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Classification of Solutions for Harmonic Functions With Neumann Boundary Value

Tao Zhang and Chunqin Zhou

Abstract. In this paper, we classify all solutions of

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial t} = -c|x|^\beta e^u & \text{on } \partial\mathbb{R}^2_+ \backslash \{0\}, \end{cases}$$

with the finite conditions

$$\int_{\partial\mathbb{R}^2_+} |x|^\beta e^u \, ds < C, \quad \sup_{\overline{\mathbb{R}^2_+}} u(x) < C.$$

Here c is a positive number and $\beta > -1$.

1 Introduction

Motivated by the blow-up analysis for solutions to Neumann boundary value problems in the presence of singular sources, we want to study the following problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial t} = -c|x|^\beta e^u & \text{on } \partial\mathbb{R}^2_+ \backslash \{0\}, \end{cases} \tag{1.1}$$

with the finite conditions

$$\int_{\partial\mathbb{R}^2_+} |x|^\beta e^u \, ds < C, \quad \sup_{\overline{\mathbb{R}^2_+}} u(x) < C, \tag{1.2}$$

where c is a positive number and $\beta > -1$. This problem is one of the blow-up limits for the corresponding Neumann boundary value problem. In this paper, we give all solutions for this problem.

Note that for $-1 < \beta < 0$, the solution to (1.1) and (1.2) can be considered in a weak sense. Recall $u \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+)$ is a weak solution to (1.1) and (1.2), if it satisfies

$$\int_{\mathbb{R}^2_+} \nabla u \nabla \varphi \, dx - c \int_{\partial\mathbb{R}^2_+} |x|^\beta e^u \varphi \, ds = 0$$

for any smooth function φ on $\overline{\mathbb{R}}^2_+$ with compact support. Since $u \in H^1_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+)$ implies $e^u \in L^p_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+)$ for all $p > 0$, then we conclude that any weak solution u of (1.1), (1.2) is

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a classical solution when $\beta \geq 0$. When $\beta \in (-1, 0)$, since $g = |x|^{\beta} e^{u} \in W^{1,s}_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+)$ for $s \in (1, \frac{2}{1-\beta})$, then by $W^{2+k,q}$ estimates for the Neumann boundary problem [ADN, KW]

$$\|u\|_{W^{2+k,q}(M)} \leq C\Big(\| \triangle u\|_{W^{k,q}(M)} + \Big\| \frac{\partial u}{\partial n} \Big\|_{W^{1+k,q}(\partial M)} + \|u\|_{W^{1+k,q}(M)} \Big),$$

where M is a domain with smooth boundary, and the norm on the boundary is defined by

$$\|g\|_{W^{1+k,q}(\partial M)} = \inf \big\{ \|G\|_{W^{1+k,q}(M)} \bigm| G \in W^{1+k,q}(M), G|_{\partial M} = g \big\},$$

we conclude that $u \in W^{2,s}_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+)$ and therefore u is continuous at the origin. Without loss of the generality, in the sequel we assume that a solution u of (1.1) and (1.2) always satisfies $u \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}}^2_+ \backslash \{0\})$ and u is continuous at the origin.

When $\beta = 0$, the problem reduces to

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial t} = -ce^{u} & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

with the energy conditions

$$\int_{\partial \mathbb{R}^2_+} e^{u}, ds < C, \quad \sup_{\overline{\mathbb{R}^2_+}} u(x) < C.$$

All solutions are given by

$$u(s,t) = \ln \frac{2t_1}{(s-s_1)^2 + (t+t_1)^2} + \ln \frac{1}{c},$$

where s_1 is any real number and t_1 is any positive number [Li]. Furthermore, under integral finiteness assumptions, $\int_{\partial \mathbb{R}^2_+} e^{u}\, ds < C$ and $\int_{\mathbb{R}^2_+} e^{2u}\, ds < C$, the result is also valid [OB, ZL, LZ].

