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A CLASS OF $\mathfrak{D}$-EXTREME MINKOWSKI-REDUCED FORMS

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Abstract

Barnes (1978, 1979) introduced the concept of a $\mathfrak{D}$-extreme form, which is a Minkowski-reduced positive definite quadratic form having prescribed diagonal coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ and providing a local minimum of the determinant of the form over all such forms. Here a class of forms which are $\mathfrak{D}$-extreme for all α and all n is described.

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1. Introduction

A positive definite or semidefinite quadratic form $f(\mathbf{x}) = \sum_1^n a_{ij}x_ix_j$ is reduced in the sense of Minkowski if, for all $j = 1, \ldots, n$ and for all integral $\mathbf{x} = (x_1, x_2, \ldots, x_n)$,

(1.1) $$\text{if g.c.d. } (x_j, x_{j+1}, \ldots, x_n) = 1, \text{ then } f(\mathbf{x}) \geqslant a_{jj}.$$

In the $\frac{1}{2}n(n+1)$-dimensional space $\mathcal{P}$ of positive definite and semidefinite forms, the set $\mathfrak{M}$ of reduced forms is defined by a finite number of inequalities, and is therefore a polyhedral cone. Among these inequalities are those determined by the set

(1.2) $$\mathbf{x} = \pm\mathbf{e}_j \quad (1 \leqslant j \leqslant n), \qquad \pm(\mathbf{e}_i - \mathbf{e}_j) \quad (1 \leqslant i < j \leqslant n)$$

where $\mathbf{e}_i$ denote the unit vectors. For these and other properties of Minkowski-reduced forms see Lekkerkerker (1969, §10) or Van der Waerden (1956).

For real $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with

(1.3) $$0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_n,$$

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the set $\mathfrak{D}(\boldsymbol{\alpha})$ of (necessarily) positive definite reduced forms is defined as the intersection of $\mathfrak{M}$ with the hyperplanes

$$a_{ii} = \alpha_i \qquad (i = 1, \ldots, n). \tag{1.4}$$

Thus $\mathfrak{D}(\boldsymbol{\alpha})$ is the set of all reduced forms with prescribed diagonal coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$. Since the finite set of inequalities determining $\mathfrak{M}$ include

$$|2a_{ij}| \leqslant a_{ii} \qquad (1 \leqslant i < j \leqslant n),$$

$\mathfrak{D}(\boldsymbol{\alpha})$ is bounded and therefore a convex polytope.

A form in $\mathfrak{D}(\boldsymbol{\alpha})$ for which the determinant $D(f)$ is a local minimum over all f in $\mathfrak{D}(\boldsymbol{\alpha})$ is called (Barnes (1978, 1979)) a $\mathfrak{D}$*-extreme* form. If the determinant is an absolute minimum over all f in $\mathfrak{D}(\boldsymbol{\alpha})$ the form is *absolutely* $\mathfrak{D}$*-extreme*. Here we show

THEOREM 1. *The form*

$$f_n(\mathbf{x}) = \sum_1^n \alpha_i x_i^2 + \sum_{1 \leqslant i < j \leqslant n} \alpha_i x_i x_j,$$

where $\boldsymbol{\alpha}$ *satisfies* (1.3), *is* $\mathfrak{D}$*-extreme for all n and all such* $\boldsymbol{\alpha}$.

The form $f_n(\mathbf{x})$ is absolutely $\mathfrak{D}$-extreme for $n = 2$ and 3 and is a natural generalization of Voronoi's principal perfect form (see Voronoi (1907))

$$\sum_1^n x_i^2 + \sum_{1 \leqslant i < j \leqslant n} x_i x_j.$$

Since the region $D(f) \geqslant$ constant is strictly convex within $\mathfrak{P}$, any $\mathfrak{D}$-extreme form is a vertex of $\mathfrak{D}(\boldsymbol{\alpha})$. In general, however, not all vertices are $\mathfrak{D}$-extreme.

For f in $\mathfrak{D}(\boldsymbol{\alpha})$ denote by $\pm \mathbf{m}_k$ $(k = 1, \ldots, t)$ all those $\mathbf{x}$ other than unit vectors for which equality holds in (1.1). Then f is called $\mathfrak{D}$*-eutactic* if its adjoint F is expressible in the form

$$F(\mathbf{x}) = \sum_1^n A_{ij} x_i x_j = \sum_1^t \rho_k (\mathbf{m}_k' \mathbf{x})^2 + \sum_1^n \sigma_i x_i^2,$$

where ρ_k, σ_i are real and $\rho_k > 0$ $(k = 1, \ldots, t)$.

