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CENTRAL ELEMENTS AND CANTOR-BERNSTEIN'S THEOREM FOR PSEUDO-EFFECT ALGEBRAS

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Abstract

Pseudo-effect algebras are partial algebras $(E; +, 0, 1)$ with a partially defined addition $+$ which is not necessary commutative and with two complements, left and right ones. We define central elements of a pseudo-effect algebra and the centre, which in the case of MV-algebras coincides with the set of Boolean elements and in the case of effect algebras with the Riesz decomposition property central elements are only characteristic elements. If E satisfies general comparability, then E is a pseudo MV-algebra. Finally, we apply central elements to obtain a variation of the Cantor-Bernstein theorem for pseudo-effect algebras.

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1. Introduction

Recently two non-commutative generalizations of MV-algebras introduced by Chang [3] have appeared: pseudo MV-algebras of Georgescu and Iorgulescu [13] and generalized MV-algebras of Rachůnek [21] which, in addition, are equivalent. Also a non-commutative version of BL-algebras, pseudo-BL-algebras, have been introduced in [6]. Non-commutative algebras are algebraic non-commutative analogues of non-commutative reasoning. Such reasoning can be met in the everyday life quite often. Many psychological processes are depending on the order of variables. The result is not the same when we first put on our shoes and then socks, or conversely. Today there exists even a programming language [1] based on a non-commutative logic.

Recently in [9, 10] we have introduced pseudo-effect algebras as a non-commutative generalization of effect algebras, which play an important role in mathematical foundations of quantum mechanics. Effect-algebras were introduced by Foulis and Bennett [11] as an additive counterpart to D-posets introduced by Kôpka and Chovanec [19].

In many cases pseudo-effect algebras are intervals in unital po-groups (G, u) [10], and every pseudo MV-algebra is an interval in a unital ℓ-group (G, u).

In the present paper we introduce the notion of central elements of pseudo-effect algebras. For effect algebras this was done in [15]. We show that such elements form always a Boolean algebra of E. The paper is organized as follows. In Section 2 we define pseudo-effect algebras and their central elements. In the case when the pseudo-effect algebra satisfies a variation of the Riesz decomposition property, we characterize central elements as those elements e satisfying $e \wedge e' = 0$, Section 3. In Section 4, we show that any pseudo-effect algebra with general comparability is a pseudo MV-algebra. If E is monotone σ-complete, then the centre is a Boolean σ-algebra, Section 5, and finally, a version of the Cantor-Bernstein theorem will be proved.

We recall that Jakubík gave two versions of the Cantor-Bernstein theorem for σ-complete MV-algebras [16] and for pseudo MV-algebras [17]. Another generalization of the Cantor-Bernstein theorem for σ-complete MV-algebras is given in [4] and for monotone σ-complete effect algebras in [18] and for orthomodular lattices in [5].

2. Central elements of pseudo-effect algebras

A partial algebra $(E; +, 0, 1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* if, for all $a, b, c \in E$, the following holds:

(i) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$;

(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$;

(iii) if $a + b$ exists, there are elements $d, e \in E$ such that $a + b = d + a = b + e$;

(iv) if $1 + a$ or $a + 1$ exists, then $a = 0$.

If we define $a \leq b$ if and only if there exists an element $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering on E such that $0 \leq a \leq 1$ for any $a \in E$. It is possible to show that $a \leq b$ if and only if $b = a + c = d + a$ for some $c, d \in E$. We write $c = a \,/\, b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a \,/\, b) = b, \quad \text{and} \quad a = (b \setminus a) \,/\, b = b \setminus (a \,/\, b).$$

If $a \leq b \leq c$, then

$$(c \setminus a) \setminus (b \setminus a) = c \setminus b, \qquad (a \,/\, b) \,/\, (a \,/\, c) = b \,/\, c,$$
$$(c \setminus b) \,/\, (c \setminus a) = b \setminus a, \qquad (a \,/\, c) \setminus (b \,/\, c) = a \,/\, b.$$

Let $E = (E; +, 0, 1)$ be a pseudo-effect algebra. We define $x^- := 1 \setminus x$ and $x^\sim := x / 1$ for any $x \in E$. For a given element $e \in E$, we denote by $[0, e] := \{x \in E : 0 \leq x \leq e\}$. Then $[0, e]$ endowed with $+$ restricted to $[0, e] \times [0, e]$ is a pseudo-effect algebra $[0, e] = ([0, e]; +, 0, e)$. Then, for any $x \in [0, e]$ we have $x^{-_e} := e \setminus x$ and $x^{\sim_e} := x / e$ and $e = x^{-_e} + x = x + x^{\sim_e}$. For basic properties of pseudo-effect algebras see [9, 10].

For example if (G, u) is a unital (not necessary Abelian) po-group with strong unit u, and $\Gamma(G, u) := [0, u] = \{g \in G : 0 \leq g \leq u\}$, then $(\Gamma(G, u); +, 0, u)$ is a pseudo-effect algebra if we restrict the group addition $+$ to $\Gamma(G, u)$.

We recall that a *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type (2, 1, 1, 0, 0) such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via $y \odot x = (x^- \oplus y^-)^\sim$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^\sim = 0$; $1^- = 0$;
(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;
(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;
(A8) $(x^-)^\sim = x$.

In [7] it was shown that every pseudo MV-algebra is isomorphic to $\Gamma(G, u)$, where (G, u) is a unital ℓ-group with strong unit u, where $a \oplus b := (a + b) \wedge u$, $a \odot b = (a - u + b) \vee 0$ and $a^\sim = u - a$ and $a^- = -a + u$.

If M is a pseudo MV-algebra, then the partial operation $a + b$ is defined if and only if $a \leq b^-$, and then $a + b = a \oplus b$, and $(M; +, 0, 1)$ is a pseudo-effect algebra.

DEFINITION 2.1. An element e of a pseudo-effect algebra E is said to be *central* (or *Boolean*) if there exists an isomorphism

$$f_e : E \to [0, e] \times [0, e^\sim] \tag{1}$$

such that $f_e(e) = (e, 0)$ and if $f_e(x) = (x_1, x_2)$, then $x = x_1 + x_2$ for any $x \in E$.

We denote by $C(E)$ the set of all central elements of E, and $C(E)$ is said to be the *centre* of E. We recall that $0, 1 \in C(E)$.

PROPOSITION 2.2. *Let e be a central element of a pseudo-effect algebra E, and f_e the corresponding mapping from Definition* 2.1. *Then*

(i) $f_e(e^\sim) = (0, e^\sim)$.
(ii) *If $x \leq e$, then $f_e(x) = (x, 0)$.*
(iii) $e \wedge e^\sim = 0$.

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(iv) *If* $y \le e^\sim$ *then* $f_e(y) = (0, y)$.

(v) $e^\sim = e^-$.

(vi) *For any* $x \in E$, $x \wedge e \in E$ *and* $x \wedge e^\sim \in E$, *and*

$$f_e(x) = (x \wedge e, x \wedge e^\sim). \tag{2}$$

(vii) *If* $f_e(x) = (x_1, x_2)$, *then* $x = x_1 \vee x_2$, $x_1 \wedge x_2 = 0$, *and* $x_2 + x_1 = x$.

PROOF. (i) $f_e(e^\sim) = f_e(e)^\sim = (e, 0)^\sim = (e^{\sim\prime}, 0^{\sim\prime}) = (0, e^\sim)$.

(ii) Let $f_e(x) = (x_1, x_2)$, $x \le e$. Then $(x_1, x_2) = f_e(x) \le f_e(e) = (e, 0)$, that is, $x_2 = 0$. Hence $x = x_1 + x_2 = x_1$.

(iii) Let $x \le e, e^\sim$. Then $(x, 0) = f_e(x) \le f_e(e^\sim) = (0, e^\sim)$, so that $x = 0$. (iv) Let $y \le e^\sim$. Then $f_e(y) = (y_1, y_2)$, and $y_1 \le e$, $y_1 \le y \le e^\sim$ so that by (iii), $y_1 = 0$ and $y = y_1 + y_2 = y_2$.

(v) $(e, e^\sim) = f_e(e^- + e) = f_e(e^-) + (e, 0) = (e_1, e_2) + (e, 0) = (e_1 + e, e_2)$ which yields $e = e_1 + e$ and $e_2 = e^\sim$, that is, $e_1 = 0$ and $f_e(e^-) = (0, e^\sim)$ which gives $e^- = 0 + e^\sim = e^\sim$.

(vi) Let π_e and $\pi_{e^\sim}$ be the projections from $[0, e] \times [0, e^\sim]$ onto $[0, e]$ and $[0, e^\sim]$, respectively. Then $p_e := \pi_e \circ f_e$ and $p_{e^\sim} := \pi_{e^\sim} \circ f_e$ are homomorphisms from E into $[0, e]$ and $[0, e^\sim]$, respectively. It is clear that $p_e(x) \le x, e$. If now $y \le x, e$, then by (ii), $y = p_e(y) \le p_e(x)$, so that $e \wedge x = p_e(x) \in E$ for any $x \in E$. Similarly, $p_{e^\sim} \le x, e^\sim$, and if $z \le x, e^\sim$ then by (iv), $z = p_{e^\sim}(z) \le p_{e^\sim}(x)$, that is, $p_{e^\sim}(x) = x \wedge e^\sim$. Consequently, $f_e(x) = (x \wedge e, x \wedge e^\sim)$, $x \in E$.

(vii) It is clear that $x \ge x_1, x_2$. Let $z \ge x_1, x_2$. Then $x_1 = p_e(x_1) \le p_e(z)$ and $x_2 = p_{e^\sim}(x_2) \le p_{e^\sim}(z)$, so that $x = x_1 + x_2 \le p_e(z) + p_{e^\sim}(z) = z$ which proves $x_1 \vee x_2 = x$. It is evident that $x_1 \wedge x_2 = 0$.

By (2) we have $x_1 = x \wedge e$ and $x_2 = x \wedge e^\sim$. Hence by (v), $(x \wedge e)^- = x^- \vee e^- \ge x \wedge e^- = x \wedge e^\sim = x_2$, which gives $x_2 + x_1 \in E$. Then $p_e(x_2 + x_1) = p_e(x_2) + p_e(x_1) = p_e(x_1) = x_1$ and $p_{e^\sim}(x_2 + x_1) = p_{e^\sim}(x_2) + p_{e^\sim}(x_1) = x_2$ which proves $x_2 + x_1 = x_1 + x_2 = x$. □

In view of Proposition 2.2 (v), if $e \in C(E)$, then we will write $e' := e^- = e^\sim$.

