Canad. Math. Bull. Vol. **51** (4), 2008 pp. 627–636

Summation of Series over Bourget Functions

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Abstract. In this paper we derive formulas for summation of series involving J. Bourget's generalization of Bessel functions of integer order, as well as the analogous generalizations by H. M. Srivastava. These series are expressed in terms of the Riemann ζ function and Dirichlet functions η, λ, β, and can be brought into closed form in certain cases, which means that the infinite series are represented by finite sums.

1 Introduction

Bessel functions play an important role in mathematics and physics. On the one hand, they are suitable for the application of the theory of complex functions, and in the theory of Fourier series they are used to replace trigonometric functions. On the other hand, they are also used for solving various problems in mathematical physics (especially with boundary value problems), acoustics, hydrodynamics, electromagnetics, nuclear physics. Numerical values of the sums of the series in terms of Bessel or related functions, as well as those in terms of the product of Bessel and trigonometric functions, particularly their closed forms, are necessary for finding solutions of some problems in the theories of telecommunications, electrostatics, etc.

Bourget functions are a generalization of Bessel functions, so they are potentially applicable. Books such as [4,5] offer a wide range of formulas which are very useful in many areas. This paper is a contribution in this regard, since the new series in terms of Bourget functions, obtainable from a general formula, are certainly valuable.

2 Preliminaries

We derived a summation formula for the series involving the trigonometric function cosine [8, Theorem 1, p. 396]. Making use of the method used in [8], we can find (see [9]) the other particular cases of that or similar type as well, writing all of them in the form of a single formula

$$\sum_{n=1}^{\infty}\frac{(s)^{n-1}f((an-b)x)}{(an-b)^{\alpha}}=\frac{c\pi}{2\Gamma(\alpha)f(\frac{\pi\alpha}{2})}x^{\alpha-1}+\sum_{i=0}^{\infty}\frac{(-1)^{i}F(\alpha-2i-\delta)}{(2i+\delta)!}x^{2i+\delta}, \tag{2.1}$$

where $\alpha>0$, $a=\left\{\begin{smallmatrix}1\\2\end{smallmatrix}\right\}$ $b=\left\{\begin{smallmatrix}0\\1\end{smallmatrix}\right\}$, $s=1$ or $s=-1$, $f=\left\{\begin{smallmatrix}\sin\\\cos\end{smallmatrix}\right\}$ $\delta=\left\{\begin{smallmatrix}1\\0\end{smallmatrix}\right\}$. We have adopted notations from [5]: to emphasize a direct relation between corresponding

Received by the editors May 23, 2006; revised November 26, 2006.
This work is supported by the Ministry of Science of Serbia.
AMS subject classification: Primary: 33C10; secondary: 11M06, 65B10.
Keywords: Riemann zeta function, Bessel function, Bourget function, Dirichlet function.

https://doi.org/10.4153/CMB-2008-062-6 Published online by Cambridge University Press

values, we deliberately do not write a comma within the same structure. For example, in the first case, if we take $a = 1$, then $b = 0$; if we take $a = 2$, then $b = 1$. Similarly, $f = \sin$ corresponds to $\delta = 1$, and $f = \cos$ corresponds to $\delta = 0$. We have retained this principle throughout the paper. The values for c and F are in the Table (1.1), where ζ, η, λ and β are the Riemann zeta function $\zeta(z) = \sum_{k=1}^{\infty} k^{-z}$, and Dirichlet functions $\eta(z) = \sum_{k=1}^{\infty}(-1)^{k-1}k^{-z} = (1 - 2^{1-z})\zeta(z)$, $\lambda(z) = \sum_{k=0}^{\infty}(2k+1)^{-z} = (1 - 2^{-z})\zeta(z)$, $\beta(z) = \sum_{k=0}^{\infty}(-1)^k(2k+1)^{-z}$. The functions ζ, η, λ are analytic in the whole complex plane except for $z = 1$, where they have a pole, The integral representation

$$\beta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{x^{z-1}e^x}{e^{2x}+1}\,dx$$

defines an analytical function for $\operatorname{Re} z \geq 1$, but it also satisfies the functional equation $\beta(z) = (\frac{\pi}{2})^{z-1}\Gamma(1-z)\cos\frac{\pi z}{2}\beta(1-z)$ extending beta to the left side of the complex plane $\operatorname{Re} z < 1$. In addition, $\zeta(-2n) = \eta(-2n) = \lambda(-2n) = 0$, whereas $\beta(-(2n-1)) = 0$, $n \in \mathbb{N}$.

