Glasgow Math. J. **44** (2002) 483–495. © 2002 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S00170895020210237 Printed in the United Kingdom

COMPARISON RESULTS FOR SOLUTIONS OF REACTION DIFFUSION PROBLEMS[1]

GÉRARD A. PHILIPPIN
Dépt de Mathématiques et de Statistique, Université Laval, Québec, Canada, G1K 7P4
e-mail: gphilip@mat.ulaval.ca

and STELLA VERNIER PIRO
Dipartimento di Matematica, Universitá di Cagliari, viale Merello 92, 09123 Cagliari, Italia
e-mail: svernier@unica.it

(Received 20 February, 2001; accepted 3 March, 2001)

Abstract. In this paper we construct upper bounds for the solutions $u(\mathbf{x}, t)$ and its gradient $|\nabla u|$ of a class of parabolic initial-boundary value problems in terms of the solution $\psi(\mathbf{x})$ of the S^t-Venant problem. These bounds are sharp in the sense that they coincide with the exact values of u and $|\nabla u|$ for appropriate geometry and appropriate initial conditions.

2000 *Mathematics Subject Classification*. 35B50, 35K55.

1. Introduction and main results. The goal of this paper is to construct sharp upper bounds for the solution $u(\mathbf{x}, t)$ of the following parabolic initial-boundary value problem

$$\Delta(u^\beta) - u_t = 0, \quad \mathbf{x} := (x_1, \dots, x_N) \in \Omega, \;\; t > 0, \tag{1.1}$$

$$u(\mathbf{x}, t) = 0, \;\; \mathbf{x} \in \partial\Omega, \;\; t > 0, \tag{1.2}$$

$$u(\mathbf{x}, 0) = g(\mathbf{x}) > 0, \;\; \mathbf{x} \in \Omega. \tag{1.3}$$

In (1.1), $\beta = \text{const.} \geq 1$ and Ω is a bounded domain in R^N, $N \geq 2$, with smooth boundary $\partial\Omega$. In (1.3), $g(\mathbf{x})$ is a given nonnegative C^1-function with $g(\mathbf{x}) = 0$, $\mathbf{x} \in \partial\Omega$. In the linear case ($\beta = 1$) $u(\mathbf{x}, t)$ may be interpreted as the temperature of a homogeneous body Ω at time t with initial temperature $g(\mathbf{x})$ and with zero temperature on the lateral surface. If $\beta > 1$, problem (1.1), (1.2), (1.3) is a model in reaction diffusion theory. Throughout the paper we assume that (1.1), (1.2), (1.3) has a classical solution. In the linear case L. E. Payne drew our attention to the following result valid for a convex domain Ω

$$u(\mathbf{x}, t) \leq k \cos\left(\frac{\pi}{2}\sqrt{1 - \frac{\psi(\mathbf{x})}{\psi_{\max}}}\right) \exp\left(-\frac{\pi^2}{4\psi_{\max}} t\right), \;\; \mathbf{x} \in \Omega, \;\; t > 0, \tag{1.4}$$

[1]Research supported by INDAM and MURST.

$$|\nabla u(\mathbf{x}, t)| \leq k \frac{\pi}{4\psi_{\max}} \exp\left(-\frac{\pi^2}{4\psi_{\max}} t\right) |\nabla \psi(\mathbf{x})|, \ \mathbf{x} \in \partial\Omega, \ t > 0. \tag{1.5}$$

In (1.4), (1.5), $\psi(\mathbf{x})$ is the solution of the S^t-Venant problem

$$\Delta\psi = -2, \ \mathbf{x} \in \Omega, \tag{1.6}$$

$$\psi = 0, \ \mathbf{x} \in \partial\Omega. \tag{1.7}$$

Moreover $\psi_{\max} := \max_\Omega \psi(\mathbf{x})$, and k is a positive constant to be chosen such that (1.4) holds initially, i.e. such that

$$g(\mathbf{x}) \leq k \ \cos\left(\frac{\pi}{2}\sqrt{1 - \frac{\psi(\mathbf{x})}{\psi_{\max}}}\right), \ \mathbf{x} \in \Omega. \tag{1.8}$$

The upper bounds (1.4), (1.5) are sharp in the sense that we have equality when Ω degenerates to an infinite slab, i.e. when Ω is located between two parallel hyperplanes, and if $g(\mathbf{x}) = k \cos\left(\frac{\pi}{2}\sqrt{1 - \frac{\psi(\mathbf{x})}{\psi_{\max}}}\right)$.

