$$
$$\overline{\gamma}_3 = \{(x, y, z) = (t, 1-t, 0) \mid t \in [0, 1]\}.$$

Then $\overline{\gamma}$ defines an element in $H_1(T_{t_0}, \mathbf{Q})$, which is also denoted by $\overline{\gamma}$. Let $\gamma$ be an element in $H_2(W_{t_0}, T_{t_0}, \mathbf{Q})$ such that $\alpha(\gamma)$ is equal to $\overline{\gamma}$. Then $\gamma$ is represented by the relative 2-cycle $\Gamma$ defined by

$$\Gamma = \{(y, z) \mid y, z \geqslant 0, y + z \leqslant 1\}.$$

### 5.3 Pairing given by period integral

We use the coordinate $(y, z)$ to compute the pairing $(\gamma, \omega)$. We assume that $t \in \mathbf{R}$ and $t < 0$. Then we have

$$(\gamma, \omega) = \int_\Gamma \frac{dy\, dz}{yz - t} = \int_0^1 \left\{ \int_0^{1-z} \frac{dy}{yz - t} \right\} dz.$$

It is equal to

$$\begin{aligned}
&\int_0^1 \frac{1}{z} \left[\log\left(1 - \frac{yz}{t}\right)\right]_{y=0}^{1-z} dz = \int_0^1 \frac{dz}{z} \log\left(1 - \frac{z}{t} + \frac{z^2}{t}\right) \\
&= \int_0^1 \left\{ \log\left(1 - \frac{z}{\alpha(t)}\right) + \log\left(1 - \frac{z}{1-\alpha(t)}\right)\right\} \frac{dz}{z} \\
&= Li_2\left(\frac{1}{\alpha(t)}\right) + Li_2\left(\frac{1}{1-\alpha(t)}\right) \\
&= -\frac{1}{2}\{\log(1-\alpha(t)) - \log(-\alpha(t))\}^2
\end{aligned}$$

where $\alpha(t) < 0, 1 - \alpha(t) > 1$ are the solutions of the equation

$$z^2 - z + t = 0.$$

Then $\alpha(t) \to 0$ for $t \to 0$. As a consequence, we have

PROPOSITION 5.2.

(1) *We have*
$$(\gamma, \omega) = -\tfrac{1}{2}\{\log(1-\alpha(t)) - \log(-\alpha(t))\}^2.$$

(2) *Let $\rho_t$ be the monodromy action of a small circle around $t = 0$. Then we have*

$$(\rho_t(\gamma), \omega) = -\tfrac{1}{2}\{\log(1-\alpha(t)) - \log(-\alpha(t)) - 2\pi\mathbf{i}\}^2,$$

*and*

$$((\rho_t-1)\cdot\gamma,\omega)=2\pi\mathbf{i}(\log(1-\alpha(t))-\log(-\alpha(t)))+2\pi^2,$$
$$((\rho_t-1)^2\cdot\gamma,\omega)=4\pi^2.$$

### 5.4 Two topological cycles and the monodromy action on the homology

**Definition of** $\gamma_1$. Let $\delta$ be a small circle around 0 in $z$-plane and $\overline{\delta}$ be its image in $\widehat{D}=\{yz=t\}$. Let $N_{\overline{\delta}}$ be its tubular neighborhood in $\widehat{\mathbf{A}^2}$ and $\gamma_1=\partial N_{\overline{\delta}}$ be its boundary. The cycle $\gamma_1$ is a $S^1$ bundle over $\overline{\delta}$. By Cauchy formula, we have

