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SOME PROPERTIES OF θ-CLOSURE

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1. Introduction. The concept of θ-closure was introduced by Veličko to study H-closed spaces and to generalize Taimanov's extension theorem [**11**], [**12**]. More recently, this notion has been used by Dickman and Porter [**1**] to characterize those Hausdorff spaces in which the Fomin H-closed extension operator commutes with the projective cover (absolute) operator and [**2**] to study extentions of functions. If X is a topological space and $A \subset X$, we let $\Sigma(A)$ and $\Gamma(A)$ represent, respectively, the family of open subsets which contain A and closed subsets which contain some element of $\Sigma(A)$. The *θ-closure of* $A \subset X$, denoted by $\mathrm{cl}_\theta(A)$ ($\mathrm{cl}_\theta(v)$ if $A = \{v\}$), is $\{x \in X$: each $V \in \Gamma(x)$ satisfies $V \cap A \neq \emptyset\}$ and A is called *θ-closed* in case $\mathrm{cl}_\theta(A) = A$. It is known that the θ-closure operator, cl_θ, is not (in general) a Kuratowski closure operator since $\mathrm{cl}_\theta(A)$ might not be θ-closed [**5**].

In this article we improve several results in [**1**] and [**2**]. We also establish that a very simple observation about θ-closures of points (Lemma 1) leads to several known results as well as a number of other interesting new results; the new results include the equality $\mathrm{cl}_\theta(A) = \bigcup_A \mathrm{cl}_\theta(x)$ for a θ-rigid subset A of a space. We prove that a space X is compact if and only if for each upper-semicontinuous multifunction, λ, on X, the multifunction, μ, defined by $\mu(x) = \mathrm{cl}_\theta(\lambda(x))$ assumes a maximal value with respect to set inclusion. We also prove a theorem on the maximality of θ-closures of points, when these θ-closures contain a compact θ-closure of a point. Utilizing the above results and motivated by the well-known equivalence relation, x is equivalent to y ($x \equiv y$) if and only if $\mathrm{cl}(x) = \mathrm{cl}(y)$, we initiate an investigation of the equivalence relation, $x \equiv y$ if and only if $\mathrm{cl}_\theta(x) = \mathrm{cl}_\theta(y)$. No separation axioms are assumed in this paper unless otherwise stated.

2. Some separation and decomposition results. In this section we improve results from [**1**] and [**2**] on separation of certain subsets by open subsets, improve a result from [**10**] on the question of when the θ-closure of a nowhere dense subset is nowhere dense, and establish that certain subsets A of topological spaces satisfy $\mathrm{cl}_\theta(A) = \bigcup_{x \in A} \mathrm{cl}_\theta(x)$.

A subset A of a space X is called *quasi H-closed* (QHC) *relative to* X [**8**] if for each covering, Ω, of A by open subsets of X, some finite $\Omega^* \subset \Omega$

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satisfies $A \subset \bigcup_{\Omega^*} \mathrm{cl}(V)$. The *$\theta$-adherence of a filterbase* Ω ($\mathrm{ad}_\theta \Omega$) on a space is defined [**11**] to be $\bigcap_\Omega \mathrm{cl}_\theta(F)$. It is shown in [**4**] that a subset A of a space X is QHC relative to X if and only if each filterbase Ω on A satisfies $A \cap \mathrm{ad}_\theta \Omega \neq \emptyset$. Let $\mathrm{ad}\Omega$ denote the adherence of a filterbase Ω on a space X and let $\mathrm{cl}(A)$ represent the closure of a subset A. We extend the definition of the operator ad, writing $\mathrm{ad}\Omega = \bigcap_\Omega \mathrm{cl}(F)$ for any family Ω of subsets of X. The equality $\mathrm{cl}_\theta(A) = \mathrm{ad}\Sigma(A)$ is utilized in [**6**] to show that $\mathrm{cl}_\theta(A)$ is QHC relative to an $H(i)$ space X for each $A \subset X$. This generalizes the result [**11**] that a θ-closed subset of an H-closed space is an H-set.

Theorem 1 significantly generalizes (2.4) of [**1**] which states that disjoint θ-closed subsets of an H-closed space are separated by disjoint open sets.

