J. Austral. Math. Soc. (Series B) 22 (1981), 394–407

THE BOUNDARY INTEGRAL EQUATION METHOD FOR THE SOLUTION OF A CLASS OF PROBLEMS IN ANISOTROPIC ELASTICITY

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(Received 30 September 1980)

(Revised 25 November 1980)

Abstract

A boundary integral procedure for the solution of an important class of problems in anisotropic elasticity is outlined. Specific numerical examples are considered in order to provide a comparison with the standard boundary integral method.

1. Introduction

The boundary integral equation method is now widely recognized as an extremely useful method for the solution of a wide class of elliptic boundary value problems. Specifically, it has been used by Rizzo and Shippy [1] to solve a number of problems in anisotropic elasticity. The procedure used by Rizzo and Shippy was to use the point force solution for an anisotropic material in Betti's reciprocal theorem in order to obtain an appropriate boundary integral equation. This was then used to obtain numerical solutions to certain problems. Here it is shown that if a particular Green's function is used in place of the point force solution then it is possible to obtain a boundary integral equation which is superior to the one used by Rizzo and Shippy for a significant class of problems. In particular, the equation derived in this paper may be used to advantage for problems involving deformations of anisotropic slabs and also for the solution of an important class of geomechanics problems.

2. Statement of the problem

Take Cartesian coordinates x_1, x_2, x_3 and assume part of the region $x_2 > 0$ (denoted by R) is filled with an anisotropic elastic material with part of the boundary of the material lying in the plane, $x_2 = 0$ (Fig. 1). The part of the boundary which lies in the $x_2 = 0$ plane will be denoted by C_1 while the remainder of the boundary will be denoted by C_2. Also the geometry of the material will be assumed to not vary in the Ox_3 direction. On C_1 it will be assumed that either the displacement vector u_k is zero or the traction vector P_i is zero. On C_2 either the displacement vector or the traction vector is specified. Furthermore the specified displacements or tractions will be required to be independent of x_3. The problem is to find the displacement and stress throughout the material.

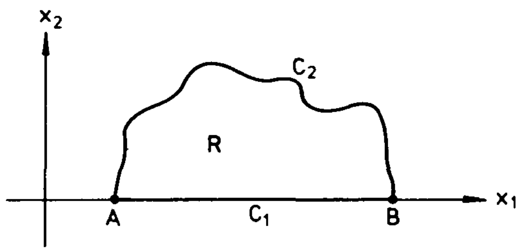

Fig. 1. General geometry of the problem.

3. Fundamental Equations

The stresses σ_{ij} are related to the elastic displacements u_k by the equations

$$\sigma_{ij} = c_{ijkl}\frac{\partial u_k}{\partial x_l}, \tag{3.1}$$

where $i, j, k, l = 1, 2, 3$ and the convention of summing over a repeated Latin suffix is used. The elastic moduli have the symmetry properties

$$c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}. \tag{3.2}$$

Substitution of (3.1) into the equilibrium equations yields

$$c_{ijkl}\frac{\partial^2 u_k}{\partial x_j \partial x_l} = 0. \tag{3.3}$$

Because of the nature of the problem under consideration it is reasonable to suppose that the u_k occurring in (3.3) are independent of x_3. The system (3.3) then becomes a special case of a more general system considered by Clements and Rizzo [2]. By employing the results in [2] it may be readily shown that an integral equation which solves the problem under consideration is

$$\lambda u_j(\mathbf{x}_0) + F^{-1}\int_C \left[P_i(\mathbf{x})\Phi_{ij}(\mathbf{x}, \mathbf{x}_0) - \Gamma_{ij}(\mathbf{x}, \mathbf{x}_0)u_i(\mathbf{x}) \right] ds(\mathbf{x}) = 0, \tag{3.4}$$

where F is an arbitrary constant, $\lambda = 1$ if $\mathbf{x}_0 \in R$, $[\mathbf{x}_0 = (a, b)]$, $C = C_1 + C_2$ and $0 < \lambda < 1$ if $\mathbf{x}_0 \in C$ and Φ_{ij} is any solution of the inhomogeneous system

$$c_{ijkl}\frac{\partial^2\Phi_{km}}{\partial x_j \partial x_l} = F\delta_{im}\delta(\mathbf{x} - \mathbf{x}_0) \quad \text{for } m = 1, 2, 3, \tag{3.5}$$

where δ_{im} and δ denote the Kronecker delta and Dirac delta function respectively. The P_i and Γ_{ij} occurring in (3.4) are given by

$$P_i = c_{ijkl}\frac{\partial \phi_k}{\partial x_l} n_j, \tag{3.6}$$

$$\Gamma_{im} = c_{ijkl}\frac{\partial^2\Phi_{km}}{\partial x_l} n_j, \tag{3.7}$$

where P_i is the traction vector.