Theorem 1 is proved using

THEOREM 2 (Barnes 1979). *A form f in* $\mathfrak{D}(\boldsymbol{\alpha})$ *is* $\mathfrak{D}$*-extreme if and only if it is a vertex of* $\mathfrak{D}(\boldsymbol{\alpha})$ *and is* $\mathfrak{D}$*-eutactic.*

We show f_n is a vertex of $\mathfrak{D}(\boldsymbol{\alpha})$ in Section 2 and that it is $\mathfrak{D}$-eutactic in Section 4, thus proving Theorem 1. Section 3 contains some necessary lemmas on determinants.

2. f_n is a vertex of $\mathfrak{D}(\boldsymbol{\alpha})$

We can express $f_n(\mathbf{x})$ in the form

$$f_n(\mathbf{x}) = \alpha_1 g_n(x_1, \dots, x_n) + (\alpha_2 - \alpha_1) g_{n-1}(x_2, \dots, x_n) + (\alpha_3 - \alpha_2) g_{n-2}(x_3, \dots, x_n) + \cdots + (\alpha_n - \alpha_{n-1}) g_1(x_n),$$

where $g_m(y_1, \dots, y_m) = \sum_1^m y_i^2 + \sum_{1 \leqslant i < j \leqslant m} y_i y_j$ is the principal perfect form of Voronoi (1907). This has the property that, for all integral $(y_1, \dots, y_m) \neq (0, \dots, 0)$, $g_m(\mathbf{y}) \geqslant 1$, with equality if and only if $\mathbf{y} = \pm \mathbf{e}_i$ $(1 \leqslant i \leqslant m)$ or $\mathbf{y} = \pm (\mathbf{e}_i - \mathbf{e}_j)$ $(1 \leqslant i < j \leqslant m)$.

Suppose g.c.d. $(x_j, x_{j+1}, \dots, x_n) = 1$, then $(x_i, \dots, x_n) \neq (0, \dots, 0)$ $(1 \leqslant i \leqslant j)$, so that

$$g_{n-i+1}(x_i, \dots, x_n) \geqslant 1 \qquad (1 \leqslant i \leqslant j),$$

and hence

$$f_n(\mathbf{x}) \geqslant \alpha_1 + (\alpha_2 - \alpha_1) + \cdots + (\alpha_j - \alpha_{j-1}) = \alpha_j = a_{jj}. \tag{2.1}$$

Thus f_n lies in $\mathfrak{D}(\boldsymbol{\alpha})$.

Also, if equality holds in (2.1), then $g_n(\mathbf{x}) = 1$, so that $\mathbf{x}$ lies in the set (1.2). Conversely, if $\mathbf{x}$ lies in the set (1.2), then equality holds in (2.1). Hence f_n satisfies the $\frac{1}{2}n(n+1)$ equalities given by equality in (1.1) at the vectors (1.2) and is thus on an edge of the cone $\mathfrak{M}$ and a vertex of $\mathfrak{D}(\boldsymbol{\alpha})$.

Moreover the set of the vectors $\pm \mathbf{m}_k$ $(k = 1, \dots, t)$ other than unit vectors for which equality holds in (1.1) is the set

$$\mathbf{x} = \pm (\mathbf{e}_i - \mathbf{e}_j) \qquad (1 \leqslant i < j \leqslant n). \tag{2.2}$$

3. Lemmas on determinants

For $0 < a_0 \leqslant a_1 \leqslant a_2 \leqslant \dots$ let $D_k = D_k(a_1, \dots, a_k)$ be the determinant of the $k \times k$ matrix with elements

$$d_{ij} = \begin{cases} 2a_i, & i = j, \\ a_m, & i \neq j, m = \min(i, j). \end{cases}$$

Similarly let $G_k = G_k(a_1, \dots, a_k)$ be the determinant of the $k \times k$ matrix with elements

$$g_{ij} = \begin{cases} 2a_i, & i = j \neq k, \\ a_k, & i = j = k, \\ a_m, & i \neq j, m = \min(i, j) \end{cases}$$

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and let $H_k = H_k(a_0, a_1, \ldots, a_k)$ be the determinant of the $k \times k$ matrix with elements

$$h_{ij} = \begin{cases} 2a_1 - a_0, & i = j = 1, \\ 2a_i, & i = j \neq 1, \\ -a_i, & j = i + 1, \\ -a_{i-1}, & j = i - 1, \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 1.