$$(\gamma_1,\omega)=(2\pi\mathbf{i})^2.$$

**Definition of** $\gamma_2$. Let $l$ be a path connecting $\alpha(t)$ and $1-\alpha(t)$ in $\{z\in\mathbf{C}\mid z\neq 0\}$ and $\bar{l}$ be its image in $\widehat{D}=\{yz=t\}$. We choose a tubular neighborhood $N_{\bar{l}}$ of $\bar{l}$ and retraction $r:N_{\bar{l}}\to\bar{l}$ such that $r^{-1}(\alpha(t))\subset E_1$ and $r^{-1}(1-\alpha(t))\subset E_2$. Note that the point $p_1$ and $p_2$ are given by $(y,z)=(1-\alpha(t),\alpha(t))$ and $(y,z)=(\alpha(t),1-\alpha(t))$. Then $\partial(r^{-1}(\alpha(t)))$ and $\partial(r^{-1}(1-\alpha(t)))$ are bounded by $T_1$ and $T_2$ in $E_1\cap X_t$ and $E_2\cap X_t$. Let $z_0$ be a point in $l$ and $\overline{z_0}$ be the corresponding point in $\widehat{D}$. Then $\partial(r^{-1}(\overline{z_0}))$ forms an $S^1$-bundle $S$ over $\bar{l}$. We set $\gamma_2=S\cup T_1\cup T_2$ and we have

$$(\gamma_2,\omega)=2\pi\mathbf{i}\int_{\alpha(t)}^{1-\alpha(t)}\frac{dz}{z}=2\pi\mathbf{i}(\log(1-\alpha(t))-\log(\alpha(t))).$$

It is equal to $2\pi\mathbf{i}(\log(1-\alpha(t))-\log(-\alpha(t)))+2\pi^2$ by choosing a proper choice of $l$.

**Action of the monodromy on the topological cycles**. By the previous subsection, we have the following proposition.

PROPOSITION 5.3. *Under the above notations, we have*

$$(\rho_t-1)\cdot\gamma=\gamma_2,\qquad (\rho_t-1)^2\cdot\gamma=-\gamma_1.$$

## §6. Lower bound of the image of cycle map

### 6.1 Monodromy weight filtration and the element $\gamma_{ijk,l}$

In this section, we prove Proposition 4.1. We choose an open set $U_0$ of $\widetilde{X}_0$ such that the pair $(U_0,U_0\cap M_l)$ is homeomorphic to $(Y_0,Y_0\cap D)$ where $Y_0$ and $D_0$ is a subvariety of $\mathbf{A}^3_{\mathbf{C}}$ defined by $Y_0=\{xyz=0\}$, $D=\{1-x-y=0\}$. Let $U$ be a sufficiently small tubular neighborhood of $U_0$. Then $U$ is homeomorphic to $\mathbf{A}^3_{\mathbf{C}}$.

We consider the restriction of the complex (15) to $U_0$, and the induced filtration $M$ on it. This filtration is also denoted by $M$. Let $X \to \Delta = \{t \in \mathbf{C} \mid |t| < \epsilon\}$ be the family of affine varieties defined in (19). We blow up the variety $X$ at $q_{23} = \{(x,y,z,t) = (1,0,0,0)\}$, $q_{31} = \{(x,y,z,t) = (0,1,0,0)\}$, $q_{12} = \{(x,y,z,t) = (0,0,1,0)\}$, and we have a family $\widetilde{X} \to \Delta$ whose closed fiber is a simple normal crossing variety. Though it is not a proper family of varieties, we consider a filtration similar to the monodromy weight filtration in $i^*\mathbf{R}\bar{j}_*\bar{j}^*\mathbf{Q}$. The associated spectral sequence will be written as $E_{U,r}^{p,q}$.

PROPOSITION 6.1.

(1) *The $E_2$-terms are given as follows.*

$$E_{U,2}^{-2,4} = \mathbf{Q}(-2), \qquad E_{U,2}^{0,2} = \mathbf{Q}(-1), \qquad E_{U,2}^{2,0} = 0.$$

*As a consequence it degenerates at $E_2$.*

(2) *The natural map $E_2^{p,q} \to E_{U,2}^{p,q}$ is surjective. As a consequence, the natural map $H^2(X_{t_0}) \to H^2(U_{t_0})$ is strictly compatible with respect to the induced filtration.*

*Proof.* We compute $E_2^{0,2}$. $E_1$-terms are similar as in Section 4.2. The suffix $l$ appearing in the symbol $m_{ij,l}$ is 1, so we denoted it by $m_{ij}$. The differentials are given by the same formula.