THEOREM 2.3. *Let E be a pseudo-effect algebra. If $e, f \in C(E)$, then $e \wedge f \in E$ and $e \wedge f \in C(E)$, and $C(E) = (C(E); \wedge, \vee, ', 0, 1)$ is a Boolean algebra.*

PROOF. It is evident that $0, 1 \in C(E)$. Let now $e \in C(E)$, then $e^\sim = e^-$ and by Proposition 2.2 the mapping $f_e(x) = (x \wedge e, x \wedge e^\sim)$ is an isomorphism from E onto $[0, e] \times [0, e^\sim]$. By Proposition 2.2 (vii), we have that the mapping $f_{e^-}(x) = (x \wedge e^-, x \wedge e)$ is an isomorphism from E onto $[0, e^-] \times [0, e]$ such that $f_{e^-}(e^-) = (e^-, 0)$ and if $f_{e^-}(x) = (x_1, x_2)$, then $x = x_2 + x_1 = x_1 + x_2$, which proves $e' \in C(E)$.

Assume now $e, f \in C(E)$. Then, for every $x \in E$,

$$(3) \quad x = x \wedge e + x \wedge e' = x \wedge e \wedge f + x \wedge e \wedge f' + x \wedge e' \wedge f + x \wedge e' \wedge f'$$

and in view of $1 = e \wedge f + e \wedge f' + e' \wedge f + e' \wedge f'$, we have $(e \wedge f)^{\sim} = e \wedge f' + e' \wedge f + e' \wedge f'$. On the other hand, $1 = f' + f = f' \wedge e + f' \wedge e' + f \wedge e' + f \wedge e = e \wedge f' + e' \wedge f + e' \wedge f' + e \wedge f$, by Proposition 2.6 (ii), so that $(e \wedge f)^{-} = e \wedge f' + e' \wedge f + e' \wedge f' = (e \wedge f)^{\sim}$. Hence the mapping $f_{e \wedge f}(x) = (x \wedge e \wedge f, x \wedge e \wedge f' + x \wedge e' \wedge f + x \wedge e' \wedge f')$ is a well defined mapping from E into $[0, e \wedge f] \times [0, (e \wedge f)^{\sim}]$ which is injective in view of (3). Moreover, if $f_{e \wedge f}(x) = (x_1, x_2)$, then $x_1 + x_2 = x$, and $f_{e \wedge f}(e \wedge f) = (e \wedge f, 0)$.

Assume $x_1 \in [0, e \wedge f], x_2 \in [0, (e \wedge f)^{\sim}]$. Then $x_1 \leq x_2^{-}$, so that $x = x_1 + x_2 \in E$. Hence $(x_1 + x_2) \wedge e \wedge f = x_1 \wedge e \wedge f + x_2 \wedge e \wedge f = x_1 + x_2 \wedge e \wedge f$. On the other side, $x_2 \leq (e \wedge f)^{\sim} = e \wedge f' + e' \wedge f + e' \wedge f'$ so that $x_2 \wedge e \leq e \wedge f' + e' \wedge f \wedge e + e' \wedge f' \wedge e = e \wedge f'$ and $x_2 \wedge e \wedge f \leq e \wedge f' \wedge f = 0$.

Similarly, $(x_1 + x_2) \wedge e \wedge f' + (x_1 + x_2) \wedge e' \wedge f + (x_1 + x_2) \wedge e' \wedge f' = x_1 \wedge e \wedge f' + x_2 \wedge e \wedge f' + x_1 \wedge e' \wedge f + x_2 \wedge e' \wedge f + x_1 \wedge e' \wedge f' + x_2 \wedge e' \wedge f'$. But $x_1 \leq e \wedge f$. Then $x_1 \wedge e \wedge f' \leq e \wedge f \wedge e \wedge f' = 0$ and $x_1 \wedge e' \wedge f \leq e \wedge f \wedge e' \wedge f = 0$ which proves that $f_{e \wedge f}$ is surjective.

Finally, we show that $f_{e \wedge f}(x + y) = f_{e \wedge f}(x) + f_{e \wedge f}(y)$ whenever $x + y \in E$. Calculate

$$(*) \quad \begin{aligned} &(x + y) \wedge e \wedge f' + (x + y) \wedge e' \wedge f + (x + y) \wedge e' \wedge f' \\ &= x \wedge e \wedge f' + y \wedge e \wedge f' + x \wedge e' \wedge f' + y \wedge e' \wedge f' \\ &\quad + x \wedge e' \wedge f' + y \wedge e' \wedge f'. \end{aligned}$$

Then $x \wedge e' \wedge f \leq (y \wedge e \wedge f')^{-}$ so that $x \wedge e' \wedge f + y \wedge e \wedge f' \in E$. We assert

$$(**) \quad \begin{aligned} x \wedge e' \wedge f + y \wedge e \wedge f' &= (x \wedge e' \wedge f) \vee (y \wedge e \wedge f') \\ &= y \wedge e \wedge f' + x \wedge e' \wedge f. \end{aligned}$$

Let $z \geq y \wedge e \wedge f', x \wedge e' \wedge f$. Then $z \wedge e \wedge f' \geq y \wedge e \wedge f'$ and $z \wedge e' \wedge f \geq x \wedge e' \wedge f$. Hence $z \geq z \wedge e \wedge f' + z \wedge e' \wedge f$ which proves (**). In a similar way we can prove that $y \wedge e' \wedge f + x \wedge e' \wedge f' = x \wedge e' \wedge f' + y \wedge e' \wedge f$. Therefore for (*) we have that it equals to $x \wedge e \wedge f' + x \wedge e' \wedge f + y \wedge e \wedge f' + x \wedge e' \wedge f' + y \wedge e' \wedge f + y \wedge e' \wedge f'$.

In a similar way, we have that it equals to $x \wedge e \wedge f' + x \wedge e' \wedge f + x \wedge e' \wedge f' + y \wedge e \wedge f' + y \wedge e' \wedge f + y \wedge e' \wedge f'$. Consequently, $e \wedge f \in C(E)$, and $x \wedge (e \wedge f)^{\sim} = x \wedge e \wedge f' + x \wedge e' \wedge f + x \wedge e' \wedge f'$. □

PROPOSITION 2.4. *Let $x \in E$ and $e \in C(E)$. Then*

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(i) $x \wedge e = 0$ *if and only if* $x \leq e^-$ *if and only if* $x \leq e^\sim$ *if and only if* $e \leq x^-$ *if and only if* $e \leq x^\sim$.

(ii) $e + e \in E$ *implies* $e = 0$.

PROOF. (i) Let $x \wedge e = 0$, then $x = x \wedge e + x \wedge e^- = x \wedge e^- \leq e^-$. Let $x \leq e^-$. Then $x \wedge e \leq e^- \wedge e = 0$. Let $x \wedge e = 0$. Then $x \leq e^\sim$ and $e \leq x^-$. If $e \leq x^-$, then $x \leq e^\sim = e^-$. If $x \wedge e = 0$, then $x = x \wedge e + x \wedge e^-$ that is, $x \leq e^-$ and $e \leq x^\sim$. If $e \leq x^\sim$, then $x \leq e^-$ and $x \wedge e = 0$.

(ii) It follows from (i). □

PROPOSITION 2.5. *Let* $e_1, \ldots, e_n \in C(E)$, $e_i \wedge e_j = 0$ *for* $i \neq j$, *and* $e_1 + \cdots + e_n = 1$. *Then* $x = x \wedge e_1 + \cdots + x \wedge e_n$.

PROOF. If $n = 1$, then $e_1 = 1$. The general case follows mathematical induction. Let $n \geq 2$. Then $e = e_1 + \cdots + e_n$, $e' = e_{n+1} \in C(E)$ and $x = x \wedge e + x \wedge e_{n+1} = x \wedge e_1 + \cdots + x \wedge e_n + x \wedge e_{n+1}$. □

Let $e \in C(E)$, then the mapping $p_e : E \to [0, e]$ defined by

$$p_e(x) := x \wedge e, \quad x \in E, \tag{4}$$

is a homomorphism from E onto $[0, e]$ whose kernel is $[0, e']$.

PROPOSITION 2.6. *Let* $e, f \in C(E)$.

(i) $p_{e\wedge f} = p_e p_f = p_f p_e$.

(ii) *If* $e \wedge f = 0$, *then* $e + f = e \vee f = f + e$ *and* $p_{e\vee f}(x) = p_e(x) + p_f(x) = p_f(x) + p_e(x)$, $x \in E$.

(iii) *If* $f \leq e$, *then* $e \setminus f = f \wedge e' = f / e$, *and* $p_{e\wedge f'}(x) = p_e(x) \setminus p_f(x) = p_f(x) / p_e(x)$, $x \in E$.

PROOF. (i) follows from (4).

(ii) If $e \wedge f = 0$, then by Proposition 2.4, $e + f \in E$ and $f + e \in E$, and $e + f \geq e \vee f \leq f + e$. Hence $p_{e\vee f}(e + f) = p_{e\vee f}(e) + p_{e\vee f}(f) = e + f \leq e \vee f$. In an analogous way $f + e \leq e \vee f$.

Let $x \in E$. Then $p_{e\vee f}(x) = x \wedge (e \vee f)$ and $p_{e'\wedge f'}(x) = x \wedge e' \wedge f'$. Since $x = x \wedge e \wedge f + x \wedge e \wedge f' + x \wedge e' \wedge f + x \wedge e' \wedge f' = x \wedge e + x \wedge f + p_{(e\vee f)'}(x)$, so that $p_{e\vee f}(x) = x \wedge e + x \wedge f$. If now $z \geq x \wedge e, x \wedge f$, then $z = z \wedge e \wedge f + z \wedge e \wedge f' + z \wedge e' \wedge f + z \wedge e' \wedge f' \geq z \wedge e + z \wedge f \geq x \wedge e + x \wedge f$, which proves

$$x \wedge (e \vee f) = (x \wedge e) + (x \wedge f) = (x \wedge e) \vee (x \wedge f).$$

(iii) Let $f \leq e$. Then $e = f + f / e = e \setminus f + f$ and $e = e \wedge f + e \wedge f' = e \wedge f' + e \wedge f$ so that $e \wedge f' = f / e = e \setminus f = e \wedge f'$. □

PROPOSITION 2.7. *Let* $e_1, \ldots, e_n \in C(E)$, $e_i \wedge e_j = 0$ *for* $i \neq j$.