The particular solution of (3.5) given in [1] will be denoted by $\Phi^{(1)}_{km}$ and $\Gamma^{(1)}_{im}$ and is given by

$$\Phi^{(1)}_{km} = \frac{1}{2\pi}\Re\left\{\sum_\alpha A_{k\alpha}N_{\alpha j}\log(z_\alpha - c_\alpha)\right\}d_{jm}, \tag{3.8}$$

$$\Gamma^{(1)}_{km} = \frac{1}{2\pi}\Re\left\{\sum_\alpha L_{ij}N_{\alpha p}(z_\alpha - c_\alpha)^{-1}\right\}n_j d_{pm}, \tag{3.9}$$

where $\Re$ denotes the real part of a complex number, $z_\alpha = x_1 + \tau_\alpha x_2$, and $c_\alpha = a + \tau_\alpha b$, where τ_α, for $\alpha = 1, 2, \ldots, N$, are the N roots with positive imaginary part of the polynomial in τ

$$|c_{i1k1} + c_{i2k1}\tau + c_{i1k2}\tau + c_{i2k2}\tau^2| = 0. \tag{3.10}$$

The $A_{k\alpha}$ occurring in (3.8) are the solutions of the system

$$(c_{i1k1} + c_{i1k2}\tau_\alpha + c_{i2k1}\tau_\alpha + c_{i2k2}\tau_\alpha^2)A_{k\alpha} = 0. \tag{3.11}$$

Also the $N_{\alpha j}$, $L_{ij\alpha}$ and d_{rj} are defined by

$$\sum_\alpha A_{k\alpha}N_{\alpha j} = \delta_{kj},$$

$$L_{ij\alpha} = (c_{ijk1} + \tau_\alpha c_{ijk2})A_{k\alpha},$$

and

$$\delta_{ij}F = -\frac{1}{2}i\sum_\alpha \left\{L_{i2\alpha}N_{\alpha r} - \bar{L}_{i2\alpha}\bar{N}_{\alpha r}\right\}d_{rj}. \tag{3.12}$$

Now it is clear that a solution to (3.5) may consist of the particular solution (3.8) plus any solution of the associated homogeneous system (3.3). Here some

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solutions of (3.5) are investigated with the aim being to obtain some simplification of (3.4) for the particular class of problems under consideration. In particular, the solution to (3.5) will be written in the form

$$\Phi_{km} = \Phi^{(1)}_{km} + \Phi^{(2)}_{km}, \qquad \Gamma_{km} = \Gamma^{(1)}_{km} + \Gamma^{(2)}_{km}, \tag{3.13}$$

where $\Phi^{(1)}_{km}$ and $\Gamma^{(1)}_{km}$ are given by (3.8) and (3.9). The extra terms $\Phi^{(2)}_{km}$ and $\Gamma^{(2)}_{km}$ will be solutions of (3.3) chosen such that either $\Phi_{km}(x_1, 0)$ or $\Gamma_{km}(x_1, 0)$ is zero. Image considerations indicate that appropriate choices are

(i) for $\Phi_{km}(x_1, 0) = 0$,

$$\Phi^{(2)}_{km} = -\frac{1}{2\pi}\Re\left\{\sum_{\alpha} A_{k\alpha}N_{\alpha q}\sum_{\beta}\bar{A}_{q\beta}\bar{N}_{\beta j}\log\left(z_\alpha - \bar{c}_\beta\right)\right\}d_{jm}, \tag{3.14}$$

(ii) for $\Gamma_{ij}(x_1, 0) = 0$,

$$\Gamma^{(2)}_{km} = -\frac{1}{2\pi}\Re\left\{\sum_{\alpha} L_{kj\alpha}M_{\alpha k}\sum_{\beta}\bar{L}_{k2\beta}\bar{N}_{\beta r}\left(z_\alpha - \bar{c}_\beta\right)^{-1}\right\}n_j d_{rm}. \tag{3.15}$$