Table 1

(1.1)

a	b	s	c	F	for
1	0	1	1	ζ	$0 < x < 2\pi$
		-1	0	η	$-\pi < x < \pi$
2	1	1	$\frac{1}{2}$	λ	$0 < x < \pi$
		-1	0	β	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

(1.2)

F	f	k
ζ, η, λ	sin	$2m-1$
	cos	$2m$
β	sin	$2m$
	cos	$2m-1$

We note that if $f = \sin$ and $\alpha = 2m$ or $f = \cos$ and $\alpha = 2m-1$, $m \in \mathbb{N}$, then the limiting value of the right-hand side of (2.1) should be taken into account (see [8]). Also, for $\alpha = k$ ($k \in \mathbb{N}$, see Table (1.2)), due to the vanishing of functions F (ζ, η, λ at negative even integers, β at negative odd integers), the right-hand side series truncates, so representation (2.1) is brought into closed form, from which we easily get all particular cases, some of which are given in [1, 4, 5].

Further, relying on (2.1), we derived the summation formula [7]

$$\text{(2.2)} \quad \sum_{n=1}^{\infty} \frac{(s)^{n-1} J_\nu((an-b)x)}{(an-b)^{\alpha}} = \frac{c\pi(\frac{x}{2})^{\alpha-1}}{2\Gamma(\frac{\alpha-\nu+1}{2})\Gamma(\frac{\alpha+\nu+1}{2})\cos(\frac{\alpha-\nu}{2}\pi)} + \sum_{i=0}^{\infty} \frac{(-1)^i F(\alpha-\nu-2i)(\frac{x}{2})^{\nu+2i}}{i!\,\Gamma(\nu+i+1)},$$

https://doi.org/10.4153/CMB-2008-062-6 Published online by Cambridge University Press

where a, b, s, c, F are read from Table (1.1), $\alpha > 0$, $\alpha > \nu > -\frac{1}{2}$, and $J_\nu(z)$ are Bessel functions of the first kind and order ν (see [10]) having the following integral representation

$$J_\nu(z) = \frac{2(\frac{z}{2})^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})} \int_0^{\pi/2} \sin^{2\nu}\theta\, \cos(z\cos\theta)\, d\theta. \tag{2.3}$$

If, in (2.2), $\alpha - \nu = 2m - 1$ and $f = \cos$ or $\alpha - \nu = 2m$ and $f = \sin$ ($m \in \mathbb{N}$), then one should take the limiting value, as in the case of (2.1). Apart from this, for $\alpha - \nu = k$ ($k \in \mathbb{N}$, see Table (1.2)), the right-hand side series in (2.2) truncates, due to the vanishing of F functions, giving rise to all closed form cases.

3 Definition of Bourget Functions

Bourget functions, first defined by J. Bourget [2] and investigated by K. K. Gorowara [3], are represented by

$$J_{p,q}(z) = \frac{1}{\pi} \int_0^\pi (2\cos\theta)^q \cos(p\theta - z\sin\theta)\, d\theta, \quad p \in \mathbb{N}_0,\ q \in \mathbb{N}. \tag{3.1}$$

For $q = 0$, they become Anger functions, defined by

$$\mathbf{J}_p(z) = \frac{1}{\pi} \int_0^\pi \cos(p\theta - z\sin\theta)\, d\theta, \quad p \in \mathbb{N}_0,$$

whence, after a rearrangement, one obtains (2.3) for $\nu = p$. So Bourget functions can be considered a generalization of Anger functions and Bessel functions for integer order.