When $\beta \neq 0$, the situation is different. Note that (1.1) is no longer translation invariant when $\beta \neq 0$. The complex analysis method in [OB] and the moving sphere method used in [LZ, ZL, PT] therefore cannot be directly utilized to classify all solutions of (1.1) and (1.2). Our research is based on the surfaces with singularities. To this point, let us first recall the definition of surfaces with singularities that was first given in [T1]. A conformal metric ds^2 on a Riemann surface Σ (possibly with boundary) has a conical singularity of order β (a real number with $\beta > -1$) at a point $p \in \Sigma \cup \partial\Sigma$ if in some neighborhood of p, $ds^2 = e^{2u}|z - z(p)|^{2\beta}|dz|^2$, where z is a coordinate of Σ defined in this neighborhood and u is smooth away from p and continuous at p. The point p is then said to be a conical singularity of *angle* $\theta = 2\pi(\beta+1)$ if $p \notin \partial\Sigma$ and a *corner* of angle $\theta = \pi(\beta+1)$ if $p \in \partial\Sigma$. For example, a football has two singularities of equal angle, while a teardrop has only one singularity. Both these examples correspond to the case $-1 < \beta < 0$; in case $\beta > 0$, the angle is larger than 2π, leading to a different geometric picture. Such singularities also appear in orbifolds and branched coverings. They can also describe the ends of complete Riemann surfaces with finite total curvature. If (Σ, ds^2) has conical singularities of order $\beta_1, \beta_2, \ldots, \beta_n$ at $p_1, p_2, \ldots, p_n$, then ds^2 is said to represent the divisor $\mathbf{A} := \sum_{i=1}^{n} \beta_i p_i$.

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Our main point is to consider surfaces with corners on their boundary. We will investigate the existence problem of conformal metrics with constant Gauss curvature and constant geodesic curvature on their boundary. Geometrically, each solution of (1.1) and (1.2) determines a metric $ds^2 = |x|^{2\beta} e^{2u} |dx|^2$ with Gaussian curvature 0 in B_1 and with geodesic curvature c on $\partial B_1 \backslash \{p_1, p_2\}$, where $p_1 = (-1, 0)$ and $p_2 = (1, 0)$. And p_1, p_2 are called the conical singularities of ds^2.

Our main theorem is the following.

Theorem 1.1 *Assume that $u \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}}^2_+ \backslash \{0\})$ and u is continuous at the origin. If u is a solution of (1.1) and (1.2), then u takes the form*

$$u(s,t) = \ln \frac{\sqrt{8}(\beta+1)\lambda^{\beta+1}}{|z^{\beta+1} - z_0|^2}.$$

Here $z_0 = (s_0, t_0)$ with $s_0 \in \mathbb{R}$ and $t_0 = -\sqrt{2} c \lambda^{\beta+1}$, when $\beta = 2k$, $k = 0, 1, 2, \ldots$, while $z_0 = (s_0, t_0)$ with

$$s_0 = \frac{\sqrt{2} c \lambda^{\beta+1}(1 - \cos(\pi\beta))}{\sin(\pi\beta)}$$

and $t_0 = -\sqrt{2} c \lambda^{\beta+1}$ when $\beta \neq k$, $k = 0, 1, 2, \ldots$.

The method to prove Theorem 1.1 is completely different from that in [Li, OB, ZL, LZ]. We will use the tricks in [JWZ], although there the authors consider the following problem:

$$\begin{cases} -\Delta u = |x|^{2\beta} e^u & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial t} = \begin{cases} c_1 |x|^\beta e^{\frac{u}{2}} & \text{on } \partial\mathbb{R}^2_+ \cap \{s > 0\}, \\ c_2 |x|^\beta e^{\frac{u}{2}} & \text{on } \partial\mathbb{R}^2_+ \cap \{s < 0\}, \end{cases} \end{cases}$$

with the energy conditions

$$\int_{\partial\mathbb{R}^2_+} |x|^{2\beta} e^u \, ds < C, \quad \int_{\mathbb{R}^2_+} |x|^\beta e^{\frac{u}{2}} \, dx < C.$$

Here c_1, c_2 are constants and $\beta > -1$. All solutions have this form. The above problems also have been treated in [JAP] under some weak conditions.

Remark 1.2 If, in Theorem 1.1, we change the finite conditions into the energy finite conditions, *i.e.*, we replace the energy conditions of (1.1) by

$$\int_{\partial\mathbb{R}^2_+} |x|^\beta e^u \, ds < C, \quad \int_{\mathbb{R}^2_+} |x|^{2\beta} e^{2u} \, dx < C,$$

then we can also get the same classification results. The most important observation is that if we set $\widetilde{u} = u + \beta \ln |x|$, then $\widetilde{u}$ is bounded from above in the region $\overline{\mathbb{R}^2_+ \backslash B^+_\epsilon(0)}$ for each small $\epsilon > 0$. The proof is similar to the proof of [JWZ, Proposition 3.1]. Then we can establish Proposition 2.1 in the following section, and consequently we can get the same classification results by a similar argument.

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2 The Decay of Solutions at Infinity

In this section, we will show the decay of solutions at infinity by using the standard potential analysis.