In the second section of this paper we construct other upper bounds for $u(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$ and for $|\nabla u|$, $\mathbf{x} \in \partial\Omega$, valid again in the linear case $\beta = 1$, but without the assumption that Ω is convex. More precisely we have the following result:

THEOREM 1. *The solution $u(\boldsymbol{x}, t)$ of (1.1), (1.2), (1.3) with $\beta = 1$ and its gradient ∇u satisfy the following inequalities*

$$u(\mathbf{x}, t) \leq k \ w(\mathbf{x}) \exp\left\{-\frac{4j^2}{N^2\sigma_0^2} t\right\}, \ \mathbf{x} \in \Omega, \ t > 0, \tag{1.9}$$

$$|\nabla u(\mathbf{x}, t)| \leq k |\nabla w(\mathbf{x})| \exp\left\{-\frac{4j^2}{N^2\sigma_0^2} t\right\}, \ \mathbf{x} \in \partial\Omega, \ t > 0, \tag{1.10}$$

with

$$w(\mathbf{x}) := \left(1 - \frac{4\psi(\mathbf{x})}{N\sigma_0^2}\right)^{\frac{2-N}{4}} J_{\frac{N-2}{2}}\left(j\sqrt{1 - \frac{4\psi(\mathbf{x})}{N\sigma_0^2}}\right). \tag{1.11}$$

In (1.11) $J_\nu(\mathbf{x})$ stands for the Bessel function of order ν and $j(> 0)$ is its first zero: $J_\nu(j) = 0$, $\psi(\mathbf{x})$ is the stress function defined by (1.6), (1.7), and σ_0 is the maximal stress defined as

$$\sigma_0 := \max_\Omega |\nabla\psi|. \tag{1.12}$$

In (1.9), (1.10), k is a positive constant to be selected such that (1.9) holds initially, i.e., such that

$$g(\mathbf{x}) \leq kw(\mathbf{x}), \ \mathbf{x} \in \Omega. \tag{1.13}$$

https://doi.org/10.1017/S0017089502030136 Published online by Cambridge University Press

We note that the upper bound for $u(\mathbf{x}, t)$ in (1.9) is constructed in such a way that it coincides to the exact value of $u(\mathbf{x}, t)$ when Ω is an N-ball of radius R with the initial data

$$g(\mathbf{x}) := \left(\frac{|\mathbf{x}|}{R}\right)^{\frac{2-N}{2}} J_{\frac{N-2}{2}}\left(j\frac{|\mathbf{x}|}{R}\right), \ \ \mathbf{x} \in \Omega. \tag{1.14}$$

In this case we have indeed

$$u(\mathbf{x}, t) = \left(\frac{|\mathbf{x}|}{R}\right)^{\frac{2-N}{2}} J_{\frac{N-2}{2}}\left(j\frac{|\mathbf{x}|}{R}\right)\exp\left\{-\frac{j^2}{R^2}t\right\}, \ \ \mathbf{x} \in \Omega, \ \ t > 0. \tag{1.15}$$

Moreover we may compute $|\mathbf{x}|/R$ and R in terms of the stress function ψ and σ_0. We have

$$\psi(\mathbf{x}) = \frac{1}{N}(R^2 - |\mathbf{x}|^2), \ \ \mathbf{x} \in \Omega, \tag{1.16}$$

$$|\nabla\psi| = \frac{2}{N}|\mathbf{x}|, \tag{1.17}$$

from which we obtain

$$\sigma_0 := \max_{\Omega} |\nabla\psi| = \frac{2R}{N}, \tag{1.18}$$

$$\frac{|\mathbf{x}|}{R} = \sqrt{1 - \frac{N\psi}{R^2}} = \sqrt{1 - \frac{4\psi}{N\sigma_0^2}}. \tag{1.19}$$

We are then lead to

$$u(\mathbf{x}, t) = \left(1 - \frac{4\psi}{N\sigma_0^2}\right)^{\frac{2-N}{4}} J_{\frac{N-2}{2}}\left(j\sqrt{1 - \frac{4\psi}{N\sigma_0^2}}\right) \exp\left\{-\frac{4j^2}{N^2\sigma_0^2}t\right\}. \tag{1.20}$$

This shows that both inequalities (1.9), (1.10) are sharp in the sense that we have equalities if Ω is an N-ball and if the initial data satisfy (1.13) with equality sign. The remainder of Section 2 deals with the case where (1.1) is replaced by the equation

$$\Delta u - u_t = -f(u), \quad \mathbf{x} \in \Omega, \ \ t > 0, \tag{1.21}$$

under some data restrictions. Section 3 addresses the following conjecture:

Conjecture. *Let $u(\mathbf{x}, t)$ be the solution of (1.1), (1.2), (1.3) in a convex domain Ω with $\beta > 1$. We then have*

$$u(\mathbf{x}, t) \leq y(\sqrt{\psi_{\max} - \psi(\mathbf{x})}) \left[k - (1-\beta)\lambda^2 t\right]^{\frac{1}{1-\beta}}, \quad \mathbf{x} \in \Omega, \ \ t > 0, \tag{1.22}$$

https://doi.org/10.1017/S0017089502030136 Published online by Cambridge University Press

$$|\nabla u(\mathbf{x},t)| \leq |\nabla y| \left[k-(1-\beta)\lambda^2 t\right]^{\frac{1}{1-\beta}}, \quad \mathbf{x}\in\partial\Omega,\ t>0. \tag{1.23}$$