If in the required solution to (3.3) the displacement vector u_i is zero on C_1 and if Φ_{km} is given by (3.13), (3.8) and (3.14) then the integrand along C_1 in (3.4) is zero and the integration need only be taken along C_2. That is, C may be replaced by C_2 in (3.4). Alternatively, if the traction vector P_i is zero on C_1 and Γ_{km} is defined by (3.13), (3.9) and (3.12) then the integrand along C_1 is again zero and hence the C in (3.4) may be replaced by C_2.

This simplification in the integral equation is not restricted to the case when either the displacement or traction vector is zero on the whole of C_1. In other relevant cases the method of superposition may be employed. The procedure for doing this will be detailed in the following section.

4. Particular problems and numerical procedure

In this section some particular two-dimensional elastic problems will be considered in order to demonstrate the usefulness of the formulas derived previously. For the present purposes it will be sufficient to consider some boundary value problems for the system of two equations governing plane deformations of a transversely isotropic material. The elastic behaviour of transversely isotropic materials is characterized by five elastic constants which will be denoted by A, N, F, C and L. If it is assumed that the x_1-axis is normal to the transverse planes then the only non-zero c_{ijkl} which are of interest are given by

$$c_{1111} = C, \quad c_{1122} = F, \quad c_{2222} = A, \quad c_{1133} = F,$$
$$c_{2233} = N, \quad c_{1331} = L, \quad c_{1212} = L, \quad c_{2323} = \tfrac{1}{2}(A - N).$$

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Thus the system of two governing equations for the displacements u_1 and u_2 are

$$C\frac{\partial^2 u_1}{\partial x_1^2} + L\frac{\partial^2 u_1}{\partial x_2^2} + (F + L)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} = 0, \tag{4.1}$$

and

$$(F + L)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + L\frac{\partial^2 u_2}{\partial x_1^2} + A\frac{\partial^2 u_2}{\partial x_2^2} = 0, \tag{4.2}$$

while (3.10) yields

$$\left[\tfrac{1}{2}(A - N)\tau^2 + L\right]\left[AL\tau^4 - (F^2 + 2FL - AC)\tau^2 + CL\right] = 0, \tag{4.3}$$

so that if τ_1 is taken to be given by

$$\tau_1^2 = -2L/(A - N), \tag{4.4}$$

then τ_2^2 and τ_3^2 are the roots of the quartic factor in (4.3). Substituting into (3.11) it follows that a suitable choice of the $A_{k\alpha}$ is

$$[A_{k\alpha}] = \begin{bmatrix} 0 & \dfrac{-i(F + L)\tau_2}{C + L\tau_2^2} & \dfrac{-i(F + L)\tau_3}{C + L\tau_2^2} \\ 0 & i & i \\ 1 & 0 & 0 \end{bmatrix} \tag{4.5}$$

and hence, from the second equation in (3.12) it follows that

$$[L_{i2\alpha}] = \begin{bmatrix} 0 & iL\left[\dfrac{C - F\tau_2^2}{C + L\tau_2^2}\right] & iL\left[\dfrac{C - F\tau_3^2}{C + L\tau_3^2}\right] \\ 0 & i\tau_2\left[A - \dfrac{F(F + L)}{C + L\tau_2^2}\right] & i\tau_3\left[A - \dfrac{F(F + L)}{C + L\tau_3^2}\right] \\ \dfrac{1}{2}\tau_1(A - N) & 0 & 0 \end{bmatrix}. \tag{4.6}$$

Formulas for the other matrices such as $N_{\alpha j}$, and $M_{\alpha j}$ may be readily derived but they are rather lengthy and nothing is to be gained by presenting them explicitly here since they are readily calculated on the computer for particular values of the constants A, N, F, C and L.

The problems will be solved by employing two methods.

Method 1

In this case the solution will be obtained by employing the integral equation (3.4) with Φ_{km} and Γ_{km} given by (3.13) and $\Phi_{km}^{(2)}$ and $\Gamma_{km}^{(2)}$ both zero. Hence, for this method the integral in (3.4) will be taken round the whole boundary $C = C_1 + C_2$.