THEOREM 1. *Two subsets of an $H(i)$ space with disjoint θ-closures are separated by disjoint open sets.*

Proof. Let X be $H(i)$ and let A, B be subsets of X satisfying $\mathrm{cl}_\theta(A) \cap \mathrm{cl}_\theta(B) = \emptyset$. If all $V \in \Sigma(A)$ and $W \in \Sigma(B)$ satisfy $V \cap W \neq \emptyset$ then $\Omega = \{V \cap W: \ V \in \Sigma(A),\ W \in \Sigma(B)\}$ is an open filterbase on X. Since X is $H(i)$ and $\mathrm{cl}_\theta(Q) = \mathrm{cl}(Q)$ for open Q, we have

$$\emptyset \neq \mathrm{ad}_\theta \Omega \subset \mathrm{ad}\Sigma(A) \cap \mathrm{ad}\Sigma(B) = \mathrm{cl}_\theta(A) \cap \mathrm{cl}_\theta(B).$$

This is a contradiction of the hypothesis. Hence the theorem is verified.

In [**2**], a subset A of a space X is called *θ-rigid* if every filterbase Ω on X satisfying $F \cap V \neq \emptyset$ for all $V \in \Gamma(A)$, $F \in \Omega$, also satisfies $A \cap \mathrm{ad}_\theta \Omega \neq \emptyset$. The following theorem is easily seen to be an improvement of the result, (6.1) from [**2**], that disjoint θ-rigid subsets of a Hausdorff space are separated by open subsets.

THEOREM 2. *If X is any space and A, B are subsets of X with A θ-rigid and $A \cap \mathrm{cl}_\theta(B) = \emptyset$, then A and B are separated by disjoint open subsets.*

Proof. If $\Omega = \{V \cap W: V \in \Sigma(A),\ W \in \Sigma(B)\}$ is a filterbase on X then, since A is θ-rigid, we have $A \cap \mathrm{ad}_\theta \Sigma(B) \neq \emptyset$, a contradiction. The proof is complete.

The following easily established lemma on θ-closures of points will be useful in the sequel.

LEMMA 1. *If x and y are points in a space, then $y \in \mathrm{cl}_\theta(x)$ if and only if $x \in \mathrm{cl}_\theta(y)$.*

It is readily seen that $\bigcup_A \mathrm{cl}_\theta(x) \subset \mathrm{cl}_\theta(A)$ for any subset A of a space. It is surprising that if A is a θ-rigid subset of a space we have equality.

THEOREM 3. *If $A \subset X$ is θ-rigid, then $\mathrm{cl}_\theta(A) = \bigcup_A \mathrm{cl}_\theta(x)$.*

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Proof. Let A be θ-rigid and let $x \in \mathrm{cl}_\theta(A)$. The constant net x is frequently in $\mathrm{cl}(V)$ for all $V \in \Sigma(A)$. Hence there is a $y \in A$ such that x is frequently in $\mathrm{cl}(V)$ for all $V \in \Sigma(y)$; therefore $x \in \mathrm{cl}_\theta(y)$. The proof is complete.

Lemma 1 may also be used to establish Theorem 4 below.

THEOREM 4. *If X is a space and $A \subset X$ is* QHC *relative to X, then* $\mathrm{cl}(A) \subset \cup_A \mathrm{cl}_\theta(x)$.

Proof. Let $y \in \mathrm{cl}(A)$. Then $\Omega = \{V \cap A : V \in \Sigma(y)\}$ is a filterbase on A. Hence

$$\emptyset \neq A \cap \mathrm{ad}_\theta \Sigma(y) = A \cap \mathrm{cl}_\theta(y).$$

For $x \in A \cap \mathrm{cl}_\theta(y)$ we have $y \in \mathrm{cl}_\theta(x)$. The proof is complete.

Our next theorem improves Theorem 5 of [**10**]. A subset V of a space X is *regular-open* if $V = \mathrm{int}(\mathrm{cl}(V))$ ($\mathrm{int}(A)$ represents the interior of A) and *regular-closed* if $X - V$ is regular-open. $\Omega \subset \Sigma(A)$ is a *base of open sets for* A if, for each $V \in \Sigma(A)$, some $W \in \Omega$ satisfies $W \subset V$. We remark that the *spiral* of a point x [**10**] is $\mathrm{cl}_\theta(x)$.