$$\begin{aligned}
H^0(T_U^{(1)})(-1) &\simeq \langle 1_{l_{ij}} \rangle_{1\leqslant i<j\leqslant 3} \oplus \langle 1_{m_{ij}} \rangle_{1\leqslant i\neq j\leqslant 3},\\
H^0(T_U^{(0)})(-1) \oplus H^2(T_U^{(2)})(0) &\simeq \langle [m_{ij}]_{g_i} \rangle_{1\leqslant i\neq j\leqslant 3} \oplus \langle [m_{ij}]_{h_{ij}} \rangle_{1\leqslant i\neq j\leqslant 3}\\
&\quad \oplus \langle 1_{q_{ij}} \rangle_{1\leqslant i<j\leqslant 3} \oplus \langle 1_{p_{123}} \rangle,\\
H^2(T_U^{(1)})(0) &\simeq \langle [m_{ij}] \rangle_{1\leqslant i\neq j\leqslant 3}.
\end{aligned}$$

The space $E_{U,2}^{0,2}$ is one dimensional generated by

$$\begin{aligned}
\gamma_{123} = {} & [m_{12}+m_{21}]_{h_{12}} + [m_{23}+m_{32}]_{h_{23}} + [m_{31}+m_{13}]_{h_{31}}\\
& + 1_{p_{123}} - 1_{q_{12}} - 1_{q_{23}} - 1_{q_{31}}.
\end{aligned}$$

In fact, the linear form

$$\begin{aligned}
\gamma_{123}^* = {} & -[m_{12}+m_{21}]_{h_{12}}^* - [m_{23}+m_{32}]_{h_{23}}^* - [m_{31}+m_{13}]_{h_{31}}^*\\
& + 1_{p_{123}}^* - 1_{q_{12}}^* - 1_{q_{23}}^* - 1_{q_{31}}^*
\end{aligned}$$

vanishes on the image of $E_1^{-1,2}$ and nonzero on $\gamma_{123}$. Therefore the restriction $E_2^{0,2} \to E_{U,2}^{0,2}$ is surjective.

We can check that $E_2^{-2,4}$ is generated by the class $[p_{123}]$. Since $d \geqslant 4$, the class $[p_{123}] - [p_{124}] + [p_{134}] - [p_{234}]$ defines an element in $E_2^{-2,4}$, which maps to $[p_{123}]$ under the natural map $E_2^{-2,4} \to E_{U,2}^{-2,4}$. ∎

Let $\iota : H_2(U_{t_0}, \mathbf{Q}) \to H_2(X_{t_0}, \mathbf{Q})$ be the homomorphism induced by the inclusion. Via this inclusion, the filtration $M$ induces that on the image of $\iota$, which is also denoted $M$.

Corollary 6.2.

(1) *Then the image of* $Gr_0^M(\iota) : Gr_0^M(H_2(U_{t_0}, \mathbf{Q})) \to Gr_0^M(H_2(U_{t_0}, \mathbf{Q}))$ *is equal to* $Gr_0^M(Im(\iota))$.

(2) *The image of* $Gr_0^M(\iota)$ *is equal to the annihilator of the kernel of*

$$Gr_0^M H^2(X_{t_0}, \mathbf{Q}(2)) \to Gr_0^M H^2(U_{t_0}, \mathbf{Q}(2))$$

*under the natural pairing*

$$Gr_0^M H^2(X_{t_0}, \mathbf{Q}(2)) \otimes Gr_0^M H_2(X_{t_0}, \mathbf{Q}) \to \mathbf{Q}(2).$$

(3) *The image* $Gr_0^M(\iota)$ *is generated by* $\gamma_{ijk,l}$ *defined by (18).*