3.1 A Different Representation of Bourget Functions

First we shall write Bourget functions in a suitable form, taking into account each of the four cases arising out of the formula (3.1) for even or odd p and q. Using the cosine addition formula, from (3.1) we have

$$\begin{aligned} J_{p,q}(z) &= \frac{1}{\pi} \int_0^\pi (2\cos\theta)^q \big(\cos p\theta \cos(z\sin\theta) + \sin p\theta \sin(z\sin\theta)\big)\, d\theta \\ &= \frac{1}{\pi} \int_0^\pi (2\cos\theta)^q \cos p\theta \cos(z\sin\theta) d\theta + \frac{1}{\pi} \int_0^\pi (2\cos\theta)^q \sin p\theta \sin(z\sin\theta) d\theta \\ &= \frac{1}{\pi} I^{\mathrm{c}} + \frac{1}{\pi} I^{\mathrm{s}}, \end{aligned}$$

where, for simplicity, I^{c} and I^{s} denote the first and second integrals respectively.

Let $p = 2n$, $n \in \mathbb{N}_0$. After introducing a substitution $\theta = \pi - t$, we find

$$I^{\mathrm{c}} = (-1)^q \int_0^\pi (2\cos t)^q \cos(2n\pi - 2nt) \cos(z\sin t)\, dt,$$

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and by applying the cosine addition formula once again, we obtain, for q odd,

$$I^{\mathrm{c}} = (-1)^q \int_0^\pi (2\cos t)^q \cos 2nt \cos(z \sin t)\, dt = (-1)^q I^{\mathrm{c}} = -I^{\mathrm{c}}.$$

Hence $I^{\mathrm{c}} = 0$ for p even and q odd, and (3.1) becomes

$$J_{p,q}(z) = \frac{1}{\pi} \int_0^\pi (2\cos\theta)^q \sin p\theta \sin(z \sin\theta)\, d\theta \quad (p+q \text{ odd}).$$

Now let $p = 2n+1$, $p \in \mathbb{N}_0$. We act in the same manner, but using I^{s} instead. If we substitute $\pi - t$ for θ, we come, for q odd, to a relation

$$I^{\mathrm{s}} = (-1)^q \int_0^\pi (2\cos t)^q \sin(2n+1)t \sin(z \sin t) dt = (-1)^q I^{\mathrm{s}} = -I^{\mathrm{s}}.$$

So $I^{\mathrm{s}} = 0$ for q odd, and (3.1) becomes

$$J_{p,q}(z) = \frac{1}{\pi} \int_0^\pi (2\cos\theta)^q \cos p\theta \cos(z \sin\theta)\, d\theta \quad (p+q \text{ even}).$$

Combining both formulas obtained for $p+q$ odd and $p+q$ even we can express Bourget functions (3.1) as a single formula comprising all four cases mentioned above, *i.e.*,

$$J_{p,q}(z) = \frac{1}{\pi} \int_0^\pi (2\cos\theta)^q f(p\theta) f(z \sin\theta)\, d\theta, \tag{3.2}$$

where $f = \left\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix}0\\ 1\end{smallmatrix}\right\}$, $p+q = 2m+\delta$, $m \in \mathbb{N}$.

3.2 Another Type of Bourget Function

H. M. Srivastava [6] defined functions analogous to (3.1):

$$I_{p,q}(z) = \frac{1}{\pi} \int_0^\pi (2\cos\theta)^q \sin(p\theta - z \sin\theta)\, d\theta, \quad p \in \mathbb{N}_0,\ q \in \mathbb{N}. \tag{3.3}$$

Because of the same structure, we shall also call the functions (3.3) Bourget functions. Following a similar procedure as that for obtaining (3.2), the integral $I_{p,q}$ can be expressed in the form of

$$I_{p,q}(z) = \frac{(-1)^{\delta+1}}{\pi} \int_0^\pi (2\cos\theta)^q f(p\theta) \overline{f}(z \sin\theta)\, d\theta, \tag{3.4}$$

where $f = \left\{\begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix}0\\ 1\end{smallmatrix}\right\}$, $p+q = 2m+\delta$, $m \in \mathbb{N}$, and $\overline{f}$ denotes a co-function of the function f. Note that taking $q = 0$ in (3.3), we obtain the Weber function $\mathbf{E}_p(z)$ of integer order $p \in \mathbb{N}_0$ (see [10]).