THEOREM 5. *Let $A \subset X$ be nowhere dense and assume that A has a base of regular open subsets. If X is T_1, then $\mathrm{cl}_\theta(A)$ is nowhere dense.*

Proof. Suppose that V is a nonempty open subset of $\mathrm{cl}_\theta(A)$ and let $\nabla(A)$ represent a base of regular-open subsets about A. Then $V \subset \mathrm{int}(\mathrm{cl}(W)) = W$ for all $W \in \nabla(A)$. Since X is T_1, we have $A = \cap_{\nabla(A)} W$. Hence $V \subset A$. This is a contradiction and the proof is complete.

A net (g, D), in a space X, *θ-converges to x* in X ($g \rightarrow_\theta x$) if g is eventually in $\mathrm{cl}(V)$ for each $V \in \Sigma(x)$ [**11**]. If Ω is an open filterbase on X and $x \in \mathrm{ad}\Omega$, the usual construction of a net from the filterbase and $\Sigma(x)$ yields a net g such that $g \rightarrow_\theta x$; the question of how, given a net g in X with $g \rightarrow_\theta x$, to construct an open filterbase Ω on X with $x \in \mathrm{ad}\Omega$ has gone unanswered. Our final result in this section answers this question.

THEOREM 6. *Let X be a space and let (g, D) be a net in X. For each $\mu \in D$, let $S(\mu) = \{g(\alpha) : \alpha \geqq \mu\}$. Then $\Omega(g) = \cup_D \Sigma(S(\mu))$ is an open filterbase on X, and $x \in \mathrm{ad}\Omega(g)$ if and only if some subnet of g θ-converges to x.*

Proof. It is clear that $\Omega(g)$ is an open filterbase on X. Suppose $x \in \mathrm{ad}\Omega(g)$. We see that

$$\mathrm{ad}\Omega(g) = \cap_D \mathrm{ad}\,\Sigma(S(\mu)) = \cap_D \mathrm{cl}_\theta(S(\mu)),$$

so that g is frequently in $\mathrm{cl}(V)$ for each $V \in \Sigma(x)$. A standard construction produces a subnet of g which θ-converges to x. Conversely, if g has a subnet which θ-converges to x, we obtain from the last stated equation that $x \in \mathrm{ad}\Omega(g)$. This completes the proof.

3. Characterizations of compactness in terms of upper-semicontinuous multifunctions and the θ-closure operator. A *multifunction* from a set X to a set Y is a function from X to $\mathscr{P}(Y) - \{\emptyset\}$, where $\mathscr{P}(Y)$ is the power set of Y. Smithson [**9**] has contributed a survey relating some of the principal results for multifunctions. If λ is a multifunction from X to Y we will write $\lambda \in \mathscr{M}(X, Y)$. If X and Y are spaces, we say that $\lambda \in \mathscr{M}(X, Y)$ is *upper-semicontinuous* (u.s.c.) at $x \in X$ if for each $W \in \Sigma(\lambda(x))$ in Y there is a $V \in \Sigma(x)$ in X satisfying $\lambda(V) \subset W$; λ is *upper-semicontinuous* (u.s.c.) on X if λ is u.s.c. at each $x \in X$. We say that $\lambda \in \mathscr{M}(X, Y)$ has a *strongly-closed graph* if $\mathrm{ad}_\theta\lambda(\Omega) \subset \{\lambda(x)\}$ for each $x \in X$ and filterbase Ω on X satisfying $\Omega \rightarrow x$ [**7**]. We are now in a position to give several new characterizations of compact spaces.

THEOREM 7. *The following statements are equivalent for a space X:*

(a) *X is compact.*

(b) *For each* u.s.c. *multifunction, λ, on X, the multifunction, μ, on X defined by*

$$\mu(x) = \mathrm{cl}_\theta(\lambda(x))$$

assumes a maximal value under set inclusion.