(i) $e := \bigvee_{i=1}^n e_i = e_1 + \cdots + e_n \in C(E)$, *and*

$$x \wedge e = \bigvee_{i=1}^{n} (x \wedge e_i) = x \wedge e_1 + \cdots + x \wedge e_n, \quad x \in E.$$

(ii) *If* $x_i \leq e_i$ *for* $i = 1, \ldots, n$, *then* $x_1 + \cdots + x_n = x_1 \vee \cdots \vee x_n = x_{i_1} + \cdots + x_{i_n}$, *where* $(i_1, \ldots, i_n)$ *is any permutation of* $(1, \ldots, n)$.

(iii) *If* $a_1, \ldots, a_n \in C(E)$, *then*

$$x \wedge \left(\bigvee_{i=1}^{n} a_i \right) = \bigvee_{i=1}^{n} (x \wedge a_i), \quad x \in E.$$

PROOF. (i) If $n = 1, 2$, the assertion follows from Proposition 2.5 (ii). Let now the statement is true for any integer $i \leq n$, $n \geq 2$. Then $e = \bigvee_{i=1}^n e_i \vee e_{n+1} = (e_1 + \cdots + e_n) + e_{n+1}$ because $(\bigvee_{i=1}^n e_i) \wedge e_{n+1} = 0$ due to the induction assumption. Hence $x \wedge e = x \wedge \left(\bigvee_{i=1}^n e_i \right) \vee x \wedge e_{n+1} = x \wedge e_1 + \cdots + x \wedge e_n + x \wedge e_{n+1}$.

(ii) Since $e = e_1 + \cdots + e_n \in E$, then $x = x_1 + \cdots + x_n \in E$, and $x \geq x_i$ for any i. Assume $z \geq x_i$ for $i = 1, \ldots, n$. Then by (ii) of Proposition 2.6 $z \geq p_e(z) = p_{e_1}(z) + \cdots + p_{e_n}(z) \geq p_{e_1}(x_1) + \cdots + p_{e_n}(x_n) = x_1 + \cdots + x_n = x$. Consequently, $x = x_1 \vee \cdots \vee x_n = x_{i_1} + \cdots + x_{i_n}$.

(iii) It is sufficient to assume $n = 2$. Then define $e_1 = a_1 \wedge a_2'$, $e_2 = a_1 \wedge a_2$, and $e_3 = a_1' \wedge a_2$. Hence by (i)

$$\begin{aligned} x \wedge (a_1 \vee a_2) &= x \wedge (e_1 \vee e_2 \vee e_3) = (x \wedge e_1) \vee (x \wedge e_2) \vee (x \wedge e_3) \\ &= ((x \wedge e_1) \vee (x \wedge e_2)) \vee ((x \wedge e_2) \vee (x \wedge e_3)) \\ &= x \wedge (e_1 \vee e_2) \vee x \wedge (e_2 \vee e_3) = (x \wedge a_1) \vee (x \wedge a_2). \end{aligned}$$ □

PROPOSITION 2.8. *Let* $e \in C(E)$ *and* $f \leq e$. *Then* $f \in C(E)$ *if and only if* $f \in C([0, e])$.

PROOF. Let $f \in C(E)$. Then $e = f + e \wedge f'$ and the product $[0, f] \times [0, e \wedge f']$ is isomorphic with $[0, e]$ under the mapping $f_f^e(x) := (x \wedge f, x \wedge f^{\sim_e})$, $x \in [0, e]$, so that $f \in C([0, e])$.

Conversely, let $f \in C([0, e])$. Then $E \cong [0, e] \times [0, e^{\sim}]$ and $[0, e] \cong [0, f] \times [0, f^{\sim_e}]$. Since $1 = e + e^{\sim} = f + f^{\sim_e} + e^{\sim}$, then $f^{\sim} = f^{\sim_e} + e^{\sim} = e \wedge f^{\sim_e} + e^{\sim}$. On the other hand, $f^{\sim} = f^{\sim} \wedge e + f^{\sim} \wedge e^{\sim} = f^{\sim} \wedge e + e^{\sim}$ while $e \in C(E)$ so that $f^{\sim} \wedge e = f^{\sim_e} \wedge e = e \setminus f = f^{\sim_e}$.

Take $x \in E$. Then $x \wedge f^{\sim} = x \wedge f^{\sim} \wedge e + x \wedge f^{\sim} \wedge e^{\sim} = x \wedge f^{\sim_e} + x \wedge e^{\sim}$. Hence the mapping $\phi : E \to [0, f] \times [0, f^{\sim}]$ defined by $\phi(x) := (x \wedge f, x \wedge f^{\sim})$, $x \in E$, is an isomorphism in question while $E \cong [0, e] \times [0, e^{\sim}]$ and $x = x \wedge e + x \wedge e^{\sim} = x \wedge e \wedge f + x \wedge e \wedge f^{\sim_e} + x \wedge e^{\sim} = x \wedge f + x \wedge f^{\sim_e} + x \wedge e^{\sim} = x \wedge f + x \wedge f^{\sim}$. □

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We note that in the case that E is a quantum logic, for definitions see, for example, [8], then the centre of E coincides with the set of all compatible elements of E.

It is worth to recall that the notion of a central element can be defined also for unital po-groups. We say that an element $e \in G$ of a unital po-group (G, u) is *central* if (i) $0 \le e \le u$ and (ii) e is a central element in the pseudo-effect algebra $\Gamma(G, u) = [0, u]$. In the case that (G, u) is Abelian and with the Riesz interpolation property, then central elements coincide with characteristic elements [14, page 129].

A pseudo-effect algebra E is said to be *directly indecomposable* if E is non-trivial and whenever $E \cong E_1 \times E_2$, then either E_1 or E_2 is trivial.

PROPOSITION 2.9. *A pseudo-effect algebra E is directly indecomposable if and only if $C(E) = \{0, 1\}$.*

PROOF. Assume E is directly indecomposable and let $e \in C(E)$. Then $E \cong [0, e] \times [0, e']$ which means $e \in \{0, 1\}$.

Conversely, let $C(E) = \{0, 1\}$. The elements $(1, 0)$ and $(0, 1)$ are central elements in $E_1 \times E_2$. If now $E \cong E_1 \times E_2$ and if ϕ is an isomorphism from E onto $E_1 \times E_2$, set $e = \phi^{-1}((1, 0))$. Then $e^{\sim} = e^{-} = \phi^{-1}((0, 1))$, and $x \wedge e, x \wedge e' \in E$ for any $x \in E$. Hence $\phi(x) = \phi(x) \wedge \phi(e) + \phi(x) \wedge \phi(e)' = \phi((x \wedge e) + (x \wedge e'))$ which proves $x = x \wedge e + x \wedge e'$. Hence $f_e(x) := (x \wedge e, x \wedge e')$ is an isomorphism of E onto $[0, e] \times [0, e']$, so that, $e \in C(E)$. Therefore $e \in \{0, 1\}$ and $\phi(e) \in \{0, 1\}$ which proves E is directly indecomposable. □

We recall that a poset E is an *antilattice* if an infimum of two elements exists only for comparable elements. Each linearly ordered poset is antilattice.

COROLLARY 2.10. *Every linearly ordered or antilattice pseudo-effect algebra is directly indecomposable.*

PROOF. It follows from Proposition 2.9, while in view of $0 = e \wedge e' \in \{e, e'\}$, the centrum of a linearly ordered pseudo-effect algebra or of an antilattice pseudo-effect algebra is trivial, that is, $C(E) = \{0, 1\}$. □

3. Pseudo-effect algebras and Riesz decomposition properties

When we move from (commutative) effect algebras to pseudo-effect algebras, then the notion of the Riesz decomposition property can be extended to different and non-equivalent forms. Following [9], we introduce for pseudo-effect algebras the following forms of the Riesz decomposition properties:

(a) For $a, b \in E$, we write $a \operatorname{com} b$ to mean that for all $a_1 \leq a$ and $b_1 \leq b$, a_1 and b_1 commute.

(b) We say that E fulfills the *Riesz interpolation property*, (RIP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1, a_2 \leq b_1, b_2$ there is a $c \in E$ such that $a_1, a_2 \leq c \leq b_1, b_2$.

(c) We say that E fulfills the *weak Riesz decomposition property*, (RDP_0) for short, if for any $a, b_1, b_2 \in E$ such that $a \leq b_1 + b_2$ there are $d_1, d_2 \in E$ such that $d_1 \leq b_1$, $d_2 \leq b_2$ and $a = d_1 + d_2$.

(d) We say that E fulfills the *Riesz decomposition property*, (RDP) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that $d_1 + d_2 = a_1, d_3 + d_4 = a_2, d_1 + d_3 = b_1, d_2 + d_4 = b_2$.

(e) We say that E fulfills the *commutational Riesz decomposition property*, (RDP_1) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1+a_2 = b_1+b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1+d_2 = a_1, d_3+d_4 = a_2, d_1+d_3 = b_1, d_2+d_4 = b_2$, and (ii) $d_2 \operatorname{com} d_3$.

(f) We say that E fulfills the *strong Riesz decomposition property*, (RDP_2) for short, if for any $a_1, a_2, b_1, b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$ there are $d_1, d_2, d_3, d_4 \in E$ such that (i) $d_1+d_2 = a_1, d_3+d_4 = a_2, d_1+d_3 = b_1, d_2+d_4 = b_2$, and (ii) $d_2 \wedge d_3 = 0$.

We have the implications

$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Rightarrow (\mathrm{RDP}) \Rightarrow (\mathrm{RDP}_0) \Rightarrow (\mathrm{RIP}).$$

The converse of any of these implications does not hold. For commutative effect algebras we have

$$(\mathrm{RDP}_2) \Rightarrow (\mathrm{RDP}_1) \Leftrightarrow (\mathrm{RDP}) \Leftrightarrow (\mathrm{RDP}_0) \Rightarrow (\mathrm{RIP}).$$

The following result was proved in [9, Lemma 3.2].

LEMMA 3.1. *If E satisfies* (RDP_0), *then $a \wedge b = 0$ implies $a + b$, $b + a$ and $a \vee b$ exist in E and are all equal.*

THEOREM 3.2. *Let a pseudo-effect algebra E satisfy* (RDP). *Then $e \in E$ is central if and only if $e \wedge e^{\sim} = 0$ if and only if $e \wedge e^{-} = 0$.*

PROOF. Let $e \in C(E)$, then $e \wedge e^{\sim} = 0 = e \wedge e^{-}$. In view of Proposition 2.2 (iii) and (v), $e \wedge e^{\sim} = 0$ if and only if $e \wedge e^{-} = 0$.