Proposition 2.1 *Let u be a solution of* (1.1) *and* (1.2). *Define*

$$d = \frac{1}{\pi}\int_{\partial\mathbb{R}_+^2} c|x|^\beta e^u\, ds.$$

Then we have $d = -\lim_{|x|\to\infty}\frac{u(x)}{\ln|x|}$. Furthermore, we have $d = 2 + 2\beta$.

Proof We use the standard potential analysis to establish this proposition. Similar arguments can be found in [CL, JWZ]. We divide the proof into three steps.

Step 1. $d = -\lim_{|x|\to\infty}\frac{u(x)}{\ln|x|}$. Let

$$w(x) = \frac{c}{2\pi}\int_{\partial\mathbb{R}_+^2}(\log|x-y| + \log|\overline{x}-y| - 2\log|y|)|y|^\beta e^{u(y)}\, dy,$$

where $\overline{x}$ is the reflection point of x about $\{t = 0\}$. It is easy to check that $w(x)$ satisfies

$$\begin{cases} \Delta w = 0, & \text{in } \mathbb{R}_+^2,\\ \dfrac{\partial w}{\partial t} = c|x|^\beta e^u, & \text{on } \partial\mathbb{R}_+^2 \setminus \{0\}, \end{cases}$$

and $\lim_{|x|\to\infty}\frac{w(x)}{\ln|x|} = d$. Consider $v(x) = u + w$. Then $v(x)$ satisfies

$$\begin{cases} \Delta v = 0, & \text{in } \mathbb{R}_+^2,\\ \dfrac{\partial v}{\partial t} = 0, & \text{on } \partial\mathbb{R}_+^2 \setminus \{0\}. \end{cases}$$

We extend $v(x)$ to $\mathbb{R}^2$ by even reflection such that $v(x)$ is harmonic in $\mathbb{R}^2\backslash\{0\}$. Since u and w are continuous at the origin, the singularity of v at 0 is removable. Hence from $\sup_{\overline{\mathbb{R}_+^2}} u(x) < C$, we know $v(x) \le C(1 + \ln(|x| + 1))$ for some positive constant C. Thus $v(x)$ is a constant. This completes the step 1.

Step 2. $d > 1 + \beta$. First, from $\int_{\partial\mathbb{R}_+^2}|x|^\beta e^u dx < +\infty$, we obtain $d \ge 1 + \beta$. Next we assume by contradiction that $d = 1 + \beta$. Let v be the Kelvin transformation of u, *i.e.*, $v(x) = u(\frac{x}{|x|^2}) - (2\beta + 2)\ln|x|$. Then v satisfies

$$\begin{cases} -\Delta v = 0, & \text{in } \mathbb{R}_+^2,\\ \dfrac{\partial v}{\partial t} = -c|x|^\beta e^v, & \text{on } \partial\mathbb{R}_+^2 \setminus \{0\}, \end{cases}$$

with the energy condition $\int_{\partial\mathbb{R}_+^2}|x|^\beta e^v\, ds < \infty$.

Let D^+ be a small positive half-disk centered at zero. Define $w(x)$ by

$$w(x) = \frac{c}{2\pi}\int_{\partial D^+\cap\{t=0\}}(\log|x-y| + \log|\overline{x}-y|)|y|^\beta e^{v(y)}\, dy,$$

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and define $g(x) = v(x) + w(x)$. It is clear that

$$\begin{cases} \Delta g = 0, & \text{in } D^+, \\ \dfrac{\partial g}{\partial t} = 0, & \text{on } \{\partial D^+ \cap \{t = 0\}\} \setminus \{0\}. \end{cases}$$

Therefore, by extending $g(x)$ to $D \setminus \{0\}$ evenly, we obtain a harmonic function $g(x)$ in $D \setminus \{0\}$.

On the other hand, we can check that $\lim_{|x|\to 0} \frac{w}{-\log|x|} = 0$, which implies

$$\lim_{|x|\to 0} \frac{g(x)}{-\log|x|} = \lim_{|x|\to 0} \frac{v(x) + w(x)}{-\log|x|} = 1 + \beta.$$

Since $g(x)$ is harmonic in $D\backslash\{0\}$, we have $g(x) = -(\beta + 1)\log|x| + g_0(x)$ with a smooth harmonic function g_0 in D. By definition, we have $w(x) < 0$ since c is positive. Thus, we have

$$\int_{\partial D^+ \cap \{t=0\}} |x|^\beta e^v \, dx = \int_{\partial D^+ \cap \{t=0\}} |x|^\beta e^{g-w} \, dx \geq \int_{\partial D^+ \cap \{t=0\}} |x|^\beta |x|^{-\beta-1} e^{g_0} \, dx = \infty,$$

which is a contradiction with $\int_{\partial \mathbb{R}^2_+} |x|^\beta e^v \, dx < \infty$. Hence we have shown that $d > 1+\beta$.