(c) *Each* u.s.c. *multifunction, λ, on X with $\lambda(x)$ θ-closed for each x assumes a maximal value under set inclusion.*

(d) *Each* u.s.c. *multifunction, λ, on X with a strongly-closed graph assumes a maximal value under set inclusion.*

Proof. The proof that (b) implies (c) is obvious. The fact that (c) and (d) are equivalent follows from Theorem 3.7 of [**6**]. Now, assume (a), let $\Omega = \{\mu(x): x \in X\}$ be ordered by inclusion and let Ω_1 be a nonempty chain in Ω. For each y such that $\mu(y) \in \Omega_1$, let

$$F(y) = \{x \in X: \mu(y) \subset \mu(x)\}.$$

Then $\{F(y)\}$ is a filterbase on the compact space X. For any such y, let $v \in \mathrm{cl}(F(y))$ and let $W \in \Sigma(\lambda(v))$. There is a $V \in \Sigma(v)$ satisfying $\lambda(V) \subset W$. Let $q \in V \cap F(y)$. Then

$$\mu(y) \subset \mu(q) = \mathrm{cl}_\theta(\lambda(q)) \subset \mathrm{cl}(W).$$

Thus $\mu(y) \subset \mu(v)$, $v \in F(y)$ and $F(y)$ is closed. Let $q \in \bigcap F(y)$. Then $\mu(q)$ is an upper bound for Ω_1. By Zorn's Lemma, Ω has a maximal element. This establishes that (a) implies (b). To complete the proof, we will verify that (a) is implied by (c). If X is not compact, there is a net, g, in X with an ordinal $\mathscr{D}$ as its index set and no convergent subnet.

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Let $\mathscr{D}$ have the order topology and, for each $k \in \mathscr{D}$, let

$$V(k) = X - \mathrm{cl}(\{g(j): j \geqq k\}).$$

Then $\{V(k): k \in \mathscr{D}\}$ is an increasing open cover of X with no finite subcover. Define $\lambda \in \mathscr{M}(X, \mathscr{D})$ by $\lambda(x) = \{j \in \mathscr{D}: j \geqq k_x\}$ where k_x is the first element k of $\mathscr{D}$ with $x \in V(k)$. Since $\mathscr{D}$ with the order topology is regular and $\lambda(x)$ is closed for each x, then $\mu(x) = \lambda(x)$ for each x. We now show that λ is u.s.c. Let $W \in \Sigma(\lambda(x))$ and let $y \in V(k_x)$. Then $k_y \leqq k_x$, so that $\lambda(y) \subset \lambda(x) \subset W$. Hence $\lambda(V(k_x)) \subset W$ and λ is u.s.c. Since μ clearly assumes no maximal value with respect to set inclusion, we see that (c) does not hold. The proof of the theorem is complete.

In a Hausdorff space the θ-closure of each point is trivially compact and maximal in the set of θ-closures of points ordered by inclusion. We may use Theorem 7 to prove that in any space, the θ-closures of points satisfy a maximality condition, when the θ-closure of some point is compact.

THEOREM 8. *Let Y be a space and let $y_0 \in Y$ with $\mathrm{cl}_\theta(y_0)$ compact. Then there is a $y \in Y$ such that* (1) $\mathrm{cl}_\theta(y_0) \subset \mathrm{cl}_\theta(y)$, *and* (2) $\mathrm{cl}_\theta(y)$ *is maximal in the set of θ-closures of points when this set is ordered by inclusion.*

Proof. Let $X = \{y \in Y: \mathrm{cl}_\theta(y_0) \subset \mathrm{cl}_\theta(y)\}$. For each $y \in X$ we have $y \in \mathrm{cl}_\theta(y_0)$ from Lemma 1. Moreover, if $v \in \mathrm{cl}(X)$ and $W \in \Sigma(v)$ then some $y \in W$ satisfies $\mathrm{cl}_\theta(y_0) \subset \mathrm{cl}_\theta(y) \subset \mathrm{cl}(W)$. Hence $\mathrm{cl}_\theta(y_0) \subset \mathrm{cl}_\theta(v)$ and X is closed in Y. Therefore X is a compact subset of Y and since the identity function from X to Y is u.s.c., the proof may be completed by appeal to the fact that (a) implies (b) in Theorem 7.

The following corollary is immediate from Theorem 8.

COROLLARY 1. *If Y is compact, then for each $y_0 \in Y$, there is a $y \in Y$ such that* (1) $\mathrm{cl}_\theta(y_0) \subset \mathrm{cl}_\theta(y)$, *and* (2) $\mathrm{cl}_\theta(y)$ *is maximal in the set of θ-closures of points, when this set is ordered by inclusion.*

Our final result in this section is another corollary which derives from Theorem 7.