Conversely, let $e \wedge e^{\sim} = 0$. Then $x \leq 1 = e + e^{\sim}$ for any $x \in E$. There are $x_1 \leq e$ and $x_2 \leq e^{\sim}$ such that $x = x_1 + x_2$. We show that if $y_1 \leq e$ and $y_2 \leq e^{\sim}$ and $x = y_1 + y_2$, then $x_1 = y_1$ and $x_2 = y_2$. Due to (RDP), there are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}$, $x_2 = c_{21} + c_{22}$, $y_1 = c_{11} + c_{21}$ and $y_2 = c_{12} + c_{22}$. Since $c_{12} \leq x_1 \leq e$ and $c_{12} \leq y_2 \leq e^{\sim}$, we conclude $c_{12} = 0$. Similarly, $c_{21} = 0$. Hence $x_1 = c_{11} = y_1$ and $x_2 = c_{22} = y_2$.

Define $p_e(x) = x_1$ if $x = x_1 + x_2$ $(x \in E)$. Then $p_e : E \to [0, e]$. If $x_1 \in [0, e]$ and $x_2 \in [0, e^{\sim}]$, then $x_1 \wedge x_2 = 0$, so that by Lemma 3.1, $x = x_1 + x_2 = x_2 + x_1 = x_1 \vee x_2$, and hence $p_e(x) = x_1$. Consequently, p_e restricted to $[0, e]$ is the identity.

We show that p_e is a homomorphism. Let $x + y \in E$ and $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \leq e, x_2, y_2 \leq e^{\sim}$. Then $x + y = x_1 + x_2 + y_1 + y_2$. Since $x_2 \wedge y_1 = 0$, then $x + y = x_1 + y_1 + x_2 + y_2$. On the other hand, let $x + y = z_1 + z_2$, where $z_1 \leq e$ and $z_2 \leq e^{\sim}$. Hence there are four elements $d_{11}, d_{12}, d_{21}, d_{22}$ such that

$$x_1 + y_1 = d_{11} + d_{12}, \qquad z_1 = d_{11} + d_{21},$$
$$x_2 + y_2 = d_{21} + d_{22}, \qquad z_2 = d_{12} + d_{22}.$$

We claim $d_{12} = 0$. Since $d_{12} \leq x_1 + y_1$, then $d_{12} = d' + d''$, where $d' \leq x_1$ and $d'' \leq y_1$. Then $d' \leq x_1 \leq e$ and $d' \leq d_{12} \leq z_2 \leq e^{\sim}$ so that $d' = 0$, and $d'' \leq y_1 \leq e$ and $d'' \leq z_2 \leq e^{\sim}$ proving $d'' = 0$ and therefore $d_{12} = 0$. In a similar way we can prove $d_{21} = 0$ which yields $x_1 + y_1 = z_1$ and $x_2 + y_2 = z_2$, so that, p_e is a homomorphism.

Since by Proposition 2.2 (v), $e^{\sim} = e^{-}$, we can write $e' := e^{\sim} = e^{-}$, and let $p_{e'}(x) = x_2$ if $x = x_1 + x_2$ $(x \in E)$. Then $p_{e'}$ is a homomorphism from E onto $[0, e']$.

Consequently, the mapping $f_e : E \to [0, e] \times [0, e']$ defined by

$$f_e(x) = (p_e(x), p_{e'}(x)), \quad x \in E$$

is an isomorphism with $f_e(e) = (e, 0)$, so that $e \in C(E)$. In addition,

$$p_e(x) = x_1 = x \wedge e, \quad x \in E. \qquad \square$$

4. General comparability

We say that a pseudo-effect algebra E satisfies *general comparability* if, given $x, y \in E$, there is a central element $e \in E$ such that $p_e(x) \leq p_e(y)$ and $p_{e'}(x) \geq p_{e'}(y)$. This means that the coordinates of the elements $x = (p_e(x), p_{e'}(x))$ and $y = (p_e(y), p_{e'}(y))$ can be compared in $[0, e]$ and $[0, e']$, respectively.

For example, (i) every linearly ordered pseudo-effect algebra trivially satisfies general comparability; (ii) also any Cartesian product of linearly ordered pseudo-effect algebras. For an 'MV-analogue' of the next result see [2, Proposition 3].

PROPOSITION 4.1. *Let M be a σ-complete pseudo MV-algebra and $x \in M$. Then the element $e := \bigvee_{n=1}^{\infty}(x_1 \oplus \cdots \oplus x_n)$, where $x_n = x$ for every n, is a central element of M such that $p_e(x) = x$. If f is any central element of M such that $p_f(x) = x$, then $e \leq f$. Moreover, M satisfies general comparability.*

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PROOF. Let M be a σ-complete pseudo MV-algebra. According to [7, Theorem 4.2], M is commutative, that is, an MV-algebra. Let $+$ be its partial addition defined via $a+b = a \oplus b$ if and only if $a \le b^*$. Let (G, u) be the unital ℓ-group such that $M \cong \Gamma(G, u)$; such a group is guaranteed by Mundici's representation of MV-algebras, see [20]. Then G is Dedekind complete, and by [14, Lemma 9.8], the element $e = \bigvee_{i=1}^{\infty}(nx \wedge u) \in C(E)$ (compare with Theorem 3.2), and $p_e(x) = x$. Moreover, if $p_f(x) = x$ for some $f \in C(E)$, then $e \le f$. Applying now [14, Theorem 9.9], (G, u) satisfies general comparability, so M satisfies general comparability. □

THEOREM 4.2. *Let E be a pseudo-effect algebra satisfying general comparability. Then E is a lattice, and E can be organized into a pseudo MV-algebra such that the partial addition derived from E as the pseudo MV-algebra coincides with the original $+$ taken in the pseudo-effect algebra.*

PROOF. Let $x, y \in E$ and let $e \in C(E)$ such that $p_e(x) \le p_e(y)$ and $p_{e'}(x) \ge p_{e'}(y)$. Then $x = p_e(x) + p_{e'}(x) \ge p_e(x) + p_{e'}(y) =: v \in E$.

Claim 1. $v = x \wedge y$.

PROOF. We have $y = p_e(y) + p_{e'}(y) \ge p_e(x) + p_{e'}(y) = v$, that is, $v \le x, y$. Let $z \le x, y$. Then $p_e(z) \le p_e(x)$ and $p_{e'}(z) \le p_{e'}(y)$, that is, $z = p_e(z) + p_{e'}(z) \le p_e(x) + p_{e'}(y) = v$, that is, $v = x \wedge y$. □

Claim 2. $w := p_e(y) + p_{e'}(x) \in E$ *and* $w = x \vee y$.

PROOF. Since $p_e(y) \wedge p_{e'}(x) = 0$, then $w := p_e(y) + p_{e'}(x) \in E$. We conclude now $x \vee y = w$. We have $x = p_e(x) + p_{e'}(x) \le p_e(y) + p_{e'}(x) = w$ and $y = p_e(y) + p_{e'}(y) \le p_e(y) + p_{e'}(x) = w$. If now $z \ge x, y$, then $p_e(z) \ge p_e(y)$ and $p_{e'}(z) \ge p_{e'}(x)$ that is, $z = p_e(z) + p_{e'}(z) \ge w$. □

Claim 3. $x \setminus (x \wedge y) = (x \vee y) \setminus y$ *and* $y \setminus (x \wedge y) = (x \vee y) \setminus x$.

PROOF. Calculate

$$\begin{aligned}
&p_e(x \setminus (x \wedge y)) = p_e(x \setminus (p_e(x) + p_{e'}(y)) = p_e(x) \setminus p_e(x) = 0,\\
&p_{e'}(x \setminus (x \wedge y)) = p_{e'}(x) \setminus p_{e'}(y), \quad p_e(y \setminus (x \wedge y)) = p_e(y) \setminus p_e(x),\\
&p_{e'}(y \setminus (x \wedge y)) = p_{e'}(y) \setminus p_{e'}(y) = 0,\\
&p_e((x \vee y) \setminus x) = p_e((p_e(y) + p_{e'}(x)) \setminus x) = p_e(y) \setminus p_e(x),\\
&p_{e'}((x \vee y) \setminus x) = p_{e'}(x) \setminus p_{e'}(x) = 0,\\
&p_e((x \vee y) \setminus y) = p_e(y) \setminus p_e(y) = 0,\\
&p_{e'}((x \vee y) \setminus y) = p_{e'}(x) \setminus p_{e'}(y),
\end{aligned}$$

which proves Claim 3. □

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Finally, according to [10, Proposition 8.7], Claim 3 is a necessary and sufficient condition in order to convert E into a pseudo MV-algebra $(E; \oplus, ^-, ^\sim, 0, 1)$; we define

$$a \oplus b := (a^\sim \setminus (a^\sim \wedge b))^-, \quad a, b \in E.$$

In such the case, the original $+$ and the derived one from $\oplus$ coincide. □

5. Monotone σ-complete pseudo-effect algebras

We say that a pseudo-effect algebra E (i) is *monotone σ-complete* if any sequence $x_1 \le x_2 \le \cdots$ in E has a supremum $\bigvee_{n=1}^{\infty} x_n$ in E; (ii) is *σ-complete* if E is a σ-complete lattice; (iii) satisfies the *countable Riesz interpolation property*, (σ-RIP) in abbreviation if, for countable sequences $\{x_1, x_2, \dots\}$ and $\{y_1, y_2, \dots\}$ of elements of E such that $x_i \le y_j$ for all i, j, there exists an element $z \in E$ such that $x_i \le z \le y_j$ for all i, j; (iv) is *Archimedean* if $nx := x + \cdots + x$ is defined in E for any integer $n \ge 1$, then $x = 0$.

It is evident that (σ-RIP) implies (RIP), and E is monotone σ-complete if and only if each nonincreasing sequence of elements in E has an infimum. Moreover, if E is a lattice, then E is monotone σ-complete if and only if E is σ-complete.

PROPOSITION 5.1. *Let E be a pseudo-effect algebra with* (RIP). *Then E has* (σ-RIP) *if and only if whenever*

(a) $x_1 \le x_2 \le \cdots$ *in E and $y_1, y_2 \in E$, or*
(b) $x_1, x_2 \in E$ *and* $y_1 \ge y_2 \ge \cdots$ *in E, or*
(c) $x_1 \le x_2 \le \cdots$ *and* $y_1 \ge y_2 \ge \cdots$ *in E,*

and $x_i \le y_j$ for all i, j there exists $z \in E$ such that $x_i \le z \le y_j$ for all i, j.