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Integral representations (3.2) and (3.4) have similar forms that enable us to represent them by means of a single formula,

$$\varphi_{p,q}(z) = \frac{\tau}{\pi}\int_0^{\pi}(2\cos\theta)^q f(p\theta)h(z\sin\theta)\,d\theta, \tag{3.5}$$

where $\varphi_{p,q} = \left\{\begin{smallmatrix} J_{p,q} \\ I_{p,q}\end{smallmatrix}\right\}$ $h = \left\{\begin{smallmatrix} f \\ \bar{f}\end{smallmatrix}\right\}$ $\tau = \left\{\begin{smallmatrix} 1 \\ (-1)^{\delta+1}\end{smallmatrix}\right\}$; $f = \left\{\begin{smallmatrix} \cos \\ \sin\end{smallmatrix}\right\}$, $\delta = \left\{\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right\}$, $p+q = 2m+\delta$, $m \in \mathbb{N}$.

If $q = 0$ in (3.5) is allowed, the Bourget functions $J_{p,q}$ and $I_{p,q}$ become the Bessel function J_p and the Weber function $\mathbf{E}_p$, respectively. If the cases for even and odd p are separated, a suitable form is obtained

$$\phi_p(z) = \frac{\tau}{\pi}\int_0^{\pi} f(p\theta)h(z\sin\theta)\,d\theta, \quad f = \left\{\begin{smallmatrix} \cos \\ \sin\end{smallmatrix}\right\}, \delta = \left\{\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right\}, p = 2m+\delta, m \in \mathbb{N}_0, \tag{3.6}$$

where $\phi_p = \left\{\begin{smallmatrix} J_p \\ \mathbf{E}_p\end{smallmatrix}\right\}$, $h = \left\{\begin{smallmatrix} f \\ \bar{f}\end{smallmatrix}\right\}$, $\tau = \left\{\begin{smallmatrix} 1 \\ (-1)^{\delta+1}\end{smallmatrix}\right\}$.

4 Series over Bourget Functions

If we put Bourget functions $J_{p,q}$ or $I_{p,q}$, denoting them by $\varphi_{p,q}$, in place of the Bessel function J_ν in (2.2), we shall deal with a new type of series, *i.e.,*

$$S_\alpha^{\varphi_{p,q}} = \sum_{n=1}^{\infty}\frac{(s)^{n-1}\varphi_{p,q}((an-b)x)}{(an-b)^\alpha}, \quad \alpha > 0. \tag{4.1}$$

First, we replace $\varphi_{p,q}$ in (4.1) with the integral from (3.5), and have

$$\frac{\tau}{\pi}\sum_{n=1}^{\infty}\frac{(s)^{n-1}}{(an-b)^\alpha}\int_0^{\pi}(2\cos\theta)^q f(p\theta)h((an-b)x\sin\theta)\,d\theta.$$

