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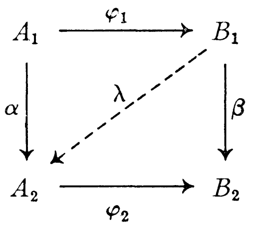
OBSTRUCTIONS TO LIFTINGS IN COMMUTATIVE SQUARES

IRWIN S. PRESSMAN

Dedicated to the memory of Jean Maranda

1. Introduction. A commutative square (1) of morphisms is said to have a *lifting* if there is a morphism $\lambda : B_1 \to A_2$ such that $\lambda\varphi_1 = \alpha$ and $\varphi_2\lambda = \beta$.

(1)



Let us assume that we are working in a fixed abelian category $\mathscr{C}$. Therefore, φ_i will have a kernel "K_i" and a cokernel "C_i" for i = 1, 2. Let $k : K_1 \to K_2$ and $c : C_1 \to C_2$ denote the canonical morphisms induced by α and β.

We shall construct a short exact sequence (s.e.s.)

$$0 \to K_2 \to H \to C_1 \to 0 \tag{2}$$

using the data of (1). We shall prove that (1) has a lifting if and only if $k = 0$, $c = 0$, and (2) represents the zero class in $\mathrm{Ext}^1(C_1, K_2)$. Furthermore, if (1) has one lifting, then the liftings will be in one-to-one correspondence with the elements of the set $|\mathrm{Hom}(C_1, K_2)|$.

The results here should be useful for certain types of problems in algebraic topology. For example, if (1) were a commutative diagram of continuous mappings of topological spaces, then the homology functors H_n would give a sequence of commutative diagrams of abelian groups. To prove the non-existence of a lifting in the category of continuous mappings, it would suffice to show that there can be no lifting for one integer n. Olum [**3**] has looked at this problem for topological spaces from a different viewpoint. The meaning of homology of a square in [**1**; **3**] is quite different from ours.

2. Splittings of short exact sequences.

Definition. A short exact sequence **E** of objects in $\mathscr{C}$

$$\mathbf{E} : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

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is said to *split* if there are morphisms $s: C \to B$ and $t: B \to A$ such that $tf = 1: A \to A$, $gs = 1: C \to C$, and $ft + sg = 1: B \to B$. The pair (t, s) is called a *splitting* of **E**. **E** may have many splittings, as the following lemma suggests.

LEMMA 1. *Let* **E** *be a split s.e.s. The splittings* (t, s) *of* **E** *are in one-to-one correspondence with the set* $|\mathrm{Hom}(C, A)|$.

Proof. Let (t, s) be a splitting of **E**, and let $u: C \to A$ be any morphism of $\mathrm{Hom}(C, A)$. It is easily checked that $(t - ug, s + fu)$ is also a splitting. The converse has a straight-forward proof which is omitted.

Suppose next that there is a commutative diagram of s.e.s.'s

$$(3)\qquad \begin{array}{ccccccccccc} \mathbf{E}^{\#} & 0 & \longrightarrow & A & \xrightarrow{f^{\#}} & B^{\#} & \xrightarrow{g^{\#}} & C^{\#} & \longrightarrow & 0 \\ \downarrow & & & \| 1 & & \downarrow h & & \downarrow k & & \\ \mathbf{E}: & 0 & \longrightarrow & A & \underset{t}{\overset{f}{\rightleftarrows}} & B & \underset{s}{\overset{g}{\rightleftarrows}} & C & \longrightarrow & 0. \end{array}$$

View $\mathbf{E}^{\#}$ as the pullback of **E** along k. If **E** has a splitting (t, s), then $\mathbf{E}^{\#}$ must also split and have a splitting $(t^{\#}, s^{\#})$.

Definition. A splitting $(t^{\#}, s^{\#})$ of $\mathbf{E}^{\#}$ is *compatible* with the splitting (t, s) of **E** if

$$th = t^{\#} \quad \text{and} \quad hs^{\#} = sk.$$

LEMMA 2. $\mathbf{E}^{\#}$ *has a unique splitting compatible with* (t, s).