Step 3. $d = 2 + 2\beta$. From $d > 1 + \beta$ we can improve the estimates for e^u to

$$e^u \leq C|x|^{-1-\beta-\varepsilon_1}, \quad \text{for } |x| \text{ near } \infty. \tag{2.1}$$

for any small $\varepsilon_1 > 0$. Then by using potential analysis, we obtain

$$-d \ln|x| - C \leq u(x) \leq -d \ln|x| + C$$

for some constant $C > 0$, see [CL].

Next we can get the derivation of gradient estimates. Similar arguments also can be found in [CK, WZ]. First, we choose some ε with $0 < \varepsilon < \varepsilon_1$ and let (r, θ) be the polar coordinate system on $\mathbb{R}^2$. From Step 1, we have

$$u(x) = -\frac{c}{2\pi} \int_{\partial \mathbb{R}^2_+} (\log|x - y| + \log|\overline{x} - y| - 2\log|y|)|y|^\beta e^{u(y)} \, dy + C.$$

Then we get

$$\begin{aligned} r u_r &= x_1 u_{x_1} + x_2 u_{x_2} \\ &= -d - \frac{c}{2\pi} \int_{\partial \mathbb{R}^2_+} \frac{y(x - y)}{|x - y|^2} |y|^\beta e^{u(y)} \, dy - \frac{c}{2\pi} \int_{\partial \mathbb{R}^2_+} \frac{y(\overline{x} - y)}{|\overline{x} - y|^2} |y|^\beta e^{u(y)} \, dy, \end{aligned}$$

and

$$\begin{aligned} u_\theta &= -x_2 u_{x_1} + x_1 u_{x_2} \\ &= \frac{c}{2\pi} \int_{\partial \mathbb{R}^2_+} \frac{\widetilde{y}(x - y)}{|x - y|^2} |y|^\beta e^{u(y)} \, dy + \frac{c}{2\pi} \int_{\partial \mathbb{R}^2_+} \frac{-\widetilde{y}(\overline{x} - y)}{|\overline{x} - y|^2} |y|^\beta e^{u(y)} \, dy, \end{aligned}$$

where $\widetilde{y} = (y_2, -y_1)$.

Now we set

$$\int_{\partial \mathbb{R}^2_+} \frac{|y|}{|x - y|} |y|^\beta e^{u(y)} \, dy = \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,$$

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where $I_1 = \int_{\Omega_1} \frac{|y|}{|x-y|} |y|^\beta e^{u(y)}\, dy$, with $\Omega_1 = \{y \mid \partial\mathbb{R}^2_+ \cap (|y| < \frac{|x|}{2})\}$, and

$$I_2 = \int_{\Omega_2} \frac{|y|}{|x-y|} |y|^\beta e^{u(y)}\, dy,$$

with $\Omega_2 = \{y \mid \partial\mathbb{R}^2_+ \cap (\frac{|x|}{2} \le |y| \le 2|x|)\}$, and

$$I_3 = \int_{\Omega_3} \frac{|y|}{|x-y|} |y|^\beta e^{u(y)}\, dy,$$

with $\Omega_3 = \{y \mid \partial\mathbb{R}^2_+ \cap (|y| > 2|x|)\}$.

When $|x|$ is large enough, we can estimate I_1, I_2, and I_3 by using $e^u \le C|x|^{-1-\beta-\varepsilon_1}$, see (2.1). First we get

$$I_1 = \int_{\Omega_1} \frac{|y|}{|x-y|} |y|^\beta e^{u(y)}\, dy \le \frac{C}{|x|} \int_{\Omega_1} |y|\,|y|^\beta e^{u(y)}\, dy \le C|x|^{-\varepsilon_1} \le C|x|^{-\varepsilon}.$$

Again by using $e^u \le C|x|^{-1-\beta-\varepsilon_1}$, we get

$$I_2 = \int_{\Omega_2} \frac{|y|}{|x-y|} |y|^\beta e^{u(y)}\, dy \le \frac{C}{x^{\varepsilon_1}} \int_{\partial\mathbb{R}^2_+ \cap \{|y| \le 4|x|\}} \frac{dy}{|y|} \le C|x|^{-\varepsilon}.$$

Similarly we can get

$$I_3 = \int_{\Omega_3} \frac{|y|}{|x-y|} |y|^\beta e^{u(y)}\, dy \le C \int_{\Omega_3} |y|^\beta e^{u(y)}\, dy \le C|x|^{-\varepsilon_1} \le C|x|^{-\varepsilon}.$$

