GENERALIZED CANONICAL SYSTEMS. FORMAL SOLUTIONS AND THE MAIN PROBLEM OF SATELLITE THEORY

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Abstract. The generalized canonical version of Hori's method is discussed using some properties of generalized canonical system and,then, applied in solving the main problem of satellite theory.

1. Some Properties of Generalized Canonical Systems

Consider the autonomous generalized canonical system governed by the Hamiltonian H,

$$H = \lambda \cdot F(x), \tag{1}$$

where $x = [x_1...x_n]^T$ is a nx1 vector of coordinates, $\lambda = [\lambda_1...\lambda_n]^T$ is a nx1 vector of momenta, $F = [F_1...F_n]^T$ is a nx1 vector of functions of the coordinates, dot denotes the scalar vector product and T the transposed vector.The functions F_k have two distinct parts : one defines an integrable dynamical system and the remaining part is related to the perturbations,

$$F_k(x) = f_k(x) + g_k(x), \ \ k = 1, ..., n. \tag{2}$$

The functions $g_k(x)$ are developable in powers of a small parameter ϵ.

Now,consider the autonomous dynamical system governed by the undisturbed Hamiltonian $H_0 = \lambda \cdot f(x)$, the general solution of which is given by

$$x = \phi(t, c_1, ..., c_n), \tag{3}$$

$$\lambda = \left(\Delta_\phi{}^{-1}\right)^T b, \tag{4}$$

where $\Delta_\phi = \left[\frac{\partial \phi}{\partial c}\right]$ is the Jacobian matrix ,being $c = [c_1...c_n]^T$ and $b = [b_1...b_n]^T$ vectors of arbitrary constants of integration.The following results are proved in Da Silva Fernandes (1991):

Proposition 1: The general solution of the autonomous dynamical system governed by the integrable part H_0 of the Hamiltonian function H,which governs a generalized canonical system,defines a Mathieu transformation between the variables of the system and the 2n arbitrary constants of integration of this general solution.This transformation involves explicitly the time t.

Proposition 2 : The dynamical system resulting from the Mathieu transformation defined by the general solution of the dynamical system governed by the Hamiltonian function H_0 is non-autonomous and corresponds to the equations of variation of the parameters of the original dynamical system.If the generalized canonical system is obtained by hamiltonization of an arbitrary non-canonical system,then the differential equations for the new momenta are unimportant.

S. Ferraz-Mello (ed.), Chaos, Resonance and Collective Dynamical Phenomena in the Solar System, 235–238.

2. Generalized Canonical Version of Hori's Method (Da Silva Fernandes and Sessin,1989)

Consider the change of variables $(\lambda, x) \to (\eta, \xi)$ defining a new dynamical system governed by the Hamiltonian H^*,

$$H^* = \eta \cdot F^*(\xi), \tag{5}$$

in such way that this new dynamical system has some advantanges for the solution.

According to Hori's method (1966) a canonical transformation $(\lambda, x) \to (\eta, \xi)$ is defined by a generating function S, developable in powers of the small parameter ϵ. The new Hamiltonian function H^* and the generating function S are obtained from a set of perturbation equations, n-th order equation of which is given by

$$\{H_0, S_m\} + \Theta_m = H_m^*, \tag{6}$$

where Θ_m is known from the preceding orders and the curly brackets denotes the Poisson brackets.

Introducing the auxiliary parameter τ through the auxiliary system defined by the undisturbed Hamiltonian H_0, general solution of which is given by Equations (3) and (4) with t replaced by τ, Equation (6) reduces to

$$\frac{d}{d\tau} S_m = \Theta_m - H_m^*. \tag{7}$$

Hori applies the classic principle of averaging for solving this equation, determining the generating function and the new Hamiltonian.

According to Proposition 1, the general solution of Hori auxiliary system defines a Mathieu transformation, which involves explicitly the auxiliary parameter τ, between the intermediary variables (η, ξ) and the arbitrary constants of integration (b, c).Consequently, two canonical transformation - Lie and Mathieu transformations - are created by the generalized canonical version of Hori's method : $(\lambda, x) \to (\eta, \xi) \to (b, c)$.

Since the Poisson brackets are invariant under a canonical transformation, the algorithm of Hori's method for generalized canonical system can be directly applied to new set (b,c) of canonical variables.This procedure simplifies the algorithm, because it is no longer necessary to invert the general solution of Hori's auxiliary system to express the new Hamiltonian function H and the generating function S as functions of the intermediary variables (η, ξ).