In (1.22), (1.23), $y(x)$ is the positive solution of the one-dimensional auxiliary problem

$$(y^\beta)_{xx}+\lambda^2 y=0, \quad x\in(0,\sqrt{\psi_{\max}}), \tag{1.24}$$

$$y_x(0)=0, \quad y(0)=1, \tag{1.25}$$

where the parameter λ is selected such that

$$y(\sqrt{\psi_{\max}})=0.$$

Moreover k is a positive constant to be chosen such that (1.22) holds initially, i.e. such that

$$g(\mathbf{x}) \leq k^{\frac{1}{1-\beta}} \quad y(\sqrt{\psi_{\max}-\psi(\mathbf{x})}), \quad \mathbf{x}\in\Omega. \tag{1.26}$$

This conjecture is supported by the fact that we have equality in (1.22) in the one-dimensional case $N=1$, if the initial data $g(\mathbf{x})$ satisfies (1.26) with equality sign. The proof will be established in the particular case $\beta=2$. The upper bounds for $u(\mathbf{x},t)$ given by (1.9) and (1.22) are constructed in analogy to earlier results established by L. E. Payne, G. A. Philippin, and J. R. L. Webb in [**3**, **4**, **6**] for solutions of elliptic boundary value problems. The proof of (1.9) (and hopefully of (1.22)) follows the same pattern as in [**4**]. We first show that the comparison function

$$\Phi(\mathbf{x},t) := w(\mathbf{x}) \exp\left\{-\frac{4j^2}{N^2\sigma_0^2}t\right\} \tag{1.27}$$

satisfies the parabolic differential inequality

$$\Delta\Phi-\Phi_t \leq 0, \quad \mathbf{x}\in\Omega, \quad t>0. \tag{1.28}$$

The inequality (1.9) will then follow by a standard comparison theorem, cf. e.g. [**7**]. For the proof of (1.28) we need the following lemma established by Weinberger in [**8**]:

Lemma 1. *The quantity*

$$\chi(\mathbf{x}) := |\nabla\psi|^2+\frac{4}{N}\psi, \ \ \mathbf{x}\in\Omega, \tag{1.29}$$

where $\psi(\mathbf{x})$ is the stress function defined in (1.6), (1.7) takes its maximum value on the boundary $\partial\Omega$ of Ω, i.e. we have

$$\frac{4}{N}\psi(\mathbf{x}) \leq \sigma_0^2-|\nabla\psi|^2, \ \ \mathbf{x}\in\Omega, \tag{1.30}$$

with equality if and only if Ω is an N-ball.

For the proof of (1.22), we hope to make use of the following lemma established by Payne in [**2**]:

LEMMA 2. *If Ω is convex, the quantity*

$$\theta := |\nabla\psi|^2 + 4\psi, \ \ \mathbf{x} \in \Omega, \tag{1.31}$$

takes its maximum value at the critical point of ψ, i.e., we have

$$|\nabla\psi|^2 \leq 4(\psi_{\max} - \psi), \ \ \mathbf{x} \in \Omega, \tag{1.32}$$

with equality if and only if Ω is an infinite slab.

2. The proof of Theorem 1. We have to check the differential inequality

$$\Delta\Phi - \Phi_t \leq 0, \ \ \mathbf{x} \in \Omega, \ \ t > 0, \tag{2.1}$$

with

$$\Phi(\mathbf{x}, t) := w(\mathbf{x})\exp\left\{-\frac{4j^2}{N^2\sigma_0^2}t\right\}, \tag{2.2}$$

$$w(\mathbf{x}) := [v(\mathbf{x})]^{\frac{2-N}{2}} J_{\frac{N-2}{2}}(jv(\mathbf{x})), \tag{2.3}$$

$$v(\mathbf{x}) := \sqrt{1 - \frac{4\psi(\mathbf{x})}{N\sigma_0^2}} \quad , \tag{2.4}$$

that will be satisfied if the following inequality holds

$$\Delta w + \frac{4j^2}{N^2\sigma_0^2} w \leq 0, \quad \ \mathbf{x} \in \Omega. \tag{2.5}$$