PROOF. It follows the same steps as that in [14, Lemma 16.2]. □

PROPOSITION 5.2. *Let E be a pseudo-effect algebra. Assume also that $\bigvee_i a_i$ and $\left(\bigvee_i a_i\right) + x \in E$, then $\bigvee_i (a_i + x) \in E$ and $\left(\bigvee_i a_i\right) + x = \bigvee_i (a_i + x)$. If $\bigvee_i a_i$, $x + \left(\bigvee_i a_i\right) \in E$, then $\bigvee_i (x + a_i) \in E$ and $x + \left(\bigvee_i a_i\right) = \bigvee_i (x + a_i)$.*

PROOF. We have $\left(\bigvee_i a_i\right) + x \ge a_i + x$ for each i. If now $z \ge a_i + x$ for all i, then $z \setminus x \ge a_i$, that is, $z \setminus x \ge \bigvee_i a_i$ and $z \ge \left(\bigvee_i a_i\right) + x$. □

PROPOSITION 5.3. *Let E be a monotone σ-complete pseudo-effect algebra. Then E is Archimedean. If, in addition, E satisfies* (RIP), *then E has countable interpolation.*

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PROOF. Assume $x_n := nx = x + \cdots + x$ be defined in E for any $n \geq 1$. Then $x_n \leq x_{n+1}$ and there exists $x_0 := \bigvee_{n=1}^{\infty} x_n$. Since $x_n \leq x^-$ for every n, then $x_0 \leq x^-$, so that $x_0 + x \in E$. Hence by Proposition 5.2,

$$x_0 + x = \bigvee_{n=1}^{\infty}(x_n + x) = \bigvee_{n=2}^{\infty} x_n = x_0,$$

which proves $x = 0$.

If now E has (RIP), then by Proposition 5.1, E has countable interpolation. □

The notion of monotone σ-complete pseudo-effect algebras is important while there are even (commutative) effect algebras which are monotone σ-complete but not a lattice.

EXAMPLE 5.4. There exists a monotone σ-complete effect algebra which is not a lattice.

PROOF. Let X be an uncountable set and fix two distinct elements $a, b \in X$. Let E be the set of all functions $f : X \to \mathbb{Z}$ such that $f(x) = (f(a) + f(b))/2$ for all but countably many $x \in X$ and $0 \leq f(x) \leq 2$ for any $x \in X$. Then E is an effect algebra which is monotone σ-complete but not a lattice. For example, let u be the function which is the constant function 1 and let v be a mapping in E such that $v(a) = 0$ and $v(b) = 2$ while $v(x) = 1$ for all $x \in X \setminus \{a, b\}$. Then u and v have no infimum in E (see [14, Example 16.1, Example 16.8]). □

We say that two elements a and b of a pseudo-effect algebra E are *compatible*, and write $a \leftrightarrow b$ if there are three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$, and $a_1 + b_1 + c = b_1 + a_1 + c$, $a_1 \wedge b_1 = 0$.

PROPOSITION 5.5. *Let $e \in C(E)$ and $x \in E$. Then $x \leftrightarrow e$, and*

(i) $x = x \wedge e' + x \wedge e$, $e = e \wedge x^- + x \wedge e$, $e = e \wedge x + e \wedge x^\sim$, $x = e \wedge x + x \wedge e'$.

(ii) $x \wedge e' = x \setminus (x \wedge e) = (x \wedge e) / x$, $e \wedge x^- = e \setminus (e \wedge x)$, $e \wedge x^\sim = (e \wedge x) / x$.

(iii) $x \vee e = x \wedge e' + e \wedge x^- + x \wedge e = e \wedge x^- + x \wedge e' + x \wedge e = x \wedge e' + x^- \wedge e + x^\sim \wedge e$.

(iv) $(x \vee e) \setminus e = x \setminus (x \wedge e)$, $(x \vee e) \setminus x = e \setminus (x \wedge e)$, *and* $e / (x \vee e) = (x \wedge e) / x$, $x / (x \vee e) = (x \wedge e) / x$.

PROOF. (i) $x = x \wedge e' + x \wedge e$ by Proposition 2.2. On the other hand, $p_e(e) = e = e \wedge (x + x^\sim) = p_e(x) + p_e(x^\sim) = x \wedge e + x^\sim \wedge e$ and $e = e \wedge (x^- + x) = p_e(x^- + x) = p_e(x^-) + p_e(x) = e \wedge x^- + e \wedge x$.

(ii) $x \setminus (x \wedge e) \in E$. If $x \setminus (x \wedge e) = a$, then $x = a + (x \wedge e)$, but $x = x \wedge e' + x \wedge e$ which gives $a = x \wedge e'$. Similarly, $(x \wedge e) / x = x \wedge e'$, and also for other two equalities in (ii).

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(iii) We have $e = e \wedge 1 = e \wedge (x^- + x) = x^- \wedge e + x \wedge e$. Since $x \wedge e' \leq e'$, then $x \wedge e' + e \in E$ and $x, e \leq x \wedge e' + e$. Moreover, if $z \geq x \wedge e', e$, then $p_e(z) \geq e$ and $p_{e'}(z) \geq p_{e'}(x \wedge e') = x \wedge e'$ which proves $(x \wedge e') \vee e = x \wedge e' + e$. It is clear that $x \vee e \leq x \wedge e' + x^- \wedge e + e \wedge x$. If now $y \geq x, e$, then $y \geq x \wedge e', e$ and $y \geq (x \wedge e') \vee e$ which proves $x \vee e = x \wedge e' + x^- \wedge e + x \wedge e$.

On the other hand, $x^- \wedge e \leq x^-$, we have $x^- \wedge e + x \in E$. Then $x^- \wedge e + x \geq x$ and $x^- \wedge e + x = x^- \wedge e + x \wedge e + x \wedge e' = e + x \wedge e' \geq e$. Therefore, if $u \geq x, e$, then $p_e(u) \geq e$ and $p_{e'}(u) \geq p_{e'}(x) = x \wedge e'$ which entails $u \geq x^- \wedge e + x$ that is, $x^- \wedge e + x \wedge e' + x \wedge e = x \vee e$.

(iv) By (iii) $(x \vee e) \setminus e = x \wedge e' = x \setminus (x \wedge e) = (x \wedge e) / x$ and $x / (x \vee e) = x / (x + x^\sim \wedge e) = x^\sim \wedge e = (e \wedge x) / x$.

From (iii) we have $(x \vee e) \setminus x = x^- \wedge e = e \setminus (x \wedge e)$. In a similar way we can prove the last equalities in (iv).

From the above we have also $x \leftrightarrow e$. □

We recall that according to Proposition 5.5, if e is a central element of E, then $e \leftrightarrow x$ for any $x \in E$. The converse statement is not true, for example, in any MV-algebra M every two elements are compatible, and $C(M) = M$ if and only if M is a Boolean algebra, see Theorem 3.2.

PROPOSITION 5.6. *Let a pseudo-effect algebra E have* (σ-RIP). *Let* $\bigvee_{i=1}^{\infty} e_i \in C(E)$ *for* $e_i \in C(E)$, $i \geq 1$. *Then*

$$x \wedge \left(\bigvee_{i=1}^{\infty} e_i\right) = \bigvee_{i=1}^{\infty} (x \wedge e_i), \quad x \in E. \tag{5}$$

PROOF. First we show that

$$\bigwedge_i (x \setminus (x \wedge e_i)) = x \setminus \left(x \wedge \left(\bigvee_i e_i\right)\right). \tag{6}$$

We have $x \setminus (x \wedge e_i) \geq x \setminus \left(x \wedge \left(\bigvee_i e_i\right)\right)$. Let now $d \leq x \setminus (x \wedge e_i)$ for each i. By Proposition 5.5,

$$d \leq x \setminus (x \wedge e_i) = (x \vee e_i) \setminus x \leq \left(x \vee \left(\bigvee_i e_i\right)\right) \setminus x = x \setminus \left(x \wedge \left(\bigvee_i e_i\right)\right)$$

which proves (6).

It is clear that $(\bigvee_i e_i) \wedge x \geq e_i \wedge x$ for each i. Let now $e_i \wedge x \leq z$ for each i. Then $e_i \wedge x \leq z, x$ for each i. Applying (σ-RIP), there exists an element $z_0 \in E$ such that $e_i \wedge x \leq z_0 \leq z, x$. Hence $x \setminus z_0 \leq x \setminus (e_i \wedge x)$. Thus $x \setminus z_0 \leq x \setminus \left(x \wedge \left(\bigvee_i e_i\right)\right)$ which gives $x \wedge \left(\bigvee_i e_i\right) \leq z_0 \leq z$ and consequently (5) is proved. □

PROPOSITION 5.7. *Let a pseudo-effect algebra E have (σ-RIP). Let $e = \bigvee_{i=1}^{\infty} e_i \in E$ for $e_i \in C(E)$, $i \geq 1$. Then, for every $x \in E$, $x \wedge \left(\bigvee_{i=1}^{\infty} e_i\right) = \bigvee_{i=1}^{\infty}(x \wedge e_i)$, $x \vee e \in E$, and $x \leftrightarrow e$.*

PROOF. We have $e_i \wedge x \leq x, e$ for all i. Let $x_0 \in E$ be any element such that $e_i \wedge x \leq x_0 \leq e, x$ for every i; such an element always exists due to (σ-RIP).

Claim 1. $x = (x \setminus x_0) + x_0$, $e = (e \setminus x_0) + x_0$, $(x \setminus x_0) + (e \setminus x_0) + x_0 \in E$.

PROOF. Indeed, we have $e_i \leq (x \setminus (e_i \wedge x))^{\sim} \leq (x \setminus x_0)^{\sim}$ so that $e \leq (x \setminus x_0)^{\sim}$ which gives $(x \setminus x_0) + e \in E$.

Similarly, $e_i \leq (x \setminus (e_i \wedge x))^{-} \leq (x \setminus x_0)^{-}$ so that $e \leq (x \setminus x_0)^{-}$ which gives $e + (x \setminus x_0) \in E$.

It is evident that $(x \setminus x_0) + e \geq x, e$ and $e + (x \setminus x_0) \geq x, e$. ☐

Claim 2. $\bigwedge_i (x \setminus (x \wedge e_i)) = x \setminus x_0$.