COROLLARY 2. *If Y is a regular space and $y_0 \in Y$, then there is a $y \in Y$ such that* (1) $y_0 \in \mathrm{cl}(y)$, *and* (2) $\mathrm{cl}(y)$ *is maximal in the set of closures of points, when this set is ordered by inclusion.*

Proof. It is well-known that in a regular space, the closure of a compact set is compact. Hence the result follows.

4. The quotient space induced by identifying those points with identical θ-closures. We define an equivalence relation, θ, on a space X by $x \, \theta \, y$ if and only if $\mathrm{cl}_\theta(x) = \mathrm{cl}_\theta(y)$. Our main result in this section is that the quotient space induced on X by θ is a T_0 space. For each $A \subset X$ let $\theta[A]$ represent the saturation of the set A by θ (i.e., $\theta[A] = \{y \in X\colon x\,\theta\,y$ for some $x \in A\}$); A is saturated with θ if $\theta[A] = A$. We obtain the following properties of θ from previous results.

THEOREM 9. *The following properties hold in a topological space X:*
(a) *Each $x \in X$ satisfies $\theta[\mathrm{cl}_\theta(x)] = \mathrm{cl}_\theta(x)$.*
(b) *Each θ-rigid subset A of X satisfies $\theta[\mathrm{cl}_\theta(A)] = \mathrm{cl}_\theta(A)$.*
(c) *Each subset A of X satisfies $\theta[A] \subset \mathrm{cl}_\theta(A)$.*
(d) *Each open subset A of X satisfies $\theta[A] \subset \mathrm{cl}(A)$.*
(e) *For each $x \in X$, $\bigcap_{\mathrm{cl}_\theta(x)} \mathrm{cl}_\theta(v) = \{y \in X\colon \mathrm{cl}_\theta(x) \subset \mathrm{cl}_\theta(y)\}$.*
(f) *For $x, y \in X$, the relations (1) $y \in \mathrm{cl}_\theta(x)$, (2) $\theta[y] \cap \mathrm{cl}_\theta(x) \neq \emptyset$, (3) $\theta[x] \cap \mathrm{cl}_\theta(y) \neq \emptyset$, (4) $\theta[x] \subset \mathrm{cl}_\theta(y)$ and (5) $\theta[y] \subset \mathrm{cl}_\theta(x)$ are equivalent.*

Proof. For the proof of (a), let $y \in \theta[\mathrm{cl}_\theta(x)]$. There is a $v \in \mathrm{cl}_\theta(x)$ with $y\,\theta\,v$. We have, from Lemma 1, that $x \in \mathrm{cl}_\theta(v)$ and, since $\mathrm{cl}_\theta(v) = \mathrm{cl}_\theta(y)$ we obtain $y \in \mathrm{cl}_\theta(x)$. Since $\mathrm{cl}_\theta(x) \subset \theta[\mathrm{cl}_\theta(x)]$ from a general property of equivalence relations, the proof of (a) is complete. The proof that (b) holds follows directly from (a), Theorem 4, and the fact that $\theta[\bigcup_\Omega F] = \bigcup_\Omega \theta[F]$ for any family, Ω, of subsets of X. It is obvious that (d) follows from (c). To verify (c), we note that for any $A \subset X$,

$$\theta[A] = \cup_A \theta[x] \subset \cup_A \theta[\mathrm{cl}_\theta(x)] = \cup_A \mathrm{cl}_\theta(x) \subset \mathrm{cl}_\theta(A).$$

Similar methods may be employed to establish (e) and (f). The proofs are omitted.

Theorem 10 is the main result in this section. We let $X(\mathrm{mod}\ \theta)$ represent the quotient space induced on X by θ.

THEOREM 10. *$X(\mathrm{mod}\ \theta)$ is T_0 for any space X.*

Proof. Suppose $x, y \in X$ with $\theta[x] \neq \theta[y]$. Without loss of generality, let $v \in \mathrm{cl}_\theta(x) - \mathrm{cl}_\theta(y)$. Then $y \notin \mathrm{cl}_\theta(v)$ and, consequently, $\theta[y] \cap \mathrm{cl}_\theta(v) = \emptyset$ from Theorem 9(f). Hence

$$\theta[y] \subset X - \mathrm{cl}_\theta(v) \quad \text{and} \quad \theta[x] \subset \mathrm{cl}_\theta(v).$$

Since $X - \mathrm{cl}_\theta(v)$ is an open subset of X and saturated with θ, we conclude that $X(\mathrm{mod}\ \theta)$ is T_0. The proof is complete.