To check (2.5) we shall make use of the following well known identities for $J_\nu(x)$:

$$xJ'_\nu(x) = \nu J_\nu(x) - xJ_{\nu+1}(x), \tag{2.6}$$

$$xJ_{\nu+1}(x) = 2\nu J_\nu(x) - xJ_{\nu-1}(x). \tag{2.7}$$

Differentiating $w(\mathbf{x})$ defined in (2.3), we obtain in view of (2.6)

$$w,_k = \left\{\frac{2-N}{2} v^{-\frac{N}{2}} J_{\frac{N-2}{2}} + jv^{\frac{2-N}{2}} J'_{\frac{N-2}{2}}\right\} v,_k = -jv^{\frac{2-N}{2}} J_{\frac{N}{2}} v,_k \,. \tag{2.8}$$

In (2.8) and in the remainder of this computation we omit the argument of the Bessel functions which is always $jv(\mathbf{x})$. Differentiating again and making use of (2.6), (2.7), we obtain

$$\begin{aligned}\Delta w &= \frac{N-2}{2} v^{-\frac{N}{2}} j J_{\frac{N}{2}} |\nabla v|^2 - j^2 v^{\frac{2-N}{2}} J'_{\frac{N}{2}} |\nabla v|^2 - j v^{\frac{2-N}{2}} J_{\frac{N}{2}} \Delta v \\ &= -j v^{-\frac{N}{2}} J_{\frac{N}{2}} \{ |\nabla v|^2 + v \Delta v \} + j^2 v^{\frac{2-N}{2}} |\nabla v|^2 J_{\frac{N+2}{2}} \\ &= j v^{-\frac{N}{2}} J_{\frac{N}{2}} \{ (N-1)|\nabla v|^2 - v\Delta v \} - j^2 v^{\frac{2-N}{2}} |\nabla v|^2 J_{\frac{N-2}{2}}. \end{aligned} \tag{2.9}$$

Next we compute from (2.4)

$$v_{,k} = -\frac{2\psi_{,k}}{N\sigma_0^2 v} \quad , \tag{2.10}$$

$$|\nabla v|^2 = \frac{4|\nabla\psi|^2}{N^2\sigma_0^4 v^2} \quad , \tag{2.11}$$

$$\Delta v = \frac{4}{N\sigma_0^2 v^3}\left[v^2 - \frac{|\nabla\psi|^2}{N\sigma_0^2} \right]. \tag{2.12}$$

Inserting (2.4), (2.11), (2.12) into (2.9) and making again use of (2.7), we obtain

$$\begin{aligned}\Delta w + \frac{4j^2}{N^2\sigma_0^2} w &= \frac{4jv^{-\frac{N}{2}}}{N^2\sigma_0^2}\left\{ NJ_{\frac{N}{2}} - jvJ_{\frac{N-2}{2}} \right\}\left\{ \frac{|\nabla\psi|^2}{v^2\sigma_0^2} - 1 \right\} \\ &= \frac{4j^2 v^{-\frac{N+2}{2}}}{N^3\sigma_0^3} J_{\frac{N+2}{2}} \{ 4\psi - N[\sigma_0^2 - |\nabla\psi|^2] \} \le 0, \;\; x \in \Omega, \end{aligned} \tag{2.13}$$

where the last inequality follows from Lemma 1. This achieves the proof of (1.9). The proof of (1.10) follows from (1.9) since we have equality in (1.9) for $\mathbf{x} \in \partial\Omega$.

It is worthwhile to mention that the inequalities (1.4), (1.5), (1.9) and (1.10) are easily modified when $u(\mathbf{x}, t)$ solves (1.1)–(1.3) with (1.1) replaced by

$$\Delta u - u_t = -f(u), \quad \mathbf{x} \in \Omega, \;\; t > 0, \tag{2.14}$$

where $f(s)$ is a differentiable function assumed to satisfy the conditions

$$f(0) = 0, \tag{2.15}$$

$$sf'(s) \ge f(s) \ge 0, s > 0. \tag{2.16}$$

Clearly (2.16) implies that the quantity $f(s)/s$ is a nondecreasing function of s. We want of course the solution $u(\mathbf{x}, t)$ of (2.14), (1.2), (1.3) to exist for all time. This will be the case if some further restrictions on f and g are imposed. Such restrictions are stated in either one of the following two Lemmas derived in [**5**].

LEMMA 3. *Let $\phi_1(\mathbf{x})$ and λ_1 be the first eigenfunction and the first eigenvalue of the clamped vibrating membrane in Ω:*

$$\Delta\phi_1 + \lambda_1\phi_1 = 0, \;\; \mathbf{x} \in \Omega, \;\; \phi_1 > 0, \;\; \mathbf{x} \in \Omega, \;\; \phi_1 = 0, \;\; \mathbf{x} \in \partial\Omega, \tag{2.17}$$

where ϕ_1 is normalized by the condition $\max_\Omega \phi_1(\mathbf{x}) = 1$. *Assume that the initial data $g(\mathbf{x})$ in (1.3) is sufficiently small in the following sense*

$$\frac{f(\Gamma_1)}{\Gamma_1} < \lambda_1, \tag{2.18}$$

with $\Gamma_1 := \max_\Omega \frac{g(\mathbf{x})}{\phi_1(\mathbf{x})}$. *We then conclude that* $u(\mathbf{x}, t)$ *solving (2.14), (1.2), (1.3) exists for all time. Moreover we have the following inequality*

$$\max_\Omega \frac{f(u(\mathbf{x}, t))}{u(\mathbf{x}, t)} \leq \frac{f(\Gamma_1)}{\Gamma_1}, \quad 0 < t < \infty. \tag{2.19}$$

