ON THE DISTANCE OF THE COMPOSITION OF TWO DERIVATIONS TO THE GENERALIZED DERIVATIONS

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Introduction. A well-known theorem of E. Posner [**10**] states that if the composition d_1d_2 of derivations d_1, d_2 of a prime ring A of characteristic not 2 is a derivation, then either $d_1 = 0$ or $d_2 = 0$. A number of authors have generalized this theorem in several ways (see e.g. [**1**], [**2**], and [**5**], where further references can be found). Under stronger assumptions when A is the algebra of all bounded linear operators on a Banach space (resp. Hilbert space), Posner's theorem was reproved in [**3**] (resp. [**12**]). Recently, M. Mathieu [**8**] extended Posner's theorem to arbitrary C*-algebras.

Let d_1, d_2 be derivations of a normed algebra A. Our purpose is to estimate the distance of d_1d_2 to the set of all generalized derivations of A. We consider the case when A is an ultraprime normed algebra (Theorem 1), the case when $d_1 = d_2$ and A is ultrasemiprime (Theorem 2), and finally, the case when A is a von Neumann algebra (Theorem 3). As a consequence of these results we obtain a partial answer to Mathieu's question [**8**]: what is the norm of the composition of two derivations in a prime C*-algebra?

Our results will follow easily from two entirely elementary observations; it is our aim in this paper to point out the kind of method that could be used.

Notation and preliminaries. Let A be a ring and let $a, b \in A$. By $M_{a,b}$ we denote the mapping $x \to axb$ on A. Recall that A is said to be *prime* if $M_{a,b} = 0$ implies $a = 0$ or $b = 0$. M. Mathieu [**6**], [**9**] introduced the notion of an ultraprime normed algebra: a complex normed algebra is *ultraprime* if there exists a constant $c > 0$ such that

$$\|M_{a,b}\| \geq c\,\|a\|\,\|b\| \qquad \text{for all } a, b \in A. \tag{1}$$

Every prime C*-algebra A is ultraprime since $\|M_{a,b}\| = \|a\|\,\|b\|$ for all $a, b \in A$ [**7**, Proposition 2.3]. Note that $B(X)$, the algebra of all bounded linear operators on a normed space X, also has this property; (moreover, the same is true for every subalgebra of $B(X)$ which contains all finite rank operators).

A ring A is said to be *semiprime* if $M_{a,a} = 0$ implies $a = 0$. M. Mathieu called a complex normed algebra A *ultrasemiprime* if there exists a constant $c > 0$ such that

$$\|M_{a,a}\| \geq c\,\|a\|^2 \qquad \text{for all } a \in A. \tag{2}$$

Every C*-algebra A is ultrasemiprime. Namely, for $a \in A$ we have $\|aa^*aa^*\| = \|(aa^*)^2\| = \|aa^*\|^2 = \|a\|^4$; hence $\|a\|^4 \leq \|aa^*a\|\,\|a\|$, which yields $\|aa^*a\| = \|a\|^3$; consequently $\|M_{a,a}\| = \|a\|^2$.

Let A be a ring. An additive mapping $\delta: A \to A$ is called a *generalized inner derivation* if $\delta(x) = ax + xb$ for some $a, b \in A$. Note that $\delta(xy) = \delta(x)y + x[y, b]$, where $[u, v]$ denotes the commutator $uv - vu$. Thus

$$\delta(xy) = \delta(x)y + xh(y) \qquad \text{for all } x, y \in A, \tag{3}$$

where h is an inner derivation of A. Now, an additive mapping $\delta: A \to A$ will be called a *generalized derivation* if there exists a derivation h of A such that δ satisfies (3). By $\Delta(A)$

we denote the set of all generalized derivations of A. In case A is a normed algebra, $\Delta_b(A)$ denotes the set of all δ in $\Delta(A)$ which are also bounded linear operators on A. Next, the set of all derivations of a ring A will be denoted by $D(A)$, and by $D_b(A)$ we denote the set of all bounded derivations of a normed algebra A.

REMARK 1. Let A be a ring, $h: A \to A$ be any function, and $\delta: A \to A$ be an additive mapping satisfying (3). We intend to show that under rather mild assumptions h must necessarily be a derivation. On the one hand we have

$$\delta(xyz) = \delta(x(yz)) = \delta(x)yz + xh(yz),$$

and on the other hand,

$$\delta(xyz) = \delta((xy)z) = \delta(x)yz + xh(y)z + xyh(z).$$

Comparing these two expressions we obtain

$$x(h(yz) - h(y)z - yh(z)) = 0 \ (x, y, z \in A).$$

Similarly, by computing $\delta(x(y+z))$ in two ways, one shows that

$$x(h(y+z) - h(y) - h(z)) = 0 \ (x, y, z \in A).$$

Thus, if A has the property that $Aa = 0$ implies $a = 0$, in particular, if A is semiprime, then h is a derivation.