The interchange of summation and integration is allowed because of uniform convergence of the right-hand side series in (4.2) with respect to $\theta \in [0,\pi]$ (see [8], where we had $h = \cos$, $x\cos\theta$ instead of $x\sin\theta$, and $\theta \in [0, \frac{\pi}{2}]$). In keeping with it, we determine convergence regions with respect to x. They are the same as those in Table (1.1). Consequently (4.1) becomes

$$S_\alpha^{\varphi_{p,q}} = \frac{\tau 2^q}{\pi}\int_0^{\pi}\cos^q\theta f(p\theta)\Big(\sum_{n=1}^{\infty}\frac{(s)^{n-1}h((an-b)x\sin\theta)}{(an-b)^\alpha}\Big)\,d\theta, \tag{4.2}$$

where $\alpha > 0$, $a = \left\{\begin{smallmatrix} 1 \\ 2\end{smallmatrix}\right\}$ $b = \left\{\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right\}$, $s = 1$ or $s = -1$, $\varphi_{p,q} = \left\{\begin{smallmatrix} J_{p,q} \\ I_{p,q}\end{smallmatrix}\right\}$ $h = \left\{\begin{smallmatrix} f \\ \bar{f}\end{smallmatrix}\right\}$ $\tau = \left\{\begin{smallmatrix} 1 \\ (-1)^{\delta+1}\end{smallmatrix}\right\}$, and independently of that, $f = \left\{\begin{smallmatrix} \cos \\ \sin\end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix} 0 \\ 1\end{smallmatrix}\right\}$, $p+q = 2m+\delta$, $m \in \mathbb{N}$. Now we apply (2.1), where we set $x\sin\theta$ in place of x, so that we obtain

$$S_\alpha^{\varphi_{p,q}} = \frac{\tau 2^q}{\pi}\int_0^{\pi}\cos^q\theta f(p\theta)\times \Big(\frac{c\pi(x\sin\theta)^{\alpha-1}}{2\Gamma(\alpha)h(\frac{\pi\alpha}{2})} + \sum_{i=0}^{\infty}\frac{(-1)^i F(\alpha-2i-d)}{(2i+d)!}(x\sin\theta)^{2i+d}\Big)\,d\theta.$$

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We have introduced a new parameter d depending on a choice of the function h in (4.2), *i.e.*, $h = \left\{ {\sin \atop \cos} \right\}$ $d = \left\{ {1 \atop 0} \right\}$. Thus the previous formula becomes

$$\begin{aligned}(4.3)\quad S_\alpha^{\varphi_{p,q}} = {} & \frac{c\tau 2^{q-1} x^{\alpha-1}}{\Gamma(\alpha) h(\frac{\pi\alpha}{2})} \int_0^\pi \cos^q\theta f(p\theta) \sin^{\alpha-1}\theta \, d\theta \\ & + \frac{\tau 2^q}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i F(\alpha-2i-d)}{(2i+d)!} x^{2i+d} \int_0^\pi \cos^q\theta f(p\theta) \sin^{2i+d}\theta \, d\theta,\end{aligned}$$

where c, F, and convergence regions are read from Table (1.1). In order to find $S_\alpha^{\varphi_{p,q}}$ we first need to calculate the right-hand side integrals. In this regard we shall consider an integral

$$(4.4)\qquad I_{\alpha-1} = \int_0^\pi \cos^q\theta \, f(p\theta) \, \sin^{\alpha-1}\theta \, d\theta,$$

where $p, q \in \mathbb{N}, \alpha > -1$, $f = \sin$ or $f = \cos$. Further, we make use of the formulas (see [4])

$$\cos p\theta = \binom{p}{0}\cos^p\theta - \binom{p}{2}\cos^{p-2}\theta\sin^2\theta + \binom{p}{4}\cos^{p-4}\theta\sin^4\theta - \cdots,$$

$$\sin p\theta = \binom{p}{1}\cos^{p-1}\theta\sin\theta - \binom{p}{3}\cos^{p-3}\theta\sin^3\theta + \binom{p}{5}\cos^{p-5}\theta\sin^5\theta - \cdots,$$

which can be written in the form of a single formula

$$f(p\theta) = \sum_{j=0}^{[p/2]} (-1)^j \binom{p}{2j+\delta} \cos^{p-2j-\delta}\theta \sin^{2j+\delta}\theta \, d\theta,$$

where $f = \left\{ {\cos \atop \sin} \right\}$ $\delta = \left\{ {0 \atop 1} \right\}$. Replacing $f(p\theta)$ in (4.4) with the right-hand side sum from the preceding formula, we have

$$(4.5)\qquad I_{\alpha-1} = \sum_{j=0}^{[p/2]} (-1)^j \binom{p}{2j+\delta} \int_0^\pi \cos^{p+q-2j-\delta}\theta \, \sin^{\alpha-1+2j+\delta}\theta \, d\theta.$$