### 6.2 The subspace of $Gr_0^M H^2(X_{t_0}, \mathbf{Q}(2))$ generated by $\gamma_{ijk,l}$

In this subsection, we compute the dimension of the subspace of $Gr_0^M H^2(X_{t_0}, \mathbf{Q}(2))$ generated by the image of $\gamma_{ijk,l}$. Eliminating elements of the form $[m_{ij,k}]_{h_{ij,k}}$ using the relation (17), we have an isomorphism

$$\operatorname{coker}(d : H^0(T^{(1)})(-1) \to H^2(T^{(2)})(0) \oplus H^0(T^{(0)})(-1)) \simeq W/K, \tag{20}$$

where

$$W = \langle [u_i]_{g_i} \rangle_i \oplus \langle [m_{ij,k}]_{g_i} \rangle_{i \neq j} \oplus \langle 1_{p_{ijk}} \rangle_{i<j<k} \oplus \langle 1_{q_{ij,k}} \rangle_{i<j,k}. \tag{21}$$

Here $u_i$ is the pull back of the line in $\overline{g_i}$, and $K$ is the space generated by elements of the form (16). By the definition of $u_i$, we have

$$[l_{ij}]_{g_i} = [u_i - \sum_l m_{ij,l}]_{g_i}.$$

Under the isomorphism (20) the class of $\gamma_{ijk,l}$ corresponds to

$$\begin{aligned}\gamma^*_{ijk,l} &= [-m_{ij,l} + m_{ik,l}]_{g_i} + [-m_{jk,l} + m_{ji,l}]_{g_j} + [-m_{ki,l} + m_{kj,l}]_{g_k} \\ &\quad + 1_{p_{ijk}} + 1_{q_{ij,l}} + 1_{q_{jk,l}} + 1_{q_{ki,l}}.\end{aligned}$$

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The projection from $W$ to $\langle [u_i]_{g_i}\rangle_{1\leqslant i\leqslant d}$ (resp. $\langle 1_{q_{ij,l}}\rangle_{1\leqslant i<j\leqslant d}$) with respect to the direct sum (21) is denoted by $\pi_u$ (resp. $\pi_{q,l}$).

LEMMA 6.3. *Suppose that* $v=\sum_{1\leqslant i<j\leqslant d} a_{ij}d(1_{l_{ij}})$ *is an element in* $\langle\gamma^*_{ijk,l}\rangle$. *Then* $v$ *can be uniquely expressed as a linear combination of* $d(1_{l_{1i}}+1_{l_{ij}}+1_{l_{j1}})$ *for* $2\leqslant i<j\leqslant d$.

*Proof.* We set $A_{ijk}=d(1_{l_{ij}}+1_{l_{jk}}+1_{l_{ki}})$. We have $\pi_u(\gamma^*_{ijk,l})=0$ and $\pi_u(d(1_{l_{ij}}))=u_i-u_j$. Therefore $v$ is a linear combination of $A_{ijk}$. Since $A_{1ij}-A_{1ik}+A_{1jk}-A_{ijk}=0$, $v$ is a linear combination of $A_{1ij}$ for $i<j$. Since

$$\pi_{q,l}(A_{1ij})=1_{q_{1i,l}}+1_{q_{ij,l}}+1_{q_{j1,l}},$$

$A_{1ij}$ $(1\leqslant i<j\leqslant d)$ are linearly independent by looking the component $\langle 1_{q_{ij}}\rangle_{2\leqslant i<j\leqslant d}$. ∎

For $1\leqslant i<j<k<m\leqslant d$ and $1\leqslant l\leqslant d$, we set

$$\begin{aligned}\widehat{\gamma_{ijkm,l}}&=\gamma^*_{ijk,l}-\gamma^*_{ijm,l}+\gamma^*_{ikm,l}-\gamma^*_{jkm,l}\\&=p_{ijk}-p_{ijm}+p_{ikm}-p_{jkm}.\end{aligned}$$

Then $\langle\gamma^*_{ijk.l}\rangle$ is generated by $\widehat{\gamma_{1jkm,1}}$ for $2\leqslant j<k<m\leqslant d$ and $\gamma^*_{1ij,l}$ for $2\leqslant i<j\leqslant d$, $1\leqslant l\leqslant d$.