Lemma 4. *Let* Ω *be convex and let* d *be the inradius of* Ω. *Suppose that the initial data* $g(\mathbf{x})$ *in (1.3) is sufficiently small in the following sense*

$$\frac{f(\Gamma_2)}{\Gamma_2} < \frac{\pi^2}{4d^2}, \tag{2.20}$$

with $\Gamma_2 := \max_\Omega \left\{ g^2 + \frac{4d^2}{\pi^2} |\nabla g|^2 \right\}^{1/2}$. *Then we can again conclude that* $u(\mathbf{x}, t)$ *exists for all time. Moreover we have the following inequality*

$$\max_\Omega \frac{f(u(\mathbf{x}, t))}{u(\mathbf{x}, t)} \leq \frac{f(\Gamma_2)}{\Gamma_2}, \quad 0 < t < \infty. \tag{2.21}$$

Lemma 3 or 4 may be used to derive the following inequality for $u(\mathbf{x}, t)$

$$\Delta u - u_t = -f(u) = -\frac{f(u)}{u} u \geq -\Lambda u, \tag{2.22}$$

with

$$\Lambda := \frac{f(\Gamma_1)}{\Gamma_1} \text{ or } \frac{f(\Gamma_2)}{\Gamma_2}. \tag{2.23}$$

Let now $U(\mathbf{x}, t)$ be the solution of

$$\Delta U - U_t + \Lambda U = 0, \quad \mathbf{x} \in \Omega, \; t > 0, \tag{2.24}$$

$$U(\mathbf{x}, t) = 0, \; \mathbf{x} \in \partial\Omega, \; t > 0, \tag{2.25}$$

$$U(\mathbf{x}, 0) = g(\mathbf{x}) > 0, \; \mathbf{x} \in \Omega. \tag{2.26}$$

Clearly we have $u(\mathbf{x}, t) \leq U(\mathbf{x}, t), \; \mathbf{x} \in \Omega, \; t > 0$. Moreover the techniques already used to obtain (1.4), (1.9) may be used again to derive upper bounds for $U(\mathbf{x}, t)$. This leads to the following results:

Theorem 2. *Let* $u(\mathbf{x}, t)$ *be the solution of (2.14), (1.2), (1.3) where* f *satisfies the conditions (2.15) and (2.16), and* g *satisfies the assumptions in Lemma 3 or 4. We then conclude that the inequalities (1.4), (1.5) remain valid if the exponential factor* $\exp\left\{-\frac{\pi^2}{4\psi_{\max}} t\right\}$ *is replaced by* $\exp\left\{\left(\Lambda - \frac{\pi^2}{4\psi_{\max}}\right)t\right\}$, *where* Λ *is given by (2.23). Moreover*

https://doi.org/10.1017/S0017089502030136 Published online by Cambridge University Press

the inequalities (1.9), (1.10) remain valid if the exponential factor $\exp\left\{-\frac{4j^2}{N^2\sigma_0^2}t\right\}$ *is replaced by* $\exp\left\{\left(\Lambda-\frac{4j^2}{N^2\sigma_0^2}\right)t\right\}$, *where* Λ *is given by (2.23).*

3. The conjecture. This section addresses the conjectured inequalities (1.22) and (1.23) with $\beta \geq 1$. These inequalities will be fully established for $\beta = 1$ and for $\beta = 2$ only. The upper bound in (1.22) is constructed in such a way that it coincides to the exact solution $\eta(x, t)$ of the one-dimensional problem

$$(\eta^\beta)_{xx} - \eta_t = 0, \;\; x \in (0, \sqrt{\psi_{\max}}\,), \;\; t > 0, \tag{3.1}$$

$$\eta_x(0, t) \;=\; \eta(\sqrt{\psi_{\max}}, t) \;=\; 0, \;\; t > 0, \tag{3.2}$$

$$\eta(x, 0) = \gamma(x) > 0, \;\; x \in (0, \sqrt{\psi_{\max}}\,), \tag{3.3}$$

with appropriate initial data $\gamma(x)$. The auxiliary problem (3.1), (3.2), (3.3) may be solved by separating the variables. To do this we write