$$D_k = a_k D_{k-1} + G_k \qquad (k \geqslant 2),$$

$$G_k = a_k D_{k-1} - a_{k-1}^2 D_{k-2} \qquad (k \geqslant 3)$$

and

$$D_k > 0, \quad G_k > 0 \qquad \textit{for } k \geqslant 1.$$

PROOF. We observe $G_1 = a_1 > 0$, $D_1 = 2a_1 > 0$, $G_2 = a_1a_2 + a_1(a_2 - a_1) > 0$ and $D_2 = a_2D_1 + G_2 = 3a_1a_2 + a_1(a_2 - a_1) > 0$.

The result then follows by induction, since

$$G_k = \begin{vmatrix} 2a_1 & a_1 & \cdots & a_1 & a_1 \\ a_1 & 2a_2 & & a_2 & a_2 \\ \vdots & & \ddots & \vdots & \\ a_1 & a_2 & \cdots & 2a_{k-1} & a_{k-1} \\ a_1 & a_2 & & a_{k-1} & a_k \end{vmatrix}$$

$$= \begin{vmatrix} 2a_1 & a_2 & \cdots & a_1 & 0 \\ a_1 & 2a_2 & & a_2 & 0 \\ \vdots & & \ddots & \vdots & \\ a_1 & a_2 & \cdots & 2a_{k-1} & -a_{k-1} \\ 0 & 0 & & -a_{k-1} & a_k \end{vmatrix}$$

$$= a_k D_{k-1} - a_{k-1}^2 D_{k-2}$$

$$= a_k G_{k-1} + a_{k-1} D_{k-2}(a_k - a_{k-1}) > 0,$$

on assuming the induction hypothesis for $k - 1$.

Similarly $D_k = 2a_k D_{k-1} - a_{k-1}^2 D_{k-2} = a_k D_{k-1} + G_k > 0$.

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LEMMA 2.

$$H_k = 2a_kH_{k-1} - a_{k-1}^2H_{k-2} \qquad (k \geqslant 3)$$
$$a_kH_{k-1} - a_{k-1}^2H_{k-2} \geqslant 0 \qquad (k \geqslant 3)$$

and

$$H_k > 0 \quad \textit{for } k \geqslant 1.$$

PROOF. We observe

$$H_1 = a_1 + (a_1 - a_0) > 0,$$
$$H_2 = a_1a_2 + 2a_2(a_1 - a_0) + a_1(a_2 - a_1) > 0,$$
$$a_3H_2 - a_2^2H_1 = a_2(a_1 - a_0)(a_3 - a_2) + a_2a_3(a_1 - a_0)$$
$$+ a_1a_2(a_3 - a_2) + a_1a_3(a_2 - a_1) \geqslant 0$$

and

$$H_3 = 2a_3H_2 - a_2^2H_1 = \left(a_3H_2 - a_2^2H_1\right) + a_3H_2 > 0.$$

Expanding similarly to Lemma 1 we have

$$H_k = 2a_kH_{k-1} - a_{k-1}^2H_{k-2}$$
$$= a_kH_{k-1} + \left(a_kH_{k-1} - a_{k-1}^2H_{k-2}\right).$$

Also

$$a_kH_{k-1} - a_{k-1}^2H_{k-2} = a_k\left(a_{k-1}H_{k-2} - a_{k-2}^2H_{k-3}\right) + a_{k-1}H_{k-2}(a_k - a_{k-1}) \geqslant 0,$$

on assuming the induction hypothesis for $k - 1$. Hence $H_k > 0$.

4. f_n is $\mathfrak{D}$-eutactic

By (2.2) the condition that f_n be $\mathfrak{D}$-eutactic is that its adjoint $F_n(\mathbf{x})$ satisfy

$$F_n(\mathbf{x}) = \sum_1^n A_{ij}x_ix_j = \sum_{1 \leqslant i < j \leqslant n} \rho_{ij}(x_i - x_j)^2 + \sum_1^n \sigma_i x_i^2$$

with all $\rho_{ij} > 0$ $(1 \leqslant i < j \leqslant n)$.