Proof. Set $t^{\#} = th$. Certainly $t^{\#}f^{\#} = thf^{\#} = tf = 1$. Now $(1 - f^{\#}t^{\#})f^{\#} = 0$ so there is a *unique* $s^{\#}$ such that $1 - f^{\#}t^{\#} = s^{\#}g^{\#}$. Moreover, $g^{\#}(1 - f^{\#}t^{\#}) = g^{\#}s^{\#}g^{\#}$ implies that $g^{\#} = g^{\#}s^{\#}g^{\#}$, and since $g^{\#}$ is right-cancellable, $g^{\#}s^{\#} = 1$. Finally,

$$(sk - hs^{\#})g^{\#} = skg^{\#} - h(1 - f^{\#}t^{\#}) = sgh - h + ft^{\#} = -fth + fth = 0,$$

so $sk - hs^{\#} = 0$, and $sk = hs^{\#}$. Therefore, $(t^{\#}, s^{\#})$ is compatible with (t, s), and is the unique splitting with this property.

This lemma has an obvious dual: one need only replace sharp ($^{\#}$) by flat ($^{\flat}$), and pullbacks by pushouts.

3. The homology of a commutative square. We shall examine the case where diagram (1) occurs with φ_1 a monomorphism and φ_2 an epimorphism. Commutative squares of this type will be called *special*. Such squares will give rise to s.e.s.'s corresponding to (2). The motivation for this came from [**4**], where special squares arose in the computation of the endomorphisms of an exact sequence of length two.

(4)
$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ Y & \xrightarrow{d} & Z \end{array}$$

Consider any square (4) in $\mathscr{C}$, and set $\partial_1 = \{a, f\} : A \to B \oplus Y$ and $\partial_2 = \langle g, -d\rangle : B \oplus Y \to Z$. (The braces $\{\ \}$ will denote the components of a morphism into a product; $\langle\ \rangle$ will be used to denote the components of a morphism from a coproduct.) The composite $\partial_2\partial_1 = 0$ if and only if (4) is commutative. Assume this to be the case, and define

$$H = \ker \partial_2/\mathrm{im}\ \partial_1.$$

H is called the *homology* of (4).

Assume now that (4) is special. Therefore, one can choose $b : B \to C$ as the cokernel of the monomorphism a, and $c : X \to Y$ as the kernel of the epimorphism d. These give the s.e.s.'s used in the pullback diagram (5) and the pushout (6).

(5)
$$\begin{array}{lccccccccc} \mathbf{E}^\sharp : & 0 & \longrightarrow & X & \xrightarrow{u} & P & \xrightarrow{v} & B & \longrightarrow & 0 \\ & & & {\scriptstyle 1}\big\| & & {\scriptstyle y}\big\downarrow & & {\scriptstyle g}\big\downarrow & & \\ \mathbf{E}' : & 0 & \longrightarrow & X & \xrightarrow{c} & Y & \xrightarrow{d} & Z & \longrightarrow & 0 \end{array}$$

(6)
$$\begin{array}{lccccccccc} \mathbf{E}'' : & 0 & \longrightarrow & A & \xrightarrow{a} & B & \xrightarrow{b} & C & \longrightarrow & 0 \\ & & & {\scriptstyle f}\big\downarrow & & {\scriptstyle z}\big\downarrow & & {\scriptstyle 1}\big\| & & \\ \mathbf{E}^\flat : & 0 & \longrightarrow & Y & \xrightarrow{q} & Q & \xrightarrow{r} & C & \longrightarrow & 0. \end{array}$$

It follows from (5) that $\partial_2 = \langle g, -d\rangle$ is an epimorphism with kernel $\{v, y\}$. Similarly, (6) shows that $\partial_1 = \{a, f\}$ is a monomorphism with cokernel $\langle z, -q\rangle$.