$$\eta(x, t) = y(x)\;\tau(t). \tag{3.4}$$

The auxiliary functions $y(x)$ and $\tau(t)$ then satisfy

$$\frac{(y^\beta)''}{y} = \frac{\dot\tau}{\tau^\beta} = -\lambda^2 = \text{const.}, \tag{3.5}$$

i.e., we have

$$(y^\beta)'' + \lambda^2 y = 0, \;\; x \in (0, \sqrt{\psi_{\max}}), \tag{3.6}$$

with

$$y'(0) = 0. \tag{3.7}$$

For convenience $y(x)$ will be normalized such that

$$y(0) = 1. \tag{3.8}$$

The parameter λ is then selected such that

$$y(\sqrt{\psi_{\max}}) = 0. \tag{3.9}$$

Moreover $\tau(t)$ satisfies the differential equation

$$\dot\tau + \lambda^2\tau^\beta = 0, \;\; t > 0. \tag{3.10}$$

Let now

$$\psi := \psi_{\max} - x^2, \;\; x \in (0, \sqrt{\psi_{\max}}), \tag{3.11}$$

https://doi.org/10.1017/S0017089502030136 Published online by Cambridge University Press

be the stress function of the one-dimensional S^t-Venant problem. Solving (3.11) for x, we obtain

$$x = \sqrt{\psi_{\max} - \psi(x)}. \tag{3.12}$$

We then construct a comparison function $z(\mathbf{x}, t)$ as follows:

$$z(\mathbf{x}, t) := y(\sqrt{\psi_{\max} - \psi(\mathbf{x})})\tau(t), \ \ \mathbf{x} \in \Omega, \ \ t > 0, \tag{3.13}$$

where $\psi(\mathbf{x})$ in (3.13) is the stress function of Ω defined in (1.6), (1.7). We want to show that $z(\mathbf{x}, t)$ satisfies the parabolic inequality

$$\Delta(z^\beta) - z_t \leq 0, \ \ \mathbf{x} \in \Omega, \ \ t > 0. \tag{3.14}$$

To this end we define

$$\sigma(\mathbf{x}) := \sqrt{\psi_{\max} - \psi(\mathbf{x})}, \ \ \mathbf{x} \in \Omega, \tag{3.15}$$

and we compute

$$\Delta(z^\beta) - z_t = \tau^\beta\{\Delta(y^\beta(\sigma)) + \lambda^2 y(\sigma)\}, \tag{3.16}$$

$$\Delta(y^\beta(\sigma(\mathbf{x}))) = (y^\beta)''|\nabla\sigma|^2 + (y^\beta)'\Delta\sigma = -\lambda^2 y|\nabla\sigma|^2 + (y^\beta)'\Delta\sigma, \tag{3.17}$$

with

$$\sigma,_k = -\frac{\psi,_k}{2\sigma}, \tag{3.18}$$

$$|\nabla\sigma|^2 = \frac{|\nabla\psi|^2}{4\sigma^2}, \tag{3.19}$$

$$\Delta\sigma = -\frac{1}{2}\left(\frac{\Delta\psi}{\sigma} - \frac{\psi,_k\,\sigma,_k}{\sigma^2}\right) = \frac{1}{\sigma}\left(1 - \frac{|\nabla\psi|^2}{4\sigma^2}\right). \tag{3.20}$$

We then obtain

$$\Delta(y^\beta(\sigma)) + \lambda^2 y(\sigma) = \frac{1}{4\sigma^2}[4(\psi_{\max} - \psi(\mathbf{x})) - |\nabla\psi|^2]\{\lambda^2 y(\sigma) + \frac{1}{\sigma}(y^\beta(\sigma))'\}. \tag{3.21}$$

Since we have by Lemma 2

$$4[\psi_{\max} - \psi(\mathbf{x})] - |\nabla\psi|^2 \leq 0, \ \ \mathbf{x} \in \Omega, \tag{3.22}$$

we conclude from (3.16), (3.21) that (3.14) will be satisfied if we have

$$\lambda^2 y(\sigma) + \frac{1}{\sigma}(y^\beta(\sigma))' \geq 0, \tag{3.23}$$

https://doi.org/10.1017/S0017089502030136 Published online by Cambridge University Press

or equivalently if the inequality

$$\beta(y(x))^{\beta-2}y'(x) + \lambda^2 x \leq 0, \;\; x \in (0, \sqrt{\psi_{\max}}) \tag{3.24}$$

is satisfied. The success of our method depends therefore on the possibility to check (3.24). This can easily be done in the linear case since we have

$$y(x) = \cos \lambda x, \tag{3.25}$$

$$\tau(t) = e^{-\lambda^2 t}, \tag{3.26}$$

in the case $\beta = 1$ with $\lambda = \frac{\pi}{2\sqrt{\psi_{\max}}}$, so that (3.24) takes the form

$$\frac{y'}{y} + \lambda^2 x = -\lambda \tan(\lambda x) + \lambda^2 x \leq 0, \tag{3.27}$$

which is clearly satisfied. This establishes Payne's result (1.4), (1.5).