REMARK 2. Suppose a ring A has a unit element 1 and take $x = 1$ in (3). Then we get $\delta(y) = \delta(1)y + h(y)$ for all $y \in A$. Hence we see that every generalized derivation of A is an inner generalized derivation if and only if every derivation of A is inner.

We now state the crucial observations, which can be proved by direct computations.

OBSERVATION 1. *Let A be a ring, $d_1, d_2 \in D(A)$ and $\delta \in \Delta(A)$. A mapping $F = d_1d_2 - \delta$ then satisfies the identity*

$$F(xyz) - F(xy)z - xF(yz) + xF(y)z = d_1(x)yd_2(z) + d_2(x)yd_1(z).$$

OBSERVATION 2. *Let A be a ring, and let $f: A \to A$, $g: A \to A$ be arbitrary functions. Then for all $x, y, z, w, u \in A$, we have*

$$2f(x)yg(z)wf(u) = \{f(x)yg(z) + g(x)yf(z)\}wf(u) + f(x)y\{g(z)wf(u) + f(z)wg(u)\} - \{f(x)(yf(z)w)g(u) + g(x)(yf(z)w)f(u)\}.$$

REMARK 3. Using the above observations it is easy to prove the following slight generalization of Posner's theorem: if $d_1, d_2 \in D(A)$ and $d_1d_2 \in \Delta(A)$, where A is a prime ring of characteristic not 2, then either $d_1 = 0$ or $d_2 = 0$. In our forthcoming paper **[2]** some similar results can be found.

The results. Let A be a normed algebra and let $d_1, d_2, d \in D_b(A)$. Our purpose is to estimate $\operatorname{dist}(d_1d_2, \Delta_b(A))$ and $\operatorname{dist}(d^2, \Delta_b(A))$, where

$$\operatorname{dist}(d_1d_2, \Delta_b(A)) = \inf\{\|d_1d_2 - \delta\|, \delta \in \Delta_b(A)\}.$$

THEOREM 1. *Let A be an ultraprime normed algebra, and let $d_1, d_2 \in D_b(A)$. If a constant $c > 0$ satisfies* (1), *then*

$$\operatorname{dist}(d_1d_2, \Delta_b(A)) \geq (c^2/6)\, \|d_1\|\, \|d_2\|.$$

Proof. Take $\delta \in \Delta_b(A)$. According to Observation 1 we have

$$\|d_1(x)yd_2(z)+d_2(x)yd_1(z)\| \le 4\,\|d_1d_2-\delta\|\,\|x\|\,\|y\|\,\|z\|$$

for all $x, y, z \in A$. Using this relation and Observation 2 we then get

$$\begin{aligned}2\,\|d_1(x)yd_2(z)wd_1(u)\| &\le (4\,\|d_1d_2-\delta\|\,\|x\|\,\|y\|\,\|z\|)\,\|w\|\,\|d_1(u)\|\\ &\quad + \|d_1(x)\|\,\|y\|\,(4\,\|d_1d_2-\delta\|\,\|z\|\,\|w\|\,\|u\|)\\ &\quad + 4\,\|d_1d_2-\delta\|\,\|x\|\,\|yd_1(z)w\|\,\|u\|\\ &\le 12\,\|d_1d_2-\delta\|\,\|d_1\|\,\|x\|\,\|y\|\,\|z\|\,\|w\|\,\|u\|.\end{aligned}$$

That is,

$$\|M_{d_1(x),d_2(z)wd_1(u)}\| \le 6\,\|d_1d_2-\delta\|\,\|x\|\,\|z\|\,\|w\|\,\|u\|.$$

Thus it follows that

$$c\,\|d_1(x)\|\,\|d_2(z)wd_1(u)\| \le 6\,\|d_1d_2-\delta\|\,\|d_1\|\,\|x\|\,\|z\|\,\|w\|\,\|u\|.$$

Hence $c\,\|d_2(z)wd_1(u)\| \le 6\|\,d_1d_2-\delta\|\,\|z\|\,\|w\|\,\|u\|$ from which

$$c^2\,\|d_2(z)\|\,\|d_1(u)\| \le 6\,\|d_1d_2-\delta\|\,\|z\|\,\|u\|$$

is derived. Consequently $c^2\,\|d_1\|\,\|d_2\| \le 6\,\|d_1d_2-\delta\|$. The proof of the theorem is complete.