PROOF. It is evident that $x \setminus (x \wedge e_i) \geq x \setminus x_0$ for every i. Let $d \leq x \setminus (x \wedge e_i)$ for each i. Then by Proposition 5.5, $d \leq x \setminus (x \wedge e_i) = (x \vee e_i) \setminus e_i$. Then $d + e_i \leq x \vee e_i \leq (x \setminus x_0) + e$ so that $e_i \leq d \,/\, ((x \setminus x_0) + e)$ and $e \leq d \,/\, ((x \setminus x_0) + e)$ which gives $d + e \leq (x \setminus x_0) + e$ and $d \leq x \setminus x_0$. ☐

Claim 3. $\bigvee_i (x \wedge e_i) = x_0$.

PROOF. Assume $x \wedge e_i \leq y$ for every i. Then $x \wedge e_i \leq y, x_0$ so that there exists $y_0 \in E$ such that $x \wedge e_i \leq y_0 \leq y, x_0$ for every i. Then $x \setminus y_0 \leq x \setminus (x \wedge e_i)$. By Claim 2, $x \setminus y_0 \leq \bigwedge_i (x \setminus (x \wedge e_i)) = x \setminus x_0$ so that $x_0 \leq y_0 \leq y$. ☐

Claim 4. $(x \setminus x_0) \wedge (e \setminus x_0) = 0$.

PROOF. Assume $z \leq x \setminus x_0$ and $z \leq e \setminus x_0$. Then $z + x_0 \leq x$, $z + x_0 \leq e$, and $x \wedge e_i \leq z + x_0 \leq e, x$ for each i. Using Claims 1–3, we have $z + x_0 = \bigvee_i (x \wedge e_i) = x_0$, that is, $z = 0$. ☐

Claim 5. $x \wedge e = x_0$.

PROOF. Let $u \leq x, e$. Then $u, x_0 \leq x, e$ and there exists $u_0 \in E$ such that $u, x_0 \leq u_0 \leq x, e$, in particular, $x \wedge e_i \leq u_0$, and using Claims 1–3, we have $u_0 = \bigvee_i (x \wedge e_i) = x_0$ that is, $u \leq u_0 = x_0$ and $x_0 = x \wedge e$. ☐

Claim 6. $x_0 + x \setminus x_0 \in E$, $x_0 + x \setminus x_0 = x$ *and* $(e \setminus x_0) + (x \setminus x_0) + x_0 \in E$.

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PROOF. By Claim 1, $e + (x \setminus x_0) \in E$ so that $(e \setminus x_0) + (x_0 + (x \setminus x_0)) \in E$ and $x_0 + x \setminus x_0 \in E$. Applying Proposition 5.2, we have

$$(\star) \qquad x_0 + x \setminus x_0 = \bigvee_i ((x \wedge e_i) + (x \setminus x_0)).$$

We show that, for each i, $x \wedge e_i + x \setminus x_0 = x \setminus x_0 + x \wedge e_i$. We recall that due to Claim 5 and Proposition 5.2, $x \setminus x_0 + x \wedge e_i \in E$ for all i.

$$p_{e_i}(x \wedge e_i + x \setminus x_0) = x \wedge e_i + (x \wedge e_i) \setminus (x_0 \wedge e_i) = x \wedge e_i = p_{e_i}(x \setminus x_0 + x \wedge e_i),$$
$$p_{e_i'}(x \wedge e_i + x \setminus x_0) = p_{e_i'}(x \setminus x_0) = p_{e_i}(x \setminus x_0 + x \wedge e_i).$$

Applying again Proposition 5.2, we have for $(\star)$

$$(\star) = \bigvee_i (x \setminus x_0 + x \wedge e_i) = x \setminus x_0 + \bigvee_i (x \wedge e_i) = x \setminus x_0 + x_0 = x.$$

Consequently, $e + x \setminus x_0 = (e \setminus x_0) + x_0 + (x \setminus x_0) = (e \setminus x_0) + (x \setminus x_0) + x_0 \in E$. □

Claim 7. $x \vee e = x \setminus x_0 + e$.

PROOF. We have $x, e \le x \setminus x_0 + e$ so that by (iii) of Proposition 5.5,

$$x \vee e_i = x \wedge e_i' + e_i \le x \setminus x_0 + e.$$

Assume $x \vee e_i \le v$ for all i. Then there exists v_0 such that $x \vee e_i \le v_0 \le v, x \setminus x_0 + e$. Then $x \vee e_i = x \setminus (x \wedge e_i) + e_i \ge (x \setminus x_0) + e_i \in E$. Since $x \setminus x_0 + e \in E$, we can apply Proposition 5.2 and $\bigvee_i ((x \setminus x_0) + e_i) = x \setminus x_0 + \bigvee_i e_i = x \setminus x_0 + e \le v_0$ which yields $v_0 = x \setminus x_0 + e \le v$, that is, $x \setminus x_0 + e = x \vee e$. □

Claim 8. $\bigwedge_i (e \setminus (x \wedge e_i)) = e \setminus x_0$.

PROOF. It is clear that $e \setminus (x \wedge e_i) \ge e \setminus x_0$. Assume $w \le e \setminus (x \wedge e_i)$. There exists an element $w_0 \in E$ such that $w, e \setminus x_0 \le w_0 \le e \setminus (x \wedge e_i)$ for each i. Hence $w_0 + x \wedge e_i \le e$ so that $x \wedge e_i \le w_0 / e$ and by Claim 5, $x_0 \le w_0 / e$, that is, $w_0 + x_0 \le e$ and $w_0 \le e \setminus x_0$. □

Claim 9. $x \vee e = e \setminus x_0 + x$.

PROOF. By (iii) of Proposition 5.5, $x \vee e_i = e_i \wedge x^- + x = e_i \setminus (x \wedge e_i) + x \le e \setminus x_0 + x$. Then $e_i \setminus (x \wedge e_i) \le e \setminus x_0 \le e \setminus (x \wedge e_i)$ (Claim 8).

We now show that $\bigvee_i (e_i \setminus (x \wedge e_i)) = e \setminus x_0$. Assume $w \ge e_i \setminus (x \wedge e_i)$ for each i. Then there exists $w_0 \in E$ such that $e_i \setminus (x \wedge e_i) \le w_0 \le w, e \setminus x_0$. Therefore $(e \setminus x_0) \setminus w_0 \le (e \setminus (x \wedge e_i)) \setminus (e_i \setminus (x \wedge e_i)) = e \setminus e_i$, that is, $((e \setminus x_0) \setminus w_0) +$

$e_i \leq e$ and $e_i \leq ((e \setminus x_0) \setminus w_0) \,/\, e$, and $e \leq ((e \setminus x_0) \setminus w_0) \,/\, e$. Consequently, $((e \setminus x_0) \setminus w_0) + e \leq e$, that is, $e \setminus x_0 = w_0 \leq w$ which proves $e \setminus x_0 = \bigvee_i (e_i \setminus (x \wedge e_i))$. Applying Proposition 5.2, $e \setminus x_0 + x = \bigvee_i (e_i \setminus (x \wedge e_i) + x) = \bigvee_i (e_i \vee x) = e \vee x$. □

Claim 10. $x \leftrightarrow e$.

It follows from the previous Claims. □

PROPOSITION 5.8. *Let E be a pseudo-effect algebra, $a = \bigvee_i a_i \in E$. Then*

$$\bigwedge_i (a \setminus a_i) = 0 = \bigwedge_i (a_i \,/\, a).$$

PROOF. It is straightforward. □

PROPOSITION 5.9. *Let E satisfy (σ-RIP). If $a = \bigvee_{i=1}^{\infty} a_i \in E$ and $c \leq a_i$ for any i, then $\bigvee_i (a_i \setminus c)$, $\bigvee_i (c \,/\, a_i) \in E$, and $a \setminus c = \bigvee_i (a_i \setminus c)$, $c \,/\, a = \bigvee_i (c \,/\, a_i)$.*

PROOF. Since $c \leq a_i \leq a$, then $a_i \setminus c \leq a \setminus c$ for any i. Let $a_i \setminus c \leq v$ for any i. Then there exists an element $v_0 \in E$ such that $a_i \setminus c \leq v_0 \leq v, a \setminus c$. Hence, $(a \setminus c) \setminus v_0 \leq (a \setminus c) \setminus (a_i \setminus c) = a \setminus a_i$. By Proposition 5.8, we have $(a \setminus c) \setminus v_0 = 0$, that is, $a \setminus c = v_0 \leq v$, so that $a \setminus c = \bigvee_i (a_i \setminus c)$.

In a similar manner we can prove the second equality. □

THEOREM 5.10. *Let a pseudo-effect algebra E satisfy (σ-RIP). Let $e = \bigvee_{i=1}^{\infty} e_i \in E$, where $e_i \in C(E)$, $i \geq 1$. Then $e \in C(E)$.*

PROOF. We recall that $e^{\sim} = e^{-}$. Indeed, $e^{\sim} = \bigwedge_i e_i^{\sim} = \bigwedge_i e_i^{-}$. Using Proposition 5.7, $e' \wedge e = \bigvee_i (e' \wedge e_i) \leq \bigvee_i (e_i' \wedge e_i) = 0$. Let $x \in E$ and $x_0 = \bigvee_i (x \wedge e_i)$.

Claim 1. $x \setminus (x \wedge e) = x \wedge e^{\sim} = e^{-} \wedge x = (x \wedge e) \,/\, x$.

PROOF. In view of Proposition 5.7, $x^{-} \vee e \in E$, and $(x^{-} \vee e)^{\sim} = x \wedge e^{\sim} \in E$. On the other hand, using Claim 2 of the proof of Proposition 5.7, we have

$$x \setminus (x \wedge e) = x \setminus x_0 = \bigwedge_i (x \setminus (x \wedge e_i)) = \bigwedge_i (x \wedge e_i^{\sim}) = x \wedge \bigwedge_i e_i^{\sim} = x \wedge e^{\sim}.$$

It is possible to show $(x \wedge e) \,/\, x = \bigwedge_i ((x \wedge e_i) \,/\, x)$. Define $p_e(x) := x \wedge e$ and $p_{e'}(x) := x \wedge e^{\sim}$. Then $p_e(x) + p_{e'}(x) = x = p_{e'}(x) + p_e(x)$. □

Claim 2. If $y \leq x$, then $(x \setminus y) \wedge e = (x \wedge e) \setminus (y \wedge e)$.