Equating coefficients gives $\rho_{ij} = -A_{ij}$ $(1 \leqslant i < j \leqslant n)$. Hence f_n is $\mathfrak{D}$-eutactic if all the off-diagonal cofactors A_{ij} of its matrix are negative. For convenience we show that the matrix B of $2f_n$ has this property. B has elements

$$b_{ij} = \begin{cases} 2\alpha_i, & i = j, \\ \alpha_m, & i \neq j,\ m = \min(i, j). \end{cases}$$

For $1 \leqslant i < j \leqslant n$ the cofactors of B are

$$B_{ij} = (-1)^{i+j} \det\begin{bmatrix} P & Q & R \\ Q' & S & T \\ R' & U & V \end{bmatrix},$$

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where the matrices P, Q, R, S, T, U, V have elements

$$p_{km} = \begin{cases} 2\alpha_k, & 1 \leqslant k \leqslant i-1, m = k, \\ \alpha_k, & 1 \leqslant k < m \leqslant i-1, \\ \alpha_m, & 1 \leqslant m < k \leqslant i-1 \end{cases}$$

$$q_{km} = \alpha_k, \quad 1 \leqslant k \leqslant i-1, 1 \leqslant m \leqslant j-i$$

$$r_{km} = \alpha_k, \quad 1 \leqslant k \leqslant i-1, 1 \leqslant m \leqslant n-j$$

$$s_{km} = \begin{cases} \alpha_{i+k-1}, & 1 \leqslant m \leqslant k \leqslant j-i, \\ 2\alpha_{i+k}, & 1 \leqslant k \leqslant j-i-1, m = k+1, \\ \alpha_{i+k}, & 1 \leqslant k \leqslant j-i-2, k+2 \leqslant m \leqslant j-i \end{cases}$$

$$t_{km} = \alpha_{i+k}, \quad 1 \leqslant k \leqslant j-1, 1 \leqslant m \leqslant n-j$$

$$u_{km} = \alpha_{i+m-1}, \quad 1 \leqslant k \leqslant n-j, 1 \leqslant m \leqslant j-i$$

$$v_{km} = \begin{cases} 2\alpha_{j+k}, & 1 \leqslant k \leqslant n-j, m = k, \\ \alpha_{j+k}, & 1 \leqslant k < m \leqslant n-j, \\ \alpha_{j+m}, & 1 \leqslant m < k \leqslant n-j. \end{cases}$$

By applying successive row and column operations then row operations, we get

$$B_{ij} = (-1)^{i+j} \det \begin{bmatrix} P & W & * \\ X & Y & * \\ O & O & Z \end{bmatrix},$$

where the O are suitably sized zero matrices, the elements in $*$ are unimportant, Y is an upper triangular matrix with diagonal elements

$$\alpha_i, -\alpha_{i+1}, -\alpha_{i+2}, \ldots, -\alpha_{j-1},$$

and W, X, Z have elements

$$w_{km} = \begin{cases} \alpha_k, & 1 \leqslant k \leqslant i-1, m = 1, \\ c_{km}, & 1 \leqslant k \leqslant i-1, 2 \leqslant m \leqslant j-i, \end{cases}$$

$$x_{km} = \begin{cases} \alpha_m, & 1 \leqslant m \leqslant i-1, k = 1, \\ 0, & 1 \leqslant m \leqslant i-1, 2 \leqslant k \leqslant j-i \end{cases}$$

$$z_{km} = \begin{cases} 2\alpha_{j+1} - \alpha_j, & k = m = 1, \\ 2\alpha_{j+k}, & 2 \leqslant k \leqslant n-j, m = k, \\ -\alpha_{j+k}, & 1 \leqslant k \leqslant n-j-1, m = k+1, \\ -\alpha_{j+m}, & 1 \leqslant m \leqslant n-j-1, k = m+1, \\ 0, & \text{otherwise}, \end{cases}$$

for some c_{km}.

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Hence

$$B_{ij} = (-1)^{i+j} G_i(\alpha_1, \ldots, \alpha_i)(-\alpha_{i+1}) \cdots (-\alpha_{j-1}) H_{n-j}(\alpha_j, \ldots, \alpha_n)$$
$$= -\alpha_{i+1}\alpha_{i+2} \cdots \alpha_{j-1} G_i(\alpha_1, \ldots, \alpha_i) H_{n-j}(\alpha_j, \ldots, \alpha_n).$$

By the results of Lemmas 1 and 2 and (1.3), we then have $B_{ij} < 0$ for $1 \leqslant i < j \leqslant n$, and hence f_n is $\mathfrak{D}$-eutactic.

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