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CHARACTERIZATIONS OF A MULTIVARIATE EXTREME VALUE DISTRIBUTION

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Abstract

Characterizations of a multivariate extreme value distribution in terms of its marginals are established.

UNIVARIATE MARGINALS; DEPENDENCE FUNCTION

In this note we shall characterize a k-dimensional extreme value distribution in terms of its univariate marginals. For details of the multivariate extreme value distributions, see Galambos (1978), Chapter 5.

Theorem. Let H be a k-dimensional extreme value distribution with univariate marginals H_i, $i = 1, \ldots, k$.

(a) We have

$$H(\boldsymbol{x}) = H_1(x_1) \cdots H_k(x_k) \quad \text{for all} \quad \boldsymbol{x} = (x_1, \ldots, x_k) \in \boldsymbol{R}^k$$

iff there exists $\boldsymbol{p} = (p_1, \ldots, p_k) \in \boldsymbol{R}^k$ such that $0 < H_i(p_i) < 1$, $i = 1, \ldots, k$ and

$$H(\boldsymbol{p}) = H_1(p_1) \cdots H_k(p_k).$$

(b) We have

$$H(\boldsymbol{x}) = \min\{H_1(x_1), \ldots, H_k(x_k)\} \quad \text{for all} \quad \boldsymbol{x} \in \boldsymbol{R}^k$$

iff there exists $\boldsymbol{p} \in \boldsymbol{R}^k$ such that

$$0 < H_1(p_1) = \cdots = H_k(p_k) < 1 \quad \text{and} \quad H(\boldsymbol{p}) = H_1(p_1).$$

Proof. (a) The proof of this part is similar to that of Theorem 2.2 of Takahashi (1987) and is omitted.

(b) Necessity is obvious so that we shall prove sufficiency. Let $D_H(\boldsymbol{y}) = H(H_1^{-1}(y_1), \ldots, H_k^{-1}(y_k))$, $\boldsymbol{y} \in (0, 1)^k$, be the dependence function of H, where H_i^{-1} is the generalized inverse of H_i, $i = 1, \ldots, k$. Then we have the following results:

(1) $D_H^s(\boldsymbol{y}^{1/s}) = D_H(\boldsymbol{y})$ for all $s > 0$. (See Lemma 5.4.1 of Galambos (1978).)

(2) If $\boldsymbol{y} \leqq \boldsymbol{y}'$, then $D_H(\boldsymbol{y}) \leqq D_H(\boldsymbol{y}')$.

(3) If H' is an extreme value distribution such that $H_i' = H_i$, $i = 1, \ldots, k$, then $H(\boldsymbol{x}) \leqq H'(\boldsymbol{x})$ for all $\boldsymbol{x} \in \boldsymbol{R}^k$ iff $D_H(\boldsymbol{y}) \leqq D_{H'}(\boldsymbol{y})$ for all $\boldsymbol{y} \in (0, 1)^k$.

Suppose the given sufficiency condition holds. Then we have

$$D_H(c\mathbf{1}) = c, \tag{4}$$

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where $c = H_1(p_1)$ and $\mathbf{1} = (1, \ldots, 1)$. The proof will be completed if we show $D_H(\mathbf{y}) = \min\{y_1, \ldots, y_k\}$. For any $y \in (0, 1)$ there exists an $s > 0$ such that $y^{1/s} = c$. Hence, by (1) and (4)

$$D_H(y\mathbf{1}) = D_H^s(y^{1/s}\mathbf{1}) = (y^{1/s})^s = y. \tag{5}$$

Let $\mathbf{y} = (y_1, \ldots, y_k)$ and $y = \min\{y_1, \ldots, y_k\}$, then by (2), (3), (5) and Theorem 5.4.1 of Galambos (1978), we have

$$y = D_H(y\mathbf{1}) \leqq D_H(\mathbf{y}) \leqq \min\{y_1, \ldots, y_k\} = y.$$

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