So we get $|\langle x, \nabla u\rangle + d| \le C|x|^{-\varepsilon}$ for $|x|$ near ∞. Consequently we have

$$\Big| u_r + \frac{d}{r} \Big| \le C|x|^{-1-\varepsilon} \quad \text{for } |x| \text{ near } \infty. \tag{2.2}$$

In a similar way, we can also get

$$|u_\theta| \le C|x|^{-\varepsilon} \quad \text{for } |x| \text{ near } \infty. \tag{2.3}$$

Therefore from (2.2) and (2.3) we can get by standard potential analysis that

$$u(x) = -d \ln |x| + C + O(|x|^{-1}) \quad \text{for } |x| \text{ near } \infty, \tag{2.4}$$

where C is some positive constant.

Next let us establish the Pohozaev identity of (1.1) and (1.2). Multiply (1.1) by $x \cdot \nabla u$ and integrate over B_R^+ to obtain $-\int_{B_R^+} (x \cdot \nabla u) \Delta u\, dx = 0$. Since

$$(x \cdot \nabla u)\Delta u = \operatorname{div}\big((x \cdot \nabla u) \nabla u \big) - \operatorname{div}\Big(\frac{x |\nabla u|^2}{2} \Big),$$

we obtain

$$\int_{\partial B_R^+ \cap \{t>0\}} x \cdot \nu \frac{|\nabla u|^2}{2} - (\nu \cdot \nabla u)(x \cdot \nabla u)\, ds \\ + \int_{\partial B_R^+ \cap \{t=0\}} x \cdot \nu \frac{|\nabla u|^2}{2} - (\nu \cdot \nabla u)(x \cdot \nabla u)\, ds = 0,$$

where ν is the outward unit normal vector to ∂B_R^+. Hence we have

$$R \int_{\partial B_R^+ \cap \{t>0\}} \frac{|\nabla u|^2}{2} - \Big| \frac{\partial u}{\partial r} \Big|^2 ds + \int_{\partial B_R^+ \cap \{t=0\}} \frac{\partial u}{\partial t} (x \cdot \nabla u)\, ds = 0.$$

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Since

$$\int_{\partial B_R^+\cap\{t=0\}} \frac{\partial u}{\partial t}(x\cdot\nabla u)\,ds = -\int_{-R}^{R} c|s|^\beta e^u s\partial_s u ds = -\int_{-R}^{R} c|s|^\beta s\partial_s e^u\,ds$$
$$= -c|s|^\beta s e^u|_{-R}^{R} + (1+\beta)\int_{-R}^{R} c|s|^\beta e^u\,ds,$$

we get the Pohozaev identity

$$R\int_{\partial B_R^+\cap\{t>0\}} \frac{|u_\theta|^2}{2R^2} - \frac{|u_r|^2}{2}\,ds = c|s|^\beta s e^u|_{-R}^{R} - (1+\beta)\int_{-R}^{R} c|s|^\beta e^u\,ds.$$

By virtue of (2.1), (2.2), and (2.3) and letting $R\to\infty$ in the Pohozaev identity, we get $d = 2+2\beta$. ■

3 Proof of Classification Results

In this section, let us prove Theorem 1.1. To this purpose, it is sufficient to investigate the existence problem of conformal metrics with constant Gauss curvature and constant geodesic curvature on their boundary. Since, geometrically, each solution of (1.1) and (1.2) determines a metric $ds^2 = |x|^{2\beta}e^{2u}|dx|^2$ with Gaussian curvature 0 in B_1 and with geodesic curvature c on $\partial B_1\backslash\{p_1,p_2\}$, where $p_1 = (-1,0)$ and $p_2 = (1,0)$, where p_1 and p_2 are the conical singularities of ds^2, it is sufficient to show the expression of ds^2 formed from solutions of (1.1) and (1.2). Therefore, next we need to prove the following theorem.

Theorem 3.1 *Let u be a solution of (1.1) and (1.2). Then $ds^2 = e^{2u}|z|^{2\beta}|dz|^2$ comes from a conformal metric with constant Gaussian curvature 0 on the unit disk B_1 and constant geodesic curvature c on ∂B_1 admitting the divisor $A = p_1\beta + p_2\beta$. More precisely, there exists $\lambda > 0$ such that the following hold.*

(i) *When $\beta = 2k$, $k = 0,1,2,\dots$, then for any positive number c, the metric is*

$$ds^2 = \frac{8(\beta+1)^2\lambda^{2\beta+2}|z|^{2\beta}|dz|^2}{(|z^{\beta+1}-z_0|^2)^2}$$

for some $z_0 = (s_0,t_0)$ with $s_0\in\mathbb{R}$ and $t_0 = -\frac{2c\lambda^{\beta+1}}{\sqrt{2}}$.