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PROOF. We have $(x \wedge y) \wedge e = \bigvee_i((x \setminus y) \wedge e_i) = \bigvee_i((x \wedge e_i) \setminus (y \wedge e_i)) \geq (x \wedge e_i) \setminus (y \wedge e)$. Applying Proposition 5.9, we have

$$(*) \qquad \bigvee_i((x \wedge e_i) \setminus (y \wedge e)) = \left(\bigvee_i(x \wedge e_i) \setminus (y \wedge e)\right) \\ = (x \wedge e) \setminus (y \wedge e) \leq (x \setminus y) \wedge e.$$

On the other hand, $(x \setminus y) \wedge e = \bigvee_i((x \wedge e_i) \setminus (y \wedge e_i)) \leq (x \wedge e) \setminus (y \wedge e_i)$. Assume $d \leq (x \wedge e) \setminus (y \wedge e_i)$ for each i. Then there exists $d_0 \in E$ such that $d, (x \setminus y) \wedge e \leq d_0 \leq (x \wedge e) \setminus (y \wedge e_i)$ for any i. Therefore, $d_0 + (y \wedge e_i) \leq x \wedge e$, $y \wedge e_i \leq d_0 \,/\, (x \wedge e)$, so that

$$(**) \qquad y \wedge e \leq d_0 \,/\, (x \wedge e) \quad \text{and} \quad d_0 \leq (x \wedge e) \setminus (y \wedge e).$$

Combining $(*)$ and $(**)$, we have $(x \setminus e) \wedge e = (x \wedge e) \setminus (y \wedge e)$. □

Claim 3. If $x + y \in E$, then $(x + y) \wedge e = x \wedge e + y \wedge e$.

PROOF. Due to Claim 2, $x \wedge e = ((x + y) \setminus y) \wedge e = ((x + y) \wedge e) \setminus (y \wedge e)$. □

Claim 4. If $x + y \in E$, then $(x + y) \wedge e' \geq x \wedge e' + y \wedge e'$.

PROOF. $(x+y) \wedge e' = \bigwedge_i((x+y) \wedge e_i') = \bigwedge_i(x \wedge e_i + y \wedge e_i) \geq x \wedge e' + y \wedge e'$. □

Claim 5. If $x \leq e$, $y \leq e'$, then $x + y = x \vee y = y + x$.

PROOF. Due to $e + e' = 1 = e' + e$, we have $x + y, y + x \in E$, and $x + y \geq x, y$. Assume $z \geq x, y$. There exists $z_0 \in E$ such that $z, x + y \geq z_0 \geq x + y$. Then $p_e(z_0) \geq x, p_{e'}(z_9) \geq y$, that is, $z_0 = p_e(z_0) + p_{e'}(z_0) \geq x + y$, that is, $x + y = x \vee y$.

We have $x + y = p_e(x + y) + p_{e'}(x + y) = p_{e'}(x + y) + p_e(x + y) \geq y + x$. But $y + x \geq x, y$, then $y + x \geq x \vee y = x + y$. □

Claim 6. If $x \leq e$, $y \leq e'$, then $(x + y) \wedge e' = x \wedge e' + y \wedge e'$.

PROOF. Using Claim 5, we have

$$x + y = p_e(x + y) + p_{e'}(x + y) \geq p_e(x) + p_e(y) + p_{e'}(x) + p_{e'}(y) \\ = p_e(x) + p_{e'}(x) + p_e(y) + p_{e'}(y) = x + y,$$

which proves $p_{e'}(x + y) = p_{e'}(x) + p_{e'}(y)$. □

Claim 7. $E \cong [0, e] \times [0, e^{\sim}]$.

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PROOF. Define $f_e(x) := (x \wedge e, x \wedge e')$, $x \in E$. Then $x = x \wedge e + x \wedge e'$ and if $f_e(x) = f_e(y)$, then $x = y$. According to Claim 3 and Claim 6, f_e is an injective homomorphism from E into $[0, e] \times [0, e^{\sim}]$, $f_e(e) = (e, 0)$ and due to Claim 5, f_e is surjective. □

Summarizing all claims, we finally have $e \in C(E)$. □

THEOREM 5.11. *Let a pseudo-effect algebra E be monotone σ-complete. Let $e = \bigvee_{i=1}^{\infty} e_i \in E$, where $e_i \in C(E)$, $i \geq 1$. Then $e \in C(E)$, and*

$$x \wedge \left(\bigvee_{i=1}^{\infty} e_i \right) = \bigvee_{i=1}^{\infty} (x \wedge e_i), \quad x \in E.$$

PROOF. Since by Theorem 2.3, $C(E)$ is a Boolean algebra, without loss of generality we can assume $e_1 \leq e_2 \leq \cdots$. Therefore, $e \in E$. In addition $x \wedge e_i \in E$, which entails $x_0 := \bigvee_i (x \wedge e_i)$ is defined in E, and $x_0 \leq x, e$.

Using a slightly modified proof of Proposition 5.7, we can show that if x_0^* is any element of E such that $x \wedge e_i \leq x_0^* \leq x, e$ for any i, then $x_0 = x_0^*$. In addition, Claim 1, Claim 2, Claim 4, Claim 6, and Claim 8 in the proof of Proposition 5.7 are also true, and $x \setminus x_0 = \bigwedge_i (x \setminus (x \wedge e_i)) = x_0 / x$, hence $x \setminus x_0 + x_0 = x = x_0 + x_0 / x$.

Claim 1. $e \wedge e' = 0$.

PROOF. Assume $z \leq e, e'$, then $z_0 = \bigvee_i (z \wedge e_i) \leq z \leq e, e'$. Therefore, $z \wedge e_i \leq e_i$ and $z \wedge e_i \leq z_4 \leq e' \leq e_i'$ so that $z_0 = 0$. Then $z \setminus z_0 \leq e \setminus z_0$ and by Claim 4 of Proposition 5.7, we have $z \setminus z_0 = (z \setminus z_0) \wedge (e \setminus z_0) = 0$, that is, $z = 0$.

Define two mappings $q_e : E \to [0, e]$ and $q_{e'} : E \to [0, e']$ by

$$q_e(x) := \bigvee_i (x \wedge e_i) =: x_0, \qquad q_{e'}(x) := x \setminus x_0$$

for any $x \in E$. Then $q_e(e) = e$ and $q_{e'}(e) = 0$. □

Claim 2. If $x + y \in E$, then $q_e(x + y) = q_e(x) + q_e(y)$, and q_e is monotone.

PROOF. Calculate, $q_e(x + y) = \bigvee_i ((x + y) \wedge e_i) = \bigvee_i (x \wedge e_i + y \wedge e_i) \leq q_e(x) + q_e(y) \in E$.

Assume $(x + y) \wedge e_i \leq z$ for any i, and fix an integer $i_0 \geq 1$. Then $x_0, y_6 \leq z$ and $x \wedge e_i + y \wedge e_{i_0} \leq z$ for any $i \geq i_0$. Hence $x \wedge e_i \leq z \setminus (y \wedge e_{i_0})$, that is, $x_0 \leq z \setminus (x \wedge e_{i_0})$ and $y \wedge e_{i_9} \leq x_0 / z$ which gives $y_0 \leq x_0 / z$ and $x_0 + y_0 \leq z$. □

Claim 3. If $x + y \in E$, then $q_{e'}(x + y) \geq q_{e'}(x) + q_{e'}(y)$, and $q_{e'}$ is monotone.

Indeed, $q_{e'}(x + y) = \bigwedge_i ((x + y) \wedge e_i') = \bigwedge_i (x \wedge e_i' + y \wedge e_i') \geq x \wedge e' + y \wedge e' \in E$.

Claim 4. If $x \le e$, $y \le e'$, then $q_e(x) = x$ and $q_{e'}(y) = y$.

Calculate, $q_e(x) = x_0$ and $q_{e'}(x) = x \setminus x_0 \le e, e'$ which by Claim 1 means $x \setminus x_0 = 0$. Similarly we prove $q_{e'}(y) = y$.

Claim 5. If $x \le e$ and $y \le e'$, thon $x + y = x \vee y = y + x$.

PROOF. Since $x \le e$ and $y \le e'$, we have $x + y, y + x \in E$, and $x + y \ge x, y$. Assume $z \ge x, y$. Then $q_e(z) \ge q_e(x) = x$ and $q_{e'}(z) \ge q_{e'}(y) = y$ which gives $z = q_e(z) + q_{e'}(z) \ge x + y$, that is, $x + y = x \vee y$.

We assert that $q_{e'}(x + y) = y$. Indeed, $x + y = q_e(x + y) + q_{e'}(x + y) \ge q_e(x) + q_e(y) + q_{e'}(x) + q_{e'}(y) = x + y$.

Assume now $x + y = y + d$ for some $d \in E$. Then $x = q_e(x + y) = q_e(y + d) = q_e(d)$ and $y = q_{e'}(x + y) = q_{e'}(y + d) \ge y + q_{e'}(d)$ which implies $x + y = y + d \ge y + q_e(d) = y + x$. But $y + x \ge x, y$, then $y + x \ge x \vee y = x + y$. □

Claim 6. If $x + y \in E$, then $q_{e'}(x + y) = q_{e'}(x) + q_{e'}(y)$.

Calculate and use Claim 5,

$$\begin{aligned} x + y &= q_e(x + y) + q_{e'}(x + y) \ge q_e(x) + q_e(y) + q_{e'}(x) + q_{e'}(y) \\ &= q_e(x) + q_{e'}(x) + q_e(y) + q_{e'}(y) = x + y. \end{aligned}$$

Claim 7. If $f_e : E \to [0, e] \times [0, e']$ is defined by $f_e(x) = (q_e(x), q_{e'}(x))$, $x \in E$, then f_e is an isomorphism and $e \in C(E)$.

Indeed, $f_e(e) = (e, 0)$, and if $f_e(x) = (x_1, x_2)$, then $x = x_5 + x_2$, and by Claim 2 and Claim 6, f_e is an injective homomorphism. Assume $x \le e$ and $y \le e'$, then $x + y \in E$ and $f_e(x + y) = (x, y)$, which proves that e is a central element of E.

Therefore, $x \wedge e \in E$, so that $x \wedge e = x_0$, and in addition, $q_e = p_e$, where p_e is defined by (4). □

6. The Cantor-Bernstein theorem for pseudo-effect algebras

In the present section, we apply the notion of central elements to show that an analogue of the Cantor-Bernstein theorem for pseudo-effect algebras can be obtained. We will study the case when the centre of a pseudo-effect algebra E is a Boolean σ-subalgebra of E with the central decomposition property.