The situation is more complicated when $\beta > 1$ because $y(x)$ cannot be expressed in terms of elementary functions. For this reason we represent $y(x)$ in a Taylor series of the form

$$y(x) = 1 + \sum_{k=1}^{\infty} a_{2k} x^{2k}. \tag{3.28}$$

Clearly this series contains only even powers of x. Let us consider the case $\beta = 2$ which is simple. In this case we have

$$y^2(x) = 1 + \sum_{k=1}^{\infty} c_{2k} x^{2k}, \tag{3.29}$$

with

$$c_{2k} = \sum_{j=0}^{k} a_{2(k-j)} a_{2j}. \tag{3.30}$$

Inserting (3.28), (3.29) into (3.6), we obtain

$$2c_2 + \lambda^2 + \sum_{k=2}^{\infty} [2k(2k+1)c_{2k} + \lambda^2 a_{2k-2}] x^{2k-2} = 0, \tag{3.31}$$

i.e. we have

$$c_{2k} = -\frac{\lambda^2}{2k(2k-1)} a_{2k-2} = \sum_{j=0}^{k} a_{2(k-j)} a_{2j}, \;\; k = 1, 2, 3, \cdots \tag{3.32}$$

The values of a_{2k} may be recursively computed from (3.32). We obtain

$$a_2 = -\frac{\lambda^2}{4}, \quad a_4 = -\frac{\lambda^4}{48}, \quad a_6 = -\frac{7}{30 \cdot 48}\lambda^6, \quad a_8 = -\frac{\lambda^8}{15 \cdot 48}, \ldots \tag{3.33}$$

i.e. we have

$$y(x) = 1 - \frac{(\lambda x)^2}{4} - \frac{(\lambda x)^4}{48} - \frac{7}{30 \cdot 48}(\lambda x)^6 - \frac{1}{15 \cdot 48}(\lambda x)^8 - \dots \tag{3.34}$$

where λ is such that $y(\sqrt{\psi_{\max}}) = 0$. Now we want to check inequality (3.24) with $\beta = 2$ that takes the form

$$2y' + \lambda^2 x = \sum_{k=2}^{\infty} 2k a_{2k} x^{2k-1} \leq 0, \quad x \in (0, \sqrt{\psi_{\max}}). \tag{3.35}$$

Clearly (3.35) will be satisfied if we can show that $a_{2k} \leq 0$, $\forall k = 2, 3, 4, \dots$ This step will be established by induction. Let us assume that $a_2, a_4, \dots, a_{2(k-1)}$ are all negative. Then from (3.32) we obtain

$$-\frac{\lambda^2}{2k(2k-1)} a_{2k-2} = \sum_{j=0}^{k} a_{2(k-j)} a_{2j} > 2a_{2k} + 2a_2 a_{2k-2}, \tag{3.36}$$

i.e.

$$a_{2k} < -a_{2k-2}\left[2a_2 + \frac{\lambda^2}{2k(2k-1)}\right] = \frac{\lambda^2}{2} a_{2k-2}\left[1 - \frac{1}{k(2k-1)}\right] < 0, \tag{3.37}$$

which completes the proof. To conclude this example we compute $\tau(t)$ from (3.10)

$$\tau(t) = \frac{1}{\lambda^2 t + k}, \quad t > 0, \tag{3.38}$$

and we select the constant $k > 0$ such that

$$kg(\mathbf{x}) \leq y(\sqrt{\psi_{\max} - \psi(\mathbf{x})}), \quad x \in \Omega. \tag{3.39}$$

It then follows from a standard comparison theorem [7] that the solution $u(\mathbf{x}, t)$ of (1.1), (1.2), (1.3) satisfies the inequality

$$u(\mathbf{x}, t) \leq \frac{y(\sqrt{\psi_{\max} - \psi(\mathbf{x})})}{\lambda^2 t + k}, \quad \mathbf{x} \in \Omega, \quad t > 0, \tag{3.40}$$

when $\beta = 2$. Finally we note that any truncation of the series (3.34) yields an upper bound in (3.40). To conclude this paper we consider the general case $\beta > 1$. Clearly we have again (3.31) where a_{2k} and c_{2k} are the Taylor coefficients of $y(x)$ and of $y^{\beta}(x)$:

$$y(x) = 1 + \sum_{k=1}^{\infty} a_{2k} x^{2k}, \tag{3.41}$$

$$y^{\beta}(x) = 1 + \sum_{k=1}^{\infty} c_{2k} x^{2k}, \tag{3.42}$$

Moreover, J. P. C. Miller has established in [**1**] that the Taylor coefficients a_{2k} and c_{2k} in (3.41), (3.42) are related as follows

$$c_{2k} = \frac{1}{k}\sum_{j=0}^{k-1}[\beta(k-j)-j]c_{2j}a_{2(k-j)},\ \ k = 1, 2, 3, \ldots \tag{3.43}$$