The equations of motion for the new variables and the time dependence of the auxiliary parameter τ can be derived using the Proposition 2 or through the Lagrange Variational Equations (Sessin, 1983) and are given by

$$\frac{dc}{dt} = \frac{\partial R^*}{\partial b}, \frac{db}{dt} = -\frac{\partial R^*}{\partial c}, \tag{8}$$

and

$$\frac{d\tau}{dt} = 1 + \left(\frac{\partial \tau}{\partial \xi}\right)^T \Delta_\phi \frac{\partial R^*}{\partial b}, \tag{9}$$

where $R^* = H^* - H_0^*$ is the "new" Hamiltonian function for the "new" set of canonical variables (b,c).

3. A General Principle for Determining the new Hamiltonian and the Generating Function

Consider the special case for which the new undisturbed Hamiltonian function H_0^* is given by

$$H_0^* = \omega(c_1)\, b_n. \tag{10}$$

In this case,the n-th order equation of algorithm of Hori's method, Equation (7), reduces to

$$\frac{d}{d\tau} S_m = \sum_{j=1}^{n} \Theta_{m,j} b_j - \left(\frac{\partial \omega}{\partial c_1}\right) \tau b_n \Theta_{m,1} - H_m^*, \tag{11}$$

A general principle,defined below by Equations (12) and (13), is applied for determining the generating function S and the new Hamiltonian R^*,

$$R^* = \sum_{i=1} \epsilon^i \left[\sum_{j=1}^{n} R_{i,j}^* b_j - \left(\frac{\partial \omega}{\partial c_1}\right) \tau b_n R_{i,1}^*\right], \tag{12}$$

$$S = \sum_{i=1} \epsilon^i \left[\sum_{j=1}^{n} S_{i,j} b_j - \left(\frac{\partial \omega}{\partial c_1}\right) \tau b_n S_{i,1}\right], \tag{13}$$

where:

$$R_{i,j}^* = < \Theta_{i,j} >_\tau, j = 1, ..., n, \tag{14}$$

$$S_{i,k} = \int \left[\Theta_{i,k} - < \Theta_{i,k} >_\tau\right] d\tau, k = 1, ..., n-1, \tag{15}$$

$$S_{i,n} = \int \left[\Theta_{i,n} - < \Theta_{i,n} >_\tau\right] d\tau + \left(\frac{\partial \omega}{\partial c_1}\right) \int S_{i,1} d\tau. \tag{16}$$

On the other hand , applying the classic averaging principle , one gets

$$\bar{R}^* = \sum_{i=1} \epsilon^i \left[\sum_{j=1}^{n} \bar{R}_{i,j}^* \bar{b}_j - \left(\frac{\partial \omega}{\partial \bar{c}_1}\right) \tau \bar{b}_n \left(\bar{R}_{i,1}^* + \frac{d}{d\tau} \bar{S}_{i,1}\right)\right], \tag{17}$$

$$\bar{S} = \sum_{i=1} \epsilon^i \left[\sum_{j=1}^{n} \bar{S}_{i,j} \bar{b}_j - \left(\frac{\partial \omega}{\partial \bar{c}_1}\right) \bar{b}_n \int \bar{S}_{i,1} d\tau\right], \tag{18}$$

where the overbar denotes a new set of arbitrary constants of integration to the general solution of Hori's auxiliary system (new set of canonical variables). The following results can be proved:

i - Only periodic terms are generated by the function S in the formal series for the coordinates . The validity of these formal series is then assured;

ii - The dynamical system governed by the Hamiltonian R^* is more tractable than the one governed by the Hamiltonian $\bar{R}^*$;

iii - an infinitesimal canonical transformation is defined between the two sets $(b_1, ..., b_n; c_1, ..., c_n)$ and $(\bar{b}_1, ..., \bar{b}_n; \bar{c}_1, ..., \bar{c}_n)$ of canonical variables.

4. The Main Problem of Satellite Theory

Consider the main problem of satellite theory (Brouwer,1959) defined in terms of the classic keplerian elements - a,e,I,Ω,ω,M.In order to solve this problem by means of the generalized canonical version of Hori's method, the auxiliary momenta $\lambda_a, \ldots, \lambda_M$ are introduced and the Hamiltonian function H is formed using the right hand side of Lagrange's equations. The undisturbed Hamiltonian $H_0 = n\lambda_M$,where n is the mean motion ,is similar to the undisturbed Hamiltonian defined in Section 3. Consequently, applying the general principle discussed in Section 3, one gets the new Hamiltonian function R^* and the generating function S (in closed form), expressed in terms of the new set of canonical variables defined by the arbitrary constants of integration of the general solution of the Hori auxiliary system. The generating function S contains mixed secular terms which disappear when the formal series for the original orbital elements is calculated. The formal solution obtained by this version of Hori's method is in agreement with the classic solution obtained by Brouwer (1959) using von Zeipel method.

Finally, it is noteworthy to mention that the generalized canonical version of Hori's method makes possible to include dissipative effects such as due the atmospheric drag.

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