LEMMA 6.4.

(1) *We have*

$$\pi_{q,l}(\gamma^*_{1ij,l'})=\delta_{l,l'}(1_{q_{1i,l}}+1_{q_{ij,l}}+1_{q_{j1,l}})$$

*and* $\pi_{q,l}(\widehat{\gamma_{ijkm,1}})=0$.

(2) *The set* $\gamma^*_{1ij,l}$ $(2\leqslant i<j\leqslant d, 1\leqslant l\leqslant d)$ *are linearly independent in* $\langle\gamma^*_{ijk,l}\rangle/\langle\widehat{\gamma_{ijkm,l}}\rangle$. *As a consequence, we have*

$$\dim(\langle\gamma^*_{ijk,l}\rangle/\langle\widehat{\gamma_{ijkp,l}}\rangle)=\frac{d(d-1)(d-2)}{2}.$$

(3) *Then the set* $\widehat{\gamma_{1ijk,1}}$ *for* $(2\leqslant i<j<k\leqslant d)$ *forms a basis of the space* $\langle\widehat{\gamma_{ijkm,l}}\rangle$. *As a consequence, we have*

$$\dim\langle\widehat{\gamma_{ijkm,l}}\rangle=\frac{(d-1)(d-2)(d-3)}{6}.$$

(4) *We have*

$$\dim\langle\gamma^*_{ijk,l}\rangle=\frac{(d-1)(d-2)(4d-3)}{6}.$$

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*Proof.* The equalities in (1) and (2) are obtained by direct calculations. The argument for linear independence is similar to the previous lemma. The statement (3) is a consequence of (1) and (2). ∎

PROPOSITION 6.5. *The set* $\{A_{1ij}\}_{1\leqslant i<j\leqslant d}$ *forms a basis of* $\langle\gamma^*_{ijk,l}\rangle\cap K$. *As a consequence,* $\dim(\langle\gamma^*_{ijk,l}\rangle\cap K)=((d-1)(d-2))/2$.

*Proof.* Since $A_{1ij}-\sum_{l=1}^{d}\gamma^*_{1ij,l}$ is annihilated by $\pi_u$ and $\pi_{q,l}$, it is an element in $\langle 1_{p_{ijk}}\rangle$.

$$\begin{aligned}A_{1ij}-\sum_{l=1}^{d}\gamma^*_{1ij,l}&=(3-d)1_{p_{1ij}}+\sum_{k\neq i,j,k}(1_{p_{1ik}}+1_{p_{ijk}}+1_{p_{j1k}})\\&=\sum_{k\neq i,j,k}(1_{p_{1ik}}+1_{p_{ijk}}+1_{p_{j1k}}-1_{p_{1ij}})\\&=-\sum_{k\neq i,j,k}\widehat{\gamma_{1ijk,1}}.\end{aligned}$$

As a consequence, $A_{1ij}$ is an element in $\langle\gamma_{ijk,l}\rangle\cap K$. Thus we have the proposition. ∎

COROLLARY 6.6. *The dimension of the subspace of* $Gr^M_0H^2(X_{t_0},\mathbf{Q}(2))$ *generated by* $\gamma_{ijk,l}$ *is equal to* $((d-1)(d-2)(2d-3))/3$.

### 6.3 The dimension of $Gr^M_{-2}H^2(X_{t_0},\mathbf{Q})$ and the proof of the main theorem

In this subsection, we prove the following proposition.

PROPOSITION 6.7. *The dimension of* $Gr^M_{-2}H^2(X_{t_0},\mathbf{Q})$ *is equal to* $((d-1)(d-2)(d-3))/6$.