If $\mathrm{cl}_\theta(x)$ is maximal in the set of θ-closures of points when this set is ordered by inclusion, it follows that $\theta[x] = \{y \in X\colon \mathrm{cl}_\theta(x) \subset \mathrm{cl}_\theta(y)\}$ and, therefore, from Theorem 9(e) we have $\theta[x]$ closed in X. Hence we obtain the following theorem and two corollaries.

THEOREM 11. *If X is a space and $\mathrm{cl}_\theta(x)$ is maximal in the set of θ-closures of points when this set is ordered by inclusion, then $\theta[x]$ is closed in X.*

COROLLARY 3. *If X is a space and $\text{cl}_\theta(x)$ is maximal for all $x \in X$ then $X(\text{mod } \theta)$ is T_1.*

COROLLARY 4. *If X is compact then $X(\text{mod } \theta)$ has at least one closed singleton.*

Proof. Use Theorems 8 and 11.

5. Some examples. In this final section, we give some examples in connection with the above results.

Example 1. *A space X with an open subset V such that $\theta[V]$ is not open in X.*

Let X be the closed interval $[0, 1]$, where the basic open sets are the usual open sets in $[0, 1)$ along with all sets of the form $[0, x) \cup (y, 1]$. Then $\theta[[0, \frac{1}{2})] = [0, \frac{1}{2}) \cup \{1\}$, which is not open.

Example 2. *A compact T_1 space with $\theta[V]$ open in X for each open V and $X(\text{mod } \theta)$ not T_1.*

For $n = 1, 2, 3, 4$, let $A(1)$ be the set of primes larger than 9, and $A(n) = [2n, 2n + 1]$ otherwise. For each n, let $\Omega(n)$ be the filter of finite complements on $A(n)$. Let $X = \cup A(n) \cup \{0, 1\}$ with the topology generated by the following collection of sets as base:

$$\{V \subset X: V \text{ is a usual open set in } \cup_{n \geqq 2} A(n)\}$$
$$\cup \{\{0\} \cup F(1) \cup F(2) \cup F(3): F(n) \in \Omega(n), n = 1, 2, 3\}$$
$$\cup \{\{1\} \cup F(1) \cup F(2) \cup F(4): F(n) \in \Omega(n), n = 1, 2, 4\}$$
$$\cup \{\{p\} \cup F(2): p \in A(1), F(2) \in \Omega(2)\}.$$

Then X is compact and T_1, but $\theta[11]$ is not closed since $0 \in \text{cl}(\theta[11]) - \theta[11]$; $\theta[V]$ is open for each open V. We note that in this space, $\text{cl}_\theta(0)$ and $\text{cl}_\theta(1)$ are maximal and distinct.

Our next example establishes that, even in a compact space, the iterate, $\text{cl}_\theta{}^n(x)$, may fail to be θ-closed for some x and every nonnegative integer n.

Example 3. Let $\mathbf{N}$ be the set of positive integers. For each $n \in \mathbf{N}$, let $J(n) = (2n, 2n + 2)$, and let $\Omega(n)$ be the filter of finite complements on $(2n - 1, 2n + 1)$. Let X be the nonnegative reals with topology generated by the base

$$\{A \subset X: A \text{ is a usual open set in } \cup_{\mathbf{N}} J(n)\} \cup \{\{2n\} \cup B: n \in \mathbf{N} \text{ and } B \in \Omega(n)\}$$
$$\cup \{[0, x) \cup (2n - 1, \infty): 0 < x < 1, n \in \mathbf{N}\}.$$

Then $\text{cl}_\theta{}^n(2)$ fails to be θ-closed for each nonnegative integer n.

Example 4. *A non-Hausdorff space in which* $\mathrm{cl}_\theta(x)$ *is maximal for each* x *in the space.*

Such a space is an infinite set with the topology of finite complements.

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