(ii) *When $\beta\neq k$, $k = 0,1,2,\dots$, then for any positive number c, the metric is*

$$ds^2 = \frac{8(\beta+1)^2\lambda^{2\beta+2}|z|^{2\beta}|dz|^2}{(|z^{\beta+1}-z_0|^2)^2}$$

for some $z_0 = (s_0,t_0)$ with $s_0 = \frac{\sqrt{2}c\lambda^{\beta+1}(1-\cos(\pi\beta))}{\sin(\pi\beta)}$ and $t_0 = -\frac{2c\lambda^{\beta+1}}{\sqrt{2}}$.

Proof We prove Theorem 3.1 by using the same argument from [JWZ]. Here we just give a sketch.

First, from Proposition 2.1, we can show that the solution u to (1.1) and (1.2) has a removable singularity at $z=\infty$ by using the Kelvin transformation as in many conformal problems. Actually, we have the following claim.

Claim 1. Let u be a solution of (1.1) and (1.2). Then the metric $ds^2 = |x|^{2\beta}e^{2u(x)}|dx|^2$ on $\mathbb{R}^2_+$ has two conical singularities at 0 and ∞ with the same order β.

Let v be the Kelvin transformation of u, *i.e.*, $v(x) \underline{=} u(x/|x|^2) - 2(\beta+1)\ln|x|$. If u is a solution of (1.1) and (1.2), then $v \in C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+} \setminus \{0\})$ and

$$\begin{cases} -\Delta v = 0, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial v}{\partial t} = -c|x|^\beta e^v, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}. \end{cases}$$

To prove the result, we first show that v is continuous at $x = 0$, *i.e.*, the singularity $z = 0$ of v is removable. Applying the asymptotic estimate (2.4), we have

$$v(x) = u\Big(\frac{x}{|x|^2}\Big) - 2(\beta+1)\ln|x| = \big(d - 2(\beta+1)\big)\ln|x| + O(1)$$

for $|x|$ near 0. Since $d = 2(1+\beta)$, we get that v is bounded near 0. Thus, by standard elliptic regularity, we conclude that v is a $C^2(\mathbb{R}^2_+) \cap C^1(\overline{\mathbb{R}^2_+})$ solution of (1.1) and (1.2).

Next note that $ds^2 = e^{2\widetilde{u}}|dx|^2$ for $\widetilde{u} = u(x) + \beta\log|x|$, where u is a solution of (1.1) and (1.2). So the metric ds^2 has a conical singularity at $z = 0$ with order β. Let $\widetilde{v}(x) = \widetilde{u}(\frac{x}{|x|^2}) - 2\log|x|$ be the Kelvin transformation of $\widetilde{u}$. Then we obtain near $z = 0$

$$\widetilde{v}(x) = u\Big(\frac{x}{|x|^2}\Big) - \beta\log|x| - 2\log|x| = \beta\log|x| + v(x).$$

Since $v(x)$ is a continuous function at $z = 0$, by the definition of a conical singularity we get that the metric $ds^2 = e^{2\widetilde{u}}dx^2$ has a conical singularity at $z = \infty$ with the same order as at $z = 0$.

Next we will introduce a kind of projective connection on $\mathbb{S}^2 = \mathbb{C} \cup \infty$ as defined in [T2]. We have the following claim.

Claim 2. Let u be a solution of (1.1) and (1.2), and $ds^2 = e^{2\widetilde{u}}|dz|^2$, where $\widetilde{u} = u + \beta\ln|z|$. Define

$$\eta(z) = \Big(\frac{\partial^2\widetilde{u}}{\partial z^2} - \Big(\frac{\partial\widetilde{u}}{\partial z}\Big)^2\Big)|dz|^2.$$

Then $\eta(z)$ can be extended to a projective connection on $\mathbb{S}^2 = \mathbb{C} \cup \infty$, still denoted by $\eta(z)$, that is compatible with the divisor $\mathbf{A}= \beta \cdot 0 + \beta \cdot \infty$.