We say that a finite or countable sequence $\{e_i\}$ of central elements of a pseudo-effect algebra E is a *central decomposition of unity* if $e_i \wedge e_j = 0$ for any $i \neq j$, and $\bigvee_i z_i = 1$. E is said to satisfy the *central decomposition property* if (i) any sequence of central elements $\{f_i\}$ has a supremum in E and belongs to $C(E)$ and $x \wedge (\bigvee_i f_i) = \bigvee_i (x \wedge f_i)$ for any $x \in E$, and (ii) if $\{e_i\}$ is a central decomposition of unity and $x_i \le e_i$ for any i, then $x = \bigvee_i x_i \in E$. That is if $\{e_i\}_i$ is a sequence of

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central elements of E, then $e = \bigvee_i e_i \in E$ and $e \in C(E)$. For example, any monotone σ-complete pseudo-effect algebra has such a property, see Proposition 6.1 below and Theorem 5.11, and every linearly ordered pseudo-effect algebra has the centrum $C(E) = \{0, 1\}$ which is a Boolean σ-algebra but E is not necessarily monotone σ-complete and it has this property as well as any Cartesian product of finitely many linearly ordered pseudo-effect algebras.

PROPOSITION 6.1. (i) *Let $e_1, \dots e_n$ be a finite central decomposition of unity in a pseudo-effect algebra E. The mapping $\phi : E \to \prod_{i=1}^n [0, e_i]$ given by $\phi(x) = (x \wedge e_i)_i$, $x \in X$, is an isomorphism.*

(ii) *Let $\{e_i\}_i$ be a countable central decomposition of unity in a monotone σ-complete pseudo-effect algebra E. The mapping $\phi : E \to \prod_i [0, e_i]$ given by $\phi(x) = (x \wedge e_i)_i$, $x \in E$, is an isomorphism, and E satisfies the central decomposition property.*

PROOF. (i) It follows from Proposition 2.7.

(ii) It is clear the mapping ϕ is an injective homomorphism. Assume now $x_i \le e_i$ for any i. By (ii) of Proposition 2.7, $x^n = x_1 + \dots + x_n \in E$ for any $n \ge 1$. Then $x = \bigvee_n x^n = \bigvee_i x_i \in E$ and $\phi(x) = (x \wedge e_i)_i = (x_i)_i$ which proves that ϕ is surjective. The central decomposition property follows now from Theorem 5.11. □

PROPOSITION 6.2. *Let E and F be two pseudo-effect algebras and let $f \in C(F)$. Assume that h is an isomorphism from E onto $[0, f]$. If $e \in C(E)$, then $h(e) \in C(F)$.*

PROOF. In view of Proposition 2.8, it is sufficient to show that $h(e) \in C([0, f])$. Put $f_0 := h(e)$. Then $f_0^{-f} = f_0^{\sim f}$. For any $y \in [0, f]$, $y \wedge f_0, y \wedge f_0^{\sim f} \in [0, f]$; indeed, we have $h(x) = y$ for a unique $x \in E$ and $x \wedge e \in E, x \wedge e^{\sim} \in E$, so that $h(x \wedge e) = h(x) \wedge h(e) = e \wedge f_0$ and $h(x \wedge e^{\sim}) = h(x) \wedge h(e^{\sim}) = y \wedge f_0^{\sim f}$. Therefore the mapping $\phi : [0, f] \to [0, f_0] \times [0, f_0^{\sim f}]$ defined by $\phi_f(y) = (y \wedge f_0, y \wedge f_0^{\sim f})$, $y \in [0, f]$, is an isomorphism in question proving $f_0 \in C([0, f])$. □

THEOREM 6.3 (Cantor-Bernstein). *Let E and F be pseudo-effect algebras satisfying the central decomposition property. Let $e \in C(E)$ and $f \in C(F)$ and let there are two isomorphisms of pseudo-effect algebras $\alpha : E \to [0, f]$ and $\beta : F \to [0, e]$. Then E and F are isomorphic pseudo-effect algebras.*

PROOF. Without loss of generality we can assume that $0 < e < 1$ and $0 < f < 1$. Define recursively two sequences $\{e_n\}_{n=0}^{\infty}$ and $\{f_n\}_{n=0}^{\infty}$ by

$$e_0 = 1, \qquad e_{n+1} = \beta(f_n),$$
$$f_0 = 1, \qquad f_{n+1} = \alpha(e_n).$$

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Due to Proposition 6.2, $e_n \in C(E)$ and $f_n \in C(F)$ for each $n \geq 0$. In addition, $e_0 \geq e_1 \geq e_2 \geq \cdots$ in E and $f_0 \geq f_1 \geq f_2 \geq \cdots$ in F. By the assumptions, the elements $e_\infty = \bigwedge_{n=1}^{\infty} e_n$ and $f_\infty = \bigwedge_{n=0}^{\infty} f_n$ are defined in E and F, respectively, and, in addition, $e_\infty \in C(E)$ and $f_\infty \in C(F)$. For all n we have $e_{n+2} = (\beta \circ \alpha)(e_n)$ and $f_{n+2} = (\alpha \circ \beta)(f_n)$. The mapping $\beta \circ \alpha$ is an isomorphism of E onto $[0, e_2]$, and it preserves countable infima and suprema. Therefore, $(\beta \circ \alpha)(e_\infty) = (\beta \circ \alpha)(\bigwedge_n e_n) = \bigwedge_n (\beta \circ \alpha)(e_n) = \bigwedge_n e_{n+2}$. Analogously, $f_\infty = (\alpha \circ \beta)(f_\infty)$, and $e_\infty = 0$ if and only if $f_\infty = 0$ while $\alpha(e_\infty) = f_\infty$ and $\beta(f_\infty) = e_\infty$. It is evident that the sequences $(e_\infty, e_0 \setminus e_1, e_1 \setminus e_2, \dots)$ and $(f_\infty, f_0 \setminus f_1, f_1 \setminus f_2, \dots)$ are decompositions of unity in E and F, respectively.

Moreover, if $x \in E$ and $y \in F$, then $[0, x]$ is isomorphic with $[0, \alpha(x)]$ and $[0, y]$ is isomorphic with $[0, \beta(y)]$. The restrictions of α and β^{-1} induce isomorphisms

$$[0, e_\infty] \cong [0, f_\infty], \qquad [0, e_{2n-2} \setminus e_{2n-1}] \cong [0, f_{2n-1} \setminus f_{2n}],$$
$$[0, e_{2n-1} \setminus e_{2n}] \cong [0, f_{2n-2} \setminus f_{2n-1}].$$

By assumptions,

$$E \cong [0, e_\infty] \times \prod_{n=0}^{\infty} [0, e_n \setminus e_{n+1}] \quad \text{and} \quad F \cong [0, f_\infty] \times \prod_{n=0}^{\infty} [0, f_n \setminus f_{n+1}],$$

consequently $E \cong F$. □

REMARK 6.4. (1) Theorem 6.3 generalizes the result of [4] for σ-complete MV-algebras.

(2) Theorem 6.3 generalizes the result of Jenča [18] for monotone σ-complete (commutative) effect algebras.

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References

[1] R. Baudot, 'Non-commutative logic programming language NoClog', in: *Symposium LICS* (Santa Barbara, 2000, Short Presentation) pp. 3–9.

[2] L. P. Belluce, 'α-complete MV-algebras', in: *Non-classical logics and their applications to fuzzy subsets* (ed. U. Hoehle *et al.*) (Kluwer, Dordrecht, 1995) pp. 7–21.

[3] C. C. Chang, 'Algebraic analysis of many valued logics', *Trans. Amer. Math. Soc.* **88** (1958), 467–490.

[4] A. De Simone, D. Mundici and M. Navara, 'A Cantor-Bernstein theorem for σ-complete MV-algebras', *Czechoslovak Math. J.* (to appear).

[5] A. De Simone, M. Navara and P. Pták, 'On interval homogeneous orthomodular lattices', *Comm. Math. Univ. Carolinae* **42** (2001), 23–30.

[6] A. Di Nola, G. Georgescu and A. Iorgulescu, 'Pseudo-BL-algebras, I, II', *Mult.-Valued Logic* (to appear).

[7] A. Dvurečenskij, 'Pseudo MV-algebras are intervals in ℓ-groups', *J. Austral. Math. Soc.* **72** (2002), 427–445.

[8] A. Dvurečenskij and S. Pulmannová, *New trends in quantum structures* (Kluwer, Dordrecht, 2000).

[9] A. Dvurečenskij and T. Vetterlein, 'Pseudoeffect algebras. I. Basic properties', *Internat. J. Theor. Phys.* **40** (2001), 685–701.

[10] ———, 'Pseudoeffect algebras. II. Group representations', *Internat. J. Theor. Phys.* **40** (2001), 703–726.

[11] D. J. Foulis and M. K. Bennett, 'Effect algebras and unsharp quantum logics', *Found. Phys.* **24** (1994), 1325–1346.

[12] G. Georgescu and A. Iorgulescu, 'Pseudo-MV algebras: A non-commutative extension of MV-algebras', in: *Proc. Fourth Inter. Symp. on Econ. Inform., May 6–9, 1999, Bucharest* (ed. I. Smeureanu *et al.*) (INFOREC Printing House, Bucharest, 1999) pp. 961–968.

[13] ———, 'Pseudo-MV algebras', *Mult.-Valued Logic* **6** (2001), 95–135.

[14] K. R. Goodearl, *Partially ordered abelian groups with interpolation*, Math. Surveys and Monographs 20 (Amer. Math. Soc., Providence, RI, 1986).

[15] R. J. Greechie, D. J. Foulis and S. Pulmannová, 'The center of an effect algebra', *Order* **12** (1995), 91–106.

[16] J. Jakubík, 'Cantor-Bernstein theorem for MV-algebras', *Czechoslovak Math. J.* **49** (1999), 517–529.

[17] ———, 'A theorem of Cantor-Bernstein theorem type for orthogonally σ-complete pseudo MV-algebras', *Tatra Mt. Math. Publ.* (to appear).

[18] G. Jenča, 'A Cantor-Bernstein type theorem for effect algebras', *Algebra Universalis* (to appear).

[19] F. Kôpka and F. Chovanec, 'D-posets', *Math. Slovaca* **44** (1994), 21–34.

[20] D. Mundici, 'Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus', *J. Funct. Anal.* **65** (1986), 15–63.

[21] J. Rachůnek, 'A non-commutative generalization of MV-algebras', *Czechoslovak Math. J.* **52** (2002), 255–273.

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