THEOREM 2. *Let A be an ultrasemiprime normed algebra, and let $d \in D_b(A)$. If a constant $c>0$ satisfies* (2), *then*

$$\mathrm{dist}(d^2, \Delta_b(A)) \ge (c/2)\,\|d\|^2.$$

Proof. Take $\delta \in \Delta_b(A)$. By Observation 1 we have

$$2\,\|d(x)yd(x)\| \le 4\,\|d^2-\delta\|\,\|x\|^2\,\|y\|\ (x, y \in A).$$

It is easy to see that this relation implies the assertion of the theorem.

Let A be a von Neumann algebra. Since every derivation of A is inner [**11**, Theorem 4.1.6] it follows from Remark 2 that every generalized derivation of A is an inner generalized derivation. In particular, $D_b(A)=D(A)$ and $\Delta_b(A)=\Delta(A)$.

THEOREM 3. *Let A be a von Neumann algebra. If $d_1, d_2 \in D(A)$, then*

$$\mathrm{dist}(d_1d_2, \Delta(A)) \le (1/2)\,\|d_1\|\,\|d_2\|.$$

In particular, for every $d \in D(A)$,

$$\mathrm{dist}(d^2, \Delta_b(A)) = (1/2)\,\|d\|^2.$$

Proof. Since A is a von Neumann algebra there exist $a_1, a_2 \in A$ such that $d_i(x) = [a_i, x]\ (x \in A)$. For arbitrary c_1, c_2 in Z, the center of A, define $\delta \in \Delta(A)$ by

$$\delta(x) = (a_1a_2+c_1c_2-c_2a_1-c_1a_2)x + x(a_2a_1+c_1c_2-c_2a_1-c_1a_2).$$

Then, $d_1d_2-\delta = -M_{a_1-c_1,a_2-c_2} - M_{a_2-c_2,a_1-c_1}$ and therefore

$$\|d_1d_2-\delta\| = \|M_{a_1-c_1,a_1-c_2} + M_{a_2-c_2,a_1-c_1}\| \le 2\,\|a_1-c_1\|\,\|a_2-c_2\|.$$

Since $\|d_i\| = 2\,\mathrm{dist}(a_i, Z)$ [**4**], [**13**] we conclude that

$$\mathrm{dist}(d_1 d_2, \Delta(A)) \le 2\,\mathrm{dist}(a_1, Z)\mathrm{dist}(a_2, Z) = (1/2)\,\|d_1\|\,\|d_2\|.$$

This inequality, together with Theorem 2 and the fact that $\|M_{a,a}\| = \|a\|^2$ for all $a \in A$, yields the second assertion.

REMARK 4. As we have mentioned above, in [**8**] M. Mathieu posed the following question: what is the norm of the composition of two derivations of a prime C*-algebra? The results above enable us to discuss this problem. Since every C*-algebra A satisfies $\|M_{a,a}\| = \|a\|^2$ for all $a \in A$, it follows from Theorem 2 that for every $d \in D_b(A)$, we have

$$(1/2)\,\|d\|^2 \le \|d^2\| \le \|d\|^2.$$

This estimate cannot be improved. Indeed, let A be the algebra of 2×2 matrices over the complex field $\mathbb{C}$, and let

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad n = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Define derivations d and g by $d(x) = [p, x]$, $g(x) = [n, x]$. Note that $d = d^3$; hence $\|d\| \le \|d\|\,\|d^2\|$ which means that $\|d^2\| \ge 1$. Using the fact that $\|d\| = 2\,\mathrm{dist}(p, \mathbb{C}1)$, or otherwise, one shows that $\|d\| = 1$. But then $\|d^2\| = \|d\|^2$. Next, we claim that $\|g^2\| = (1/2)\,\|g\|^2$. Namely, observe that $\|g\| = 2$; since $g^2 = -2M_{n,n}$ we have $\|g^2\| = 2$.

Now let A be a prime C*-algebra. As we have mentioned above, in this case $\|M_{a,b}\| = \|a\|\,\|b\|$ for all $a, b \in A$. Thus it follows from Theorem 1 that for all $d_1, d_2 \in D_b(A)$,

$$(1/6)\,\|d_1\|\,\|d_2\| \le \|d_1 d_2\| \le \|d_1\|\,\|d_2\|.$$

We leave as an open question whether or not the constant 1/6 can be improved.

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