Because $\partial_2\partial_1 = 0$, ∂_1 factors uniquely through $\{v, y\}$. This is seen in diagram (7), all of whose rows and columns are s.e.s.'s.

$$
(7)\qquad
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\exists! e} & P & \xrightarrow{h} & H & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \{v,\, y\}} & & \downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & A & \xrightarrow{\partial_1} & B \oplus Y & \xrightarrow{\langle z,\, -q \rangle} & Q & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \partial_2} & & \downarrow{\scriptstyle w} & & \\
 & & & & Z & = & Z & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

From (5) and (7) we note that

$$bv = rzv = r\langle z, -q\rangle\{v, y\} = rkh$$

is an epimorphism, so rk must also be an epimorphism. This gives rise to the commutative diagram (8) whose rows and columns are s.e.s.'s.

$$
(8)\qquad
\begin{array}{cccccccccc}
 & & & & & 0 & & 0 & & \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & A & = & A & & \\
 & & & & & \downarrow{\scriptstyle e} & & \downarrow{\scriptstyle a} & & \\
\mathbf{E}^{\#} : & 0 & \longrightarrow & X & \xrightarrow{u} & P & \xrightarrow{v} & B & \longrightarrow & 0 \\
\downarrow & & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle b} & & \\
\mathbf{G} : & 0 & \longrightarrow & X & \xrightarrow{hu} & H & \xrightarrow{rk} & C & \longrightarrow & 0 \\
 & & & & & \downarrow & & \downarrow & & \\
 & & & & & 0 & & 0 & &
\end{array}
$$

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The s.e.s. **G** corresponds to the s.e.s. (2), referred to earlier.

$$\mathbf{G} : 0 \longrightarrow X \xrightarrow{hu} H \xrightarrow{rk} C \longrightarrow 0.$$

THEOREM 1. *The s.e.s.* **G** *splits if and only if there is a lifting* λ *of the commutative special square* (4).

Proof. Assume first that (t, s) is a splitting of **G**, and that $(t^\#, s^\#)$ is the unique compatible splitting of $\mathbf{E}^\#$ given by (5) and Lemma 2. Set $\lambda = ys^\#$. Recall from the proof of Lemma 2 that $t^\# = th$. Since $a = ve$ in (8), and $he = 0$ in (7), we have

$$s^\# a = s^\# ve = (1 - ut^\#)e = e - uthe = e.$$

It follows from figures (5) and (6) that

$$\lambda a = ys^\# a = ye = f, \quad \text{and} \quad d\lambda = dys^\# = gvs^\# = g.$$

Therefore, λ is in fact a lifting of (4).

Conversely, let us suppose that (4) has a lifting μ. We could then choose [**2**, p. 72–73]

$$\mathbf{E}^\# : 0 \to X \xrightarrow{u = \{0, 1\}} B \oplus X \xrightarrow{v = \langle 1, 0 \rangle} B \to 0,$$

and

$$\mathbf{E}^\flat : 0 \to Y \xrightarrow{q = \{0, 1\}} C \oplus Y \xrightarrow{r = \langle 1, 0 \rangle} C \to 0.$$

Then $y = \langle \mu, c \rangle$ and $z = \{b, \mu\}$, so $\{v, y\} : P \to B \oplus Y$ in (7) becomes $\{\langle 1, 0 \rangle, \langle \mu, c \rangle\} : B \oplus X \to B \oplus Y$ and $e = \{a, 0\} : A \to B \oplus X$. This allows us to set $h = b \oplus 1 : B \oplus X \to C \oplus X$. Similarly, $\langle z, -q \rangle$ becomes $\langle \{b, \mu\}, \{0, -1\} \rangle : B \oplus Y \to C \oplus Y$ and $k = 1 \oplus c : C \oplus X \to C \oplus Y$. It follows from this that $hu = \{0, 1\} : X \to C \oplus X$ and $rk = \langle 1, 0 \rangle : C \oplus X \to C$. Therefore, **G** is the split s.e.s.

$$0 \to X \xrightarrow{\{0, 1\}} C \oplus X \xrightarrow{\langle 1,0 \rangle} C \to 0$$

if (4) has a lifting. (The congruence class of the s.e.s. **G** in (8) is independent of the choice of pullback P in (5) and pushout Q in (6). We shall omit the proof of this fact.)

COROLLARY. *If* **G** *splits,* (4) *has* $|\mathrm{Hom}(C, X)|$ *liftings.*

Proof. Since **G** splits, there is at least one lifting λ of (4). If $\theta : C \to X$ is any morphism of $\mathrm{Hom}(C, X)$, then $\lambda + c\theta b$ will also be a lifting of (4). If $\lambda + c\theta b = \lambda + c\rho b$, then $\theta = \rho$.