Method 2

Here the integral equation (3.4) will again be used with Φ_{km} and Γ_{km} given by (3.13). However, in this case $\Phi_{km}^{(2)}$ and $\Gamma_{km}^{(2)}$ will be obtained through (3.15) and the integral will only be taken along C_2.

Three particular problems will be considered.

Problem 1: Test Problem

Consider the region shown in Fig. 2 with the following boundary conditions on the four sides.

$$\begin{aligned} &AB: \quad P_i = 0. \\ &\left.\begin{matrix} BC: \\ CD: \\ DA: \end{matrix}\right\} \quad \frac{u_k}{lp_0} \text{ given by (4.8) below.} \end{aligned} \tag{4.7}$$

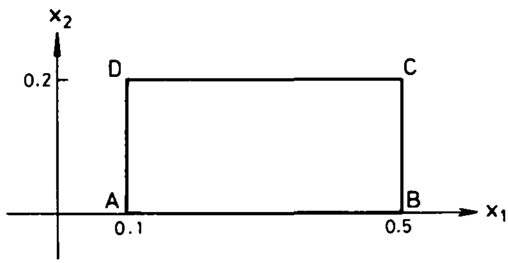

Fig. 2. Geometry for the test problem.

The problem is to use Methods 1 and 2 to find a numerical solution to (4.1) and (4.2) which satisfies the above boundary conditions. These results may then be compared with those obtained from the analytical solution which is

$$\frac{u_k}{lp_0} = -\mathfrak{R}\left[\frac{1}{\pi i}\sum_{\alpha=1}^{2} A_{k\alpha}M_{\alpha 2}\left\{\left(\frac{z_\alpha}{l} - \frac{a_2}{l}\right)\log\left(\frac{z_\alpha}{l} - \frac{a_2}{l}\right) - \left(\frac{z_\alpha}{l} - \frac{a_1}{l}\right)\log\left(\frac{z_\alpha}{l} - \frac{a_1}{l}\right)\right\}\right], \tag{4.8}$$

and

$$\frac{P_i}{p_0} = -\mathfrak{R}\left[\frac{1}{\pi i}\sum_{\alpha=1}^{2} L_{ij\alpha}M_{\alpha 2}\log\left\{\frac{z_\alpha - a_2}{z_\alpha - a_1}\right\}\right]n_j. \tag{4.9}$$

Problem 2: Deformations of a slab on a rigid foundation

Consider the elastic slab on a rigid foundation with a load on the opposite face as shown in Fig. 3. The boundary conditions are

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$$
\begin{aligned}
AB &: \frac{P_i}{p_0} = \begin{cases} 1 & \text{for } 0.35 < |x_1/l| < 0.85, \\ 0 & \text{for } 0.85 < |x_1/l| < 1.1 \text{ and } 0.1 < |x_1/l| < 0.35, \end{cases} \\
BC &: P_i = 0, \\
CD &: u_k = 0,
\end{aligned}
\tag{4.10}
$$

and

$$DA : P_i = 0.$$

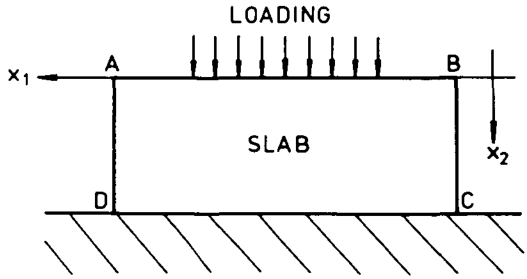

Fig. 3. Geometry for a slab on a rigid foundation.

The problem is to use Methods 1 and 2 to find a numerical solution to (4.1) and (4.2) which satisfies the above boundary conditions. No simple analytical solution to this problem exists.