Combining (3.43) with

$$c_{2k} = -\frac{\lambda^2}{2k(2k-1)}a_{2k-2}, \tag{3.44}$$

and solving for a_{2k}, we obtain

$$a_{2k} = \frac{\lambda^2}{k\beta}\sum_{j=1}^{k}\frac{\beta(k-j)-j}{2j(2j-1)}a_{2j-2}a_{2(k-j)},\ \ k = 1, 2, 3, \ldots, \tag{3.45}$$

from which we compute recursively

$$\begin{aligned}
a_2 &= -\frac{\lambda^2}{2\beta},\\
a_4 &= \frac{\lambda^4}{4!\beta^2}[1-3(\beta-1)],\\
a_6 &= -\frac{\lambda^6}{6!\beta^3}[1-3(\beta-1)+30(\beta-1)^2],\\
a_8 &= \frac{\lambda^8}{8!\beta^4}[1-66(\beta-1)-201(\beta-1)^2-630(\beta-1)^3],
\end{aligned} \tag{3.46}$$

etc.

Now we want to check inequality (3.24) that can be rewritten as

$$\frac{\beta}{\beta-1}(y^{\beta-1})' + \lambda^2 x \leq 0,\ \ x \in (0, \sqrt{\psi_{\max}}). \tag{3.47}$$

To this end we write

$$y^{\beta-1}(x) = 1 + \sum_{k=1}^{\infty} d_{2k}x^{2k}, \tag{3.48}$$

where the coefficients d_{2k} are related to a_{2k} according to Miller's formula

$$d_{2k} = \frac{1}{k}\sum_{j=0}^{k-1}[(\beta-1)(k-j)-j]d_{2j}a_{2(k-j)},\ \ k = 1, 2, 3, \ldots. \tag{3.49}$$

Using (3.49) and the values a_2, a_4, a_6, a_8 already computed we obtain

https://doi.org/10.1017/S0017089502030136 Published online by Cambridge University Press

$$
\begin{aligned}
d_2 &= -\frac{\lambda^2(\beta-1)}{2\beta},\\
d_4 &= -\frac{\lambda^4(\beta-1)}{2\cdot 3!\beta^2},\\
d_6 &= -\frac{2\lambda^6(\beta-1)(\beta+\frac{1}{3})}{5!\beta^3},\\
d_8 &= -\frac{4\lambda^8(\beta-1)(3\beta+1)(15\beta+2)}{8!\beta^4},
\end{aligned}
\tag{3.50}
$$

etc.

The condition (3.47) then takes the form

$$\frac{\beta}{\beta-1}(y^{\beta-1})' + \lambda^2 x = \frac{\beta}{\beta-1}\{4d_4x^3 + 6d_6x^5 + 8d_8x^7 + \cdots\} \leq 0, \tag{3.51}$$

and will be satisfied for istance if $d_4, d_6, d_8, \ldots$ are all nonpositive. This seems to be the case from (3.50), but remains open.

Acknowledgements. We are indebted to L. E. Payne who communicated to us the inequalities (1.4), (1.5).

REFERENCES

1. P. Henrici, *Applied and computational complex analysis*, Vol. I (Wiley Interscience).

2. L. E. Payne, Bounds for the maximum stress in the Saint-Venant torsion problem, *Indian J. Mech. and Math.* special issue (1968), 51–59.

3. L. E. Payne, Bounds for solutions of a class of quasilinear elliptic boundary value problems in terms of the torsion function, *Proc. Roy. Soc., Edinburgh Sect. A* **88** (1981), 251–261.

4. L. E. Payne and G. A. Philippin, Comparison theorems for a class of nonlinear elliptic boundary value problems, *Nonlinear Analysis* **9** (1985), 787–797.

5. L. E. Payne and G. A. Philippin, Decay bounds for solutions of second order parabolic problems and their derivatives, *Math. Models and Methods in Appl. Sci.*, **5**, (1995), 95–110; Corrigendum and addendum, 865–866.

6. L. E. Payne and J. L. R. Webb, Comparison results in second order quasilinear Dirichlet problems, *Proc. Roy. Soc., Edinburgh Sect. A* **118** (1991), 91–103.

7. M. H. Protter and H. F. Weinberger, *Maximum principles in differential equations* (Prentice Hall, Englewood Cliffs, NJ, 1967).

8. H. F. Weinberger, Remark on the preceding paper of Serrin, *Arch. Rat. Mech. Anal.* **43** (1971), 319–320.