If μ is any other lifting of (4), then $(\mu - \lambda)a = \mu a - \lambda a = f - f = 0$, so $\mu - \lambda = \psi b$ for a unique morphism $\psi : C \to Y$. Similarly, $d\psi b = d(\mu - \lambda) = 0$, so $d\psi = 0$ because b is an epimorphism. Therefore, there is a unique morphism $\theta : C \to X$ such that $\psi = c\theta$. That is, $\mu = \lambda + c\theta b$.

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4. The obstructions. If one follows the notation of § 1, the commutative square (1) gives rise to the canonical commutative diagram (9), where $u_i t_i = \varphi_i$ for $i = 1, 2$. J_i denotes the image of φ_i.

$$\begin{array}{ccccccccc} K_1 & \xrightarrow{s_1} & A_1 & \xrightarrow{t_1} & J_1 & \xrightarrow{u_1} & B_1 & \xrightarrow{v_1} & C_1 \\ \downarrow k & \mathbf{I} & \downarrow \alpha & \mathbf{II} & \downarrow j & \mathbf{III} & \downarrow \beta & \mathbf{IV} & \downarrow c \\ K_2 & \xrightarrow{s_2} & A_2 & \xrightarrow{t_2} & J_2 & \xrightarrow{u_2} & B_2 & \xrightarrow{v_2} & C_2 \end{array} \qquad (9)$$

Let us suppose that square **II** has a lifting $\eta : J_1 \to A_2$. This implies that $\alpha = \eta t_1$, so $s_2 k = \alpha s_1 = \eta t_1 s_1 = 0$. Since s_2 is a monomorphism, it follows that $k = 0$. Conversely, if $k = 0$, then the second square must have a *unique* lifting η. Dually, the third square has a lifting ν if and only if $c = 0$.

LEMMA 3. *Square* **II** (*respectively*, **III**) *has a unique lifting if and only if* $k = 0$ (*respectively*, $c = 0$).

If both k and c are zero and $j = t_2\eta = \nu u_1$, then there is a commutative diagram (10).

$$\begin{array}{ccccccccccc} & & 0 & \longrightarrow & J_1 & \xrightarrow{u_1} & B_1 & \xrightarrow{v_1} & C_1 & \longrightarrow & 0 \\ & & & & \downarrow \eta & \mathbf{V} & \downarrow \nu & & & & \\ 0 & \longrightarrow & K_2 & \xrightarrow{s_2} & A_2 & \xrightarrow{t_2} & J_2 & \longrightarrow & 0 & & \end{array} \qquad (10)$$

The central square **V** of (10) is a *special* square in the sense of § 3. **V** has a lifting if and only if the short exact sequence (2) splits, where H in (2) is the homology of **V**, and the sequence is obtained in the usual manner. Let us denote the class of this s.e.s. in the abelian group $\mathrm{Ext}^1(C_1, K_2)$ by $[\mathbf{G}_V]$.

Let us introduce the following abelian group elements as our *obstructions* to finding a lifting:

OB 1: the element k in the group $\mathrm{Hom}(K_1, K_2)$.

OB 2: the element c in the group $\mathrm{Hom}(C_1, C_2)$.

OB 3: $[\mathbf{G}_V]$ in the group $\mathrm{Ext}^1(C_1, K_2)$.

THEOREM 2. *The commutative square* (1) *has a lifting* λ *if and only if* OB 1, OB 2, *and* OB 3 *are all zero. If there is one lifting, then there are precisely* $|\mathrm{Hom}(C_1, K_2)|$ *liftings.*

REFERENCES

1. Peter J. Hilton, *Homotopy theory and duality* (Gordon and Breach, New York, 1965).
2. Saunders MacLane, *Homology* (Springer, Berlin, 1963).
3. Paul Olum, *Homology of squares and factoring of diagrams*, pp. 480–489, Lecture Notes in Mathematics, No. 99 (Springer-Verlag, New York, 1969).
4. Irwin S. Pressman, *Endomorphisms of exact sequences*, Bull. Amer. Math. Soc. 77 (1971), 239–242.

The Ohio State University,
Columbus, Ohio