It is necessary at this point to further detail the implementation of Method 2 for this problem. Here the superposition procedure is employed. That is, the desired solution is written as the sum of two solutions in the form

$$u_k = u_k^{(1)} + u_k^{(2)}, \qquad P_i = P_i^{(1)} + P_i^{(2)}, \tag{4.11}$$

where

$$\frac{u_k^{(1)}}{lp_0} = -\Re\Bigg[\frac{1}{\pi i}\sum_{\alpha=1}^{2} A_{k\alpha}M_{\alpha 2}\bigg\{\Big(\frac{z_\alpha}{l} - \frac{0.85}{l}\Big)\log\Big(\frac{z_\alpha}{l} - \frac{0.85}{l}\Big) - \Big(\frac{z_\alpha}{l} - \frac{0.35}{l}\Big)\log\Big(\frac{z_\alpha}{l} - \frac{0.35}{l}\Big)\bigg\}\Bigg], \tag{4.12}$$

and

$$\frac{P_i^{(1)}}{p_0} = -\Re\left[\frac{1}{\pi i}\sum_{\alpha=1}^{2} L_{ij\alpha}M_{\alpha 2}\log\left\{\frac{z_\alpha - 0.85}{z_\alpha - 0.35}\right\}\right]n_j. \tag{4.13}$$

This solution satisfies the conditions on AB. In order to satisfy the remaining boundary conditions in the other three sides Method 2 is employed to obtain $\phi^{(2)}$ and $P_i^{(2)}$ in such a way as to compensate for the effect of $\phi^{(1)}$ and $P_1^{(1)}$. That is,

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the boundary conditions for $\phi^{(2)}$ on the three sides are

$$\begin{aligned} BC:&\quad P_i^{(2)} = -P_i^{(1)},\\ CD:&\quad u_k^{(2)} = -u_k^{(1)},\\ \text{and}\qquad DA:&\quad P_i^{(2)} = -P_i^{(1)}. \end{aligned} \tag{4.14}$$

The sum of the two solutions then gives the solution which satisfies the given boundary conditions.

Problem 3: Deformations of a supported slab

Consider the elastic slab resting on two supports with a load on the opposite face as shown in Fig. 4. The boundary conditions are

$$\begin{aligned} AB &: \frac{P_i}{p_0} = \begin{cases} 1 & \text{for } 0.35 < |x_1/l| < 0.85,\\ 0 & \text{for } 0.85 < |x_1/l| < 1.1 \text{ and } 0.1 < |x_1/l| < 0.35, \end{cases}\\ BC &: P_i = 0,\\ CD &: u_k = 0 \quad \text{for } 1.0 < |x_1/l| < 1.1 \text{ and } 0.1 < |x_1/l| < 0.2,\\ &\quad P_1 = 0 \quad \text{for } 0.2 < |x_1/l| < 1.0,\\ \text{and}\quad DA &: P_i = 0. \end{aligned} \tag{4.15}$$

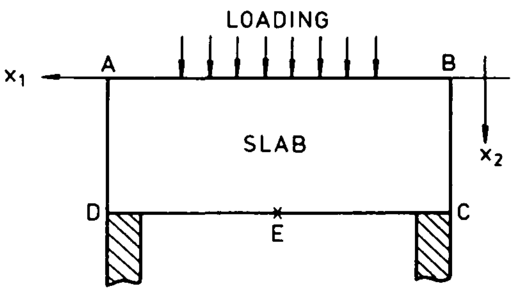

Fig. 4. Geometry for a supported slab.

Again the problem is to use Methods 1 and 2 to find a numerical solution to (4.1) and (4.2) which satisfies the above boundary conditions. No simple analytical solution to this problem exists.

As in Problem 2 the superposition principle may be used to solve the problem. The procedure is a simple modification of the one outlined for Problem 2.

Now consider the numerical procedure.

Method 1. Letting $\phi_1 = 1$ and $\phi_2 = 0$ with $F = 1$ in equations (3.4) a value for λ is obtained as

$$\lambda^{(1)} = \int_C \Gamma_{11}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}); \qquad 0 = \int_C \Gamma_{12}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}). \tag{4.16a}$$

Similarly letting $\phi_1 = 0$ and $\phi_2 = 1$,

$$0 = \int_C \Gamma_{21}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}); \qquad \lambda^{(2)} = \int_C \Gamma_{22}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}). \tag{4.16b}$$

Substituting these values into (3.4) yields

$$\int_C P_i(x)\Phi_{ij}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}) = \int_C \Gamma_{ij}(\mathbf{x}, \mathbf{x}_0)[\phi_i(\mathbf{x}) - \phi_i(\mathbf{x}_0)]\, ds(\mathbf{x}). \tag{4.17}$$

The numerical technique used to solve equation (4.17) consists of replacing the integration by summation so that a system of linear equations is obtained. This is then solved by standard matrix inversion techniques.