In fact, from the assumption, we know that $\widetilde{u}$ satisfies

$$\begin{cases} -\Delta\widetilde{u} = 0, & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial\widetilde{u}}{\partial t} = -ce^{\widetilde{u}}, & \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}, \\ \displaystyle\int_{\partial\mathbb{R}^2_+} e^{\widetilde{u}}ds < \infty. \end{cases} \tag{3.1}$$

Let $f(z) = \frac{\partial^2\widetilde{u}}{\partial z^2} - \big(\frac{\partial\widetilde{u}}{\partial z}\big)^2$. Then from (3.1), $f(z)$ is holomorphic on $\mathbb{R}^2_+$ and $\operatorname{Im} f = \frac{1}{2}\big(\frac{\partial\widetilde{u}}{\partial s}\frac{\partial\widetilde{u}}{\partial t} - \frac{\partial^2\widetilde{u}}{\partial s\partial t}\big)$. On the other hand, since on $\partial\mathbb{R}^2_+ \setminus \{0\}$, $\frac{\partial\widetilde{u}}{\partial t} = -ce^{\widetilde{u}}$, we have

$$\frac{\partial^2\widetilde{u}}{\partial s\partial t} = -ce^{\widetilde{u}}\frac{\partial\widetilde{u}}{\partial s} = \frac{\partial\widetilde{u}}{\partial s}\frac{\partial\widetilde{u}}{\partial t}.$$

This implies that $f(z)$ is real on $\partial\mathbb{R}^2_+ \setminus \{0\}$, and we may extend $f(z)$ to a holomorphic function on $\mathbb{C} \setminus \{0\}$ by $f(z) = \overline{f(\overline{z})}$ for $z \in \mathbb{R}^2_-$. Thus we extend η to $\mathbb{C}$ such that η is holomorphic on $\mathbb{C} \setminus \{0\}$.

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Since

$$\frac{\partial^2 \widetilde{u}}{\partial z^2} - (\frac{\partial \widetilde{u}}{\partial z})^2 = \frac{\partial^2 u}{\partial z^2} - (\frac{\partial u}{\partial z})^2 - \frac{\beta}{z}\frac{\partial u}{\partial z} - \frac{\beta(\beta+2)}{4z^2}.$$

in $\overline{\mathbb{R}^2_+} \setminus \{0\}$, then similar to the argument in [JWZ], we can show by using Claim 1 that $\eta(z)$ is a projective connection on $\mathbb{S}^2 = \mathbb{C} \cup \infty$ and has a regular singularity of weight $\rho = -\frac{1}{4}\beta(\beta+2)$ at $z = 0$ and at $z = \infty$.

In view of Claim 2, we can give the proof of Theorem 3.1. From Claim 2 we know that $\eta(z)$ is a projective connection on $S^2 = \mathbb{C} \cup \{\infty\}$ with regular singularities at $z = 0$ and $z = \infty$. It follows from [T2, Proposition 2] that

$$\eta(z) = -\frac{\beta(\beta+2)}{4} \cdot \frac{|dz|^2}{z^2}$$

in the standard coordinate z.

Setting $h = e^{-\widetilde{u}}$, we have

$$\frac{\partial^2 h}{\partial z^2} = \frac{\beta(\beta+2)}{4} \cdot \frac{h}{z^2}, \quad \text{for any } z \in \mathbb{R}^2_+, \tag{3.2}$$

and the boundary condition is

$$\frac{\partial h}{\partial \overline{z}} - \frac{\partial h}{\partial z} = ic, \quad \text{on } \partial\mathbb{R}^2_+ \setminus \{0\}. \tag{3.3}$$

All solutions of (3.2) are of the form $h(z,\overline{z}) = f(\overline{z})z^{-\frac{\beta}{2}} + g(\overline{z})z^{1+\frac{\beta}{2}}$, for any $z \in \mathbb{R}^2_+$. Since h is real and analytic, we have

$$h(z,\overline{z}) = a(\overline{z}z)^{-\frac{\beta}{2}} + pz^{1+\frac{\beta}{2}}\overline{z}^{-\frac{\beta}{2}} + \overline{p}\overline{z}^{1+\frac{\beta}{2}}z^{-\frac{\beta}{2}} + b(z\overline{z})^{1+\frac{\beta}{2}}, \quad \text{for any } z \in \mathbb{R}^2_+.$$

Here, $a, b \in \mathbb{R}$ and $p \in \mathbb{C}$. Since $\widetilde{u} = u + \beta \ln|x|$ near 0 for some continuous function u, it is clear that $a \neq 0$. Then rewriting $h(z,\overline{z})$, we have

$$h = a \cdot \Big(\frac{|1 + \overline{\mu}z^{\beta+1}|^2 + \nu|z|^{2\beta+2}}{|z|^\beta}\Big),$$

for some parameters $\mu = \frac{p}{a} \in \mathbb{C}$ and $\nu = \frac{ab - p\overline{p}}{a^2} \in \mathbb{R}$. Therefore, a conformal metric should be

$$ds^2 = \frac{|dz|^2}{h^2} = \frac{1}{a^2} \cdot \frac{|z|^{2\beta}|dz|^2}{(|1 + \overline{\mu}z^{\beta+1}|^2 + \nu|z|^{2\beta+2})^2}.$$