*Proof.* Since the monodromy weight spectral sequence degenerates at $E_2$-term, the 0th, 1st and 2nd cohomology of the following complex is isomorphic to $Gr^M_{-2}H^2(X_{t_0},\mathbf{Q})$, $Gr^M_{-1}H^3(X_{t_0},\mathbf{Q})=0$ and $Gr^M_0H^4(X_{t_0},\mathbf{Q})\simeq\mathbf{Q}(-2)$. Since $E_2$-term is a cohomology of the complex

$$0\to H^0(T^{(0)})(-2)\to H^2(T^{(1)})(-1)\to H^4(T^{(2)})(0)\to 0$$

and by the expression of strata in Section 4, we have

$$\dim(H^0(T^{(0)})(-2))=\frac{d(d-1)(d-2)}{6}+d\cdot\frac{d(d-1)}{2},$$

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$$\dim(H^2(T^{(1)})(-1)) = \frac{d(d-1)}{2} + d^2(d-1),$$
$$\dim(H^4(T^{(2)})(0)) = d + d \cdot \frac{d(d-1)}{2}.$$

Therefore we have the dimension of $Gr^M_{-2}H^2(X_{t_0}, \mathbf{Q})$. ∎

*Proof of Theorem 1.1.* The image of the cycle map $c^{2,1}_{et}(X)$ contains the image

$$\begin{aligned}\mathrm{Im}(\langle\gamma_{ijk,l}\rangle) &\to H^1(\pi_1(\Delta^*, t_0), H_2(X_{t_0}, \mathbf{Q})) \\ &= \mathrm{Coker}(H^2(X_{t_0}, \mathbf{Q}) \xrightarrow{N} H^2(X_{t_0}, \mathbf{Q})).\end{aligned}$$

Since the monodromy action is strictly compatible with respect to the monodromy weight spectral sequence, the graded piece of the above cokernel is equal to

$$\begin{aligned}&Gr^M_0 H^1(\pi_1(\Delta^*, t_0), H^2(X_{t_0}, \mathbf{Q})) \\ &= \mathrm{Coker}(Gr^M_{-1}H^2(X_{t_0}, \mathbf{Q}) \xrightarrow{N} Gr^M_0 H^2(X_{t_0}, \mathbf{Q})).\end{aligned}$$

Now we consider the following homomorphism of vector spaces.

$$\begin{array}{ccc} & & W/K \\ & & \bigcup \\ Gr^M_{-1}H^2(X_{t_0}) & \xrightarrow{N} & Gr^M_0 H^2(X_{t_0}) \\ & & \uparrow \\ & & \langle\gamma_{ijk,l}\rangle. \end{array}$$

Let $\overline{\langle\gamma_{ijk,l}\rangle}$ be the image of $\langle\gamma_{ijk,l}\rangle$ in $\mathrm{Coker}(Gr^M_{-1}H^2(X_{t_0}) \to W/K)$. Then we have

$$\begin{aligned}\dim \overline{\langle\gamma_{ijk,l}\rangle} &= \dim(\langle\gamma_{ijk,l}\rangle + Gr^M_{-1}H^2(X_{t_0})) - \dim Gr^M_{-1}H^2(X_{t_0}) \\ &\geqslant \dim\langle\gamma_{ijk,l}\rangle - \dim Gr^M_{-1}H^2(X_{t_0}) \\ &= \frac{(d-1)(d-2)(2d-3)}{3} - \frac{(d-1)(d-2)(d-3)}{6} \\ &= \frac{(d-1)^2(d-2)}{2}.\end{aligned}$$

Thus we have the theorem. ∎

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## References

[B] S. Bloch, *Algebraic cycles and higher K -theory*, Adv. Math. **61**(3) (1986), 267–304.

[CMS] A. Collino, S. Müller-Stach and S. Saito, *On $K_1$ and $K_2$ of algebraic surfaces. Special issue in honor of Hyman Bass on his seventieth birthday. Part I*, K-Theory **30**(1) (2003), 37–69.

[M] S. J. Müller-Stach, *Constructing indecomposable motivic cohomology classes on algebraic surfaces*, J. Algebraic Geom. **6**(3) (1997), 513–543.

[PSt] C. Peters and J. H. M. Steenbrink, *Mixed Hodge Structures, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*, A Series of Modern Surveys in Mathematics, Springer, 2010.

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