Following Symm [3], the boundary C is divided into N segments from $\mathbf{q}_{k-1}$ to $\mathbf{q}_k$, $k = 1, 2, \ldots, N$, with $\mathbf{q}_0 = \mathbf{q}_N$. The midpoint of this segment is $\bar{\mathbf{q}}_k$. If the integrals in (4.17) are replaced by sums then (4.18) is obtained

$$\begin{aligned}\sum_{m=1}^{N} \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} & P_i(\mathbf{x})\Phi_{ij}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}) \\ &= \sum_{m=1}^{N} \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} \Gamma_{ij}(\mathbf{x}, \mathbf{x}_0)\phi_i(\mathbf{x})\, ds(\mathbf{x}) \\ &\quad - \phi_i(\mathbf{x}_0) \sum_{m=1}^{N} \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} \Gamma_{ij}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}).\end{aligned} \tag{4.18}$$

The segments on which $P_i(\mathbf{x})$, $i = 1, 2$, are known, are renumbered $1, 2, \ldots, r$ and the segments on which $\phi_i(\mathbf{x})$, $i = 1, 2$, are known, renumbered $r + 1, \ldots, N$. Taking x_0 to be each of the "midpoints" $\bar{\mathbf{q}}_k$ in turn will yield $2N$ linear algebraic equations for $\phi_i(\bar{\mathbf{q}}_1), \ldots, \phi_i(\bar{\mathbf{q}}_r)$, $P_i(\bar{\mathbf{q}}_{r+1}), \ldots, P_i(\bar{\mathbf{q}}_N)$, $i = 1, 2$. The integral equation (4.18) then becomes

$$\begin{aligned}\sum_{m=r+1}^{N} P_i(\bar{\mathbf{q}}_m) \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} \Phi_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\, ds(\mathbf{x}) &- \sum_{\substack{m=1\\ m\neq l}}^{r} \phi_i(\bar{\mathbf{q}}_m) \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} \Gamma_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\, ds(\mathbf{x}) \\ &= \sum_{\substack{m=r+1\\ m\neq l}}^{N} \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} \Gamma_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\phi_i(\mathbf{x})\, ds(\mathbf{x}) \\ &\quad - \phi_i(\bar{\mathbf{q}}_l) \sum_{\substack{m=1\\ m\neq l}}^{N} \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} \Gamma_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\, ds(\mathbf{x}).\end{aligned} \tag{4.19}$$

Equation (4.19) may be rewritten in matrix form as

$$\sum_{i=1}^{2} A_{ij}X_i = B_j, \qquad j = 1, 2, \tag{4.20}$$

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where

$$A_{ij} = [S_{kl}]^{ij}, \qquad X_i = [x_k]^i, \qquad B_j = [r_l]^j. \tag{4.21}$$

The elements of these matrices are

$$S_{kl}^{ij} = \begin{cases} \displaystyle\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} \Gamma_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\, ds(\mathbf{x}) & \text{if } l \neq k \text{ and } k \leqslant r, \\ \displaystyle\sum_{\substack{m=1 \\ m \neq l}}^{N} \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} \Gamma_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\, ds(\mathbf{x}) & \text{if } l = k \text{ and } k \leqslant r, \\ \displaystyle\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} \Phi_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\, ds(\mathbf{x}) & \text{if } k > r, \end{cases} \tag{4.22}$$

$$x_k^i = \begin{cases} \phi_i(\bar{\mathbf{q}}_k) & \text{if } k \leqslant r, \\ P_i(\bar{\mathbf{q}}_k) & \text{if } k > r, \end{cases} \tag{4.23}$$

and

$$\begin{aligned} r_l^i = & \sum_{\substack{m=r+1 \\ m \neq l}}^{N} \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} \Gamma_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\phi_i(\mathbf{x})\, ds(\mathbf{x}) \\ & - \sum_{m=1}^{r} \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} P_i(\mathbf{x})\Phi_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\, ds(\mathbf{x}) \\ & - \xi\phi_i(\bar{\mathbf{q}}_l) \sum_{\substack{m=1 \\ m \neq l}}^{N} \int_{\mathbf{q}_{m-1}}^{\mathbf{q}_m} \Gamma_{ij}(\mathbf{x}, \bar{\mathbf{q}}_l)\, ds(\mathbf{x}), \end{aligned} \tag{4.24}$$

where

$$\xi = \begin{cases} 1 & \text{if } l > r, \\ 0 & \text{if } l \leqslant r. \end{cases}$$

When using equation (3.8) for Φ_{ij} it is necessary to be careful when evaluating the integral of Φ_{ij} as the function Φ_{ij} has a logarithmic singularity at $x = x_0$.