Setting $w = \frac{1}{z}$, we have

$$ds^2 = \frac{1}{a^2} \cdot \frac{|w|^{2\beta}|dw|^2}{(|\overline{\mu} + w^{\beta+1}|^2 + \nu)^2}.$$

On the other hand, if we assume (r,θ) is the polar coordinate system in $\mathbb{R}^2$, then we have $h(r,\theta) = ar^{-\beta} + pre^{i\theta(1+\beta)} + \overline{p}re^{-i\theta(1+\beta)} + br^{2+\beta}$. And its boundary condition (3.3) can be rewritten as

$$-\frac{\partial h}{\partial \theta}(e^{i\theta} + e^{-i\theta}) + ir\frac{\partial h}{\partial r}(e^{i\theta} - e^{-i\theta}) = -2rc,$$

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for $\theta = 0$ and $\theta = \pi$. Therefore, by using the partial derivative $\frac{\partial h}{\partial \theta}$ at $\theta = 0$ and $\theta = \pi$, respectively, we obtain $2(\beta + 1)(\overline{p} - p) = 2ic$, and $2(\beta + 1)(\overline{p}e^{-i\beta\pi} - pe^{i\beta\pi}) = 2ic$. Then there are two cases.

Case 1. β is an integer. Since c is a positive number, only when $\beta = 2k$, $k = 0, 1, 2, \ldots,$ can one determine $\operatorname{Im}\{p\}$, namely $\operatorname{Im}\{p\} = -\frac{c}{2(\beta+1)}$. Now we set $\frac{\operatorname{Im}\{p\}}{a} = -\frac{2c\lambda^{\beta+1}}{\sqrt{2}}$. Then we have

$$a = \frac{\sqrt{2}}{4(\beta + 1)\lambda^{\beta+1}}$$

and consequently,

$$ds^2 = \frac{8(\beta + 1)^2 \lambda^{2(\beta+1)} |w|^{2\beta} |dw|^2}{(|w^{\beta+1} - w_0|^2 + v)^2},$$

where $w_0 = (x_0, t_0)$ for some real number x_0 and $t_0 = -\frac{2c\lambda^{\beta+1}}{\sqrt{2}}$. Set

$$u = \log \frac{2\sqrt{2}(\beta + 1)\lambda^{\beta+1}}{|w^{\beta+1} - w_0|^2 + v}.$$

Then it follows from the definition of the conformal metric that u is a solution of (1.1). Hence, we have $v = 0$. This implies

$$ds^2 = \frac{8(\beta + 1)^2 \lambda^{2(\beta+1)} |w|^{2\beta} |dw|^2}{(|w^{\beta+1} - w_0|^2)^2}.$$

Case 2. $\beta \neq k, k = 0, 1, 2, \ldots$. In this case, one can find a unique complex number p. If we set $\frac{\operatorname{Im}\{p\}}{a} = -\frac{2c\lambda^{\beta+1}}{\sqrt{2}}$, then we have $a = \frac{\sqrt{2}}{4(\beta+1)\lambda^{\beta+1}}$, and consequently we have

$$ds^2 = \frac{8(\beta + 1)^2 \lambda^{2(\beta+1)} |w|^{2\beta} |dw|^2}{(|w^{\beta+1} - w_0|^2 + v)^2},$$

where $w_0 = (x_0, t_0)$ is a fixed point for

$$x_0 = \frac{\sqrt{2}c\lambda^{\beta+1}(1 - \cos(\pi\beta))}{\sin(\pi\beta)} \quad \text{and} \quad t_0 = -\frac{2c\lambda^{\beta+1}}{\sqrt{2}}.$$

Then as in the first case, we can get

$$ds^2 = \frac{8(\beta + 1)^2 \lambda^{2(\beta+1)} |w|^{2\beta} |dw|^2}{(|w^{\beta+1} - w_0|^2)^2}.$$

We complete the proof. ■

Since the domain $\overline{\mathbb{R}}^2_+ \backslash \{0\}$ is simply connected, we consider $z^{1+\beta}$ as a well-defined function, even for non-integer β. In polar coordinates, we have

$$u = \ln \frac{\sqrt{8}(\beta + 1)\lambda^{\beta+1}}{(r^{\beta+1}\cos(1 + \beta)\theta - s_0)^2 + (r^{1+\beta}\sin(1 + \beta)\theta - t_0)^2}.$$

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