When the integration is taking place along the segment containing the current value of x_0 the singularity is struck. This can be overcome by considering the segment as two separate sections either side of x_0. The integral then becomes

$$\begin{aligned} & 2\pi\int_{\mathbf{q}_{k-1}}^{\mathbf{q}_k} \Phi_{ij}(\mathbf{x}, \bar{\mathbf{q}}_k)\, ds(\mathbf{x}) \\ & = \Re\left[\sum_{\alpha=1}^{2} A_{i\alpha}N_{\alpha m}|\mathbf{q}_k - \mathbf{q}_{k-1}|\left(\log\frac{1}{2}\{(x_k - x_{k-1}) + \tau_\alpha(y_k - y_{k-1})\} - 1\right)\right], \end{aligned} \tag{4.25}$$

where

$$\mathbf{q}_k = (x_k, y_k).$$

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If the integral is taken over any other segment then it can be approximated by Simpson's Rule.

Method 2. Consider again equations (4.16)(a) and (b). If Γ_{ij} and Φ_{ij} are chosen as given in equations (3.13), and (3.15), then

$$\lambda^{(1)} = \int_{C_1} \Gamma_{11}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}) + \int_{C_2} \Gamma_{11}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}), \tag{4.26}$$

and

$$\lambda^{(2)} = \int_{C_1} \Gamma_{22}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}) + \int_{C_2} \Gamma_{22}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}),$$

where C_1 and C_2 are different segments of the boundary C as given in Fig. 1.

However $\Gamma_{11}(\mathbf{x}, \mathbf{x}_0)$ and $\Gamma_{22}(\mathbf{x}, \mathbf{x}_0)$ are selected so that they are zero along C_1. The values of λ are therefore

$$\lambda^{(1)} = \int_{C_2} \Gamma_{11}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}),$$

and

$$\lambda^{(2)} = \int_{C_2} \Gamma_{22}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}). \tag{4.27}$$

Since $P_i(\mathbf{x}) = 0$ on C_1, $\Gamma_{ij}(\mathbf{x}, \mathbf{x}_0)$ is chosen to be zero along C_1 and so the integral equation of 3.4 becomes

$$\int_{C_2} P_i(\mathbf{x})\Phi_{ij}(\mathbf{x}, \mathbf{x}_0)\, ds(\mathbf{x}) = \int_{C_2} \Gamma_{ij}(\mathbf{x}, \mathbf{x}_0)[\phi_i(\mathbf{x}) - \phi_i(\mathbf{x}_0)]\, ds(\mathbf{x}), \tag{4.28}$$

and the method proceeds in exactly the same way as Method 1, except that the boundary is now C_2 instead of C.

5. Numerical results

In order to obtain some numerical results it is necessary to consider a particular transversely isotropic material. Here, for illustrative purposes only, the constants for a crystal of titanium will be used. These constants are $A = 16.2$, $N = 9.2$, $F = 6.9$, $C = 18.1$ and $L = 4.67$. If each of these values is multiplied by 10^{11} then the units for the constants are dynes/cm^2.

Problem 1 admits the analytic solution given by (4.8) and (4.9), which can be compared with the numerical solutions obtained from Methods 1 and 2. These results are presented in Tables 1 and 2 with numerical values given for every fourth segment in Table 1 and every eighth segment in Table 2.

TABLE 1
Numerical and analytic solutions using a 24 point boundary

POINT		NUMERICAL SOLUTION				ANALYTIC SOLUTION	
		METHOD I		METHOD II			
(x, y)		P_1	P_2	P_1	P_2	P_1	P_2
1.100	.063	−.01695	.02514	−.02357	−.00654	−.02613	−.00002
1.100	.438	−.13756	.04706	−.13815	.04848	−.13857	.04730
.663	.500	.02905	−.00903	.02767	−.00797	.02752	−.00794
.163	.500	.01623	−.00241	.01532	−.00307	.01371	−.00294
.100	.063	.01091	−.01345	.00629	.00453	.00815	.00055

TABLE 2
Numerical and analytic solutions using a 48 point boundary

POINT		NUMERICAL SOLUTION				ANALYTIC SOLUTION	
		METHOD I		METHOD II			
(x, y)		P_1	P_2	P_1	P_2	P_1	P_2
1.100	.031	−.00676	.02360	−.01299	−.00725	−.01313	−.00102
1.100	.469	−.14198	.05237	−.14246	.05283	−.14307	.05233
.631	.500	.02707	−.00782	.02654	−.00750	.02623	−.00741
.131	.500	.01518	−.00255	.01467	−.00279	.01318	−.00278
.100	.031	.00638	−.01289	.00273	.00454	.00408	.00071

Both methods give reasonably close answers when the error introduced by the integration method is taken into account, however Method 2 is superior to Method 1, in the accuracy of the solutions. The boundary was discretised into 24 and 48 points for the test case and convergence to the analytic result is evident as more boundary points are taken for both methods.

The size of the matrices A_{ij} given in (4.20) depends on the number of segments used for the boundary on which the integration takes place.

Because for some mixed boundary value problems, the determinants of the individual matrices A_{ij} can be extremely small in magnitude, a partitioned matrix Q, made up of the four A_{ij} matrices is used to solve the system given in (4.20).

Discretising the boundary into 48 points gives a 96×96 coefficient matrix Q when Method 1 is employed, but this reduces to a 64×64 matrix when Method 2 is employed. This results in a 39% decrease in running time for Method 2 compared with Method 1.

Problems 2 and 3 have no analytic solution, so the only comparison possible is between the two numerical solutions. The results are given using 24 points around the boundary. Since the results are symmetric or asymmetric only results for one side and half the top are presented.

TABLE 3

Results from a slab on a solid foundation (Fig. 3)

POSITION	SOLUTION	METHOD 2	SOLUTION	METHOD 1	DIFFERENCE	
(x, y)	X_1	X_2	X_1^*	X_2^*	$X_1^* - X_1$	$X_2^* - X_2$
1.1,0.063	.00069	.00280	.00239	.00353	.00170	.00073
1.1,0.188	.00520	.00290	.00566	.00370	.00046	.00080
1.1,0.313	.00573	.00366	.00600	.00381	.00027	.00015
1.1,0.438	.00278	.00294	.00310	.00261	.00032	−.00033
1.038,0.5	−.20765	−.39383	−.22265	−.39453	−.01500	−.00070
0.913,0.5	−.18607	−.38931	−.18843	−.39210	−.00236	−.00279
0.788,0.5	−.14364	−.56311	−.14572	−.55400	−.00208	+ .00911
0.663,0.5	−.04683	−.67385	−.05463	−.65970	−.00780	+ .01415

TABLE 4

Results from a simply supported slab (Fig. 4).

X is the unknown, either ϕ or P.

POSITION	SOLUTION	METHOD 2	SOLUTION	METHOD 1	DIFFERENCE	
(x, y)	X_1	X_2	X_1^*	X_2^*	$X_1^* - X_1$	$X_2^* - X_2$
0.538,.5	−.00350	.05263	−.00344	.05686	.00006	.00423
0.413,0.5	−.00910	.04366	−.00895	.04779	.00015	.00413
0.288,0.5	−.01015	.02629	−.01000	.02962	.00015	.00333
0.163,0.5	1.05812	−1.98597	1.06367	−2.01966	.00555	.03369
0.1,0.438	−.00610	.00624	−.00625	.00611	.00015	.00013
0.1,0.313	−.00405	.01309	−.00404	.04468	.00004	.00159
0.1,0.188	+ .00169	.01603	.00082	.01928	.00087	.00325
0.1,0.063	.01337	.01865	.00879	.02107	.00458	.00242

It should be noted that the numerical procedure used here to solve the integral equation (4.17) may be improved upon in several ways. For example, piecewise quadratic polynomial representations (see for example Cruse in reference [4], Fairweather et al. [6]) for the solution of such integral equations may be employed to yield improved accuracy with cruder discretizations.

Finally, it is of interest to note that the type of approach employed in this paper is essentially in the same spirit as some work by Cruse in references [4], [5] who successfully employed a somewhat similar procedure for the solution of certain problems in isotropic elasticity.

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