The integral in (4.5) is of the type

$$J = \int_0^\pi \cos^a\theta \sin^b\theta \, d\theta = \int_0^{\pi/2} \cos^a\theta \sin^b\theta \, d\theta + \int_{\pi/2}^{\pi} \cos^a\theta \sin^b\theta \, d\theta.$$

Introducing a substitution $\theta = t + \frac{\pi}{2}$ in the second integral, we have

$$J = \int_0^{\pi/2} \cos^a\theta \sin^b\theta \, d\theta + (-1)^a \int_0^{\pi/2} \cos^b t \sin^a t \, dt.$$

Because of [1, (6.2.1), p. 258]

$$\int_0^{\pi/2} \cos^a t \sin^b t \, dt = \frac{1}{2} \mathrm{B}\Big(\frac{a+1}{2}, \frac{b+1}{2}\Big),$$

and considering that $\mathrm{B}(\mu, \nu) = \mathrm{B}(\nu, \mu)$, we further have

$$J = \frac{1+(-1)^a}{2} \mathrm{B}\Big(\frac{a+1}{2}, \frac{b+1}{2}\Big).$$

By virtue of this, (4.5) now becomes

$$I_{\alpha-1} = \sum_{j=0}^{[p/2]} (-1)^j \binom{p}{2j+\delta} \frac{1+(-1)^{p+q-2j-\delta}}{2} \mathrm{B}\Big(\frac{p+q-2j+\delta+1}{2}, \frac{\alpha+2j+\delta}{2}\Big),$$

where $\delta = 0$ if we set $f = \cos$ in (4.4), and $\delta = 1$ if $f = \sin$. If we replace $\alpha - 1$ with $2i + d$ in (4.4), we will obtain a quite similar formula for I_{2i+d}, so that we finally find

$$S_\alpha^{\varphi_{p,q}} = \frac{c\tau 2^{q-1}}{\Gamma(\alpha) h\big(\frac{\pi\alpha}{2}\big)} x^{\alpha-1} I_{\alpha-1} + \frac{\tau 2^q}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i F(\alpha-2i-d)}{(2i+d)!} x^{2i+d} I_{2i+d}, \tag{4.6}$$

where $\varphi_{p,q} = \left\{ {J_{p,q} \atop I_{p,q}} \right\}$ $h = \left\{ {f \atop f} \right\}$ $\tau = \left\{ {1 \atop (-1)^{\delta+1}} \right\}$; $f = \left\{ {\cos \atop \sin} \right\}$ $\delta = \left\{ {0 \atop 1} \right\}$; $h = \left\{ {\cos \atop \sin} \right\}$ $d = \left\{ {0 \atop 1} \right\}$. The rest of the parameters we find in Table (1.1).

4.1 An Important Particular Case

We deal with a special case, placing $q = 0$ in (4.6) and taking into account the choice of parameters in (3.6). It is easy to calculate the integrals

$$I_{\alpha-1} = \int_0^\pi f(p\theta) \sin^{\alpha-1}\theta \, d\theta = \frac{\pi\Gamma(\alpha) f^3(\frac{p\pi}{2})}{2^{\alpha-1}\Gamma(\frac{1-p+\alpha}{2})\Gamma(\frac{1+p+\alpha}{2})},$$

$$I_{2i+d} = \int_0^\pi f(p\theta) \sin^{2i+d}\theta \, d\theta = \frac{(2i+d)!\pi f^3(\frac{p\pi}{2})}{2^{2i+d}\Gamma(1+i+\frac{d-p}{2})\Gamma(1+i+\frac{d+p}{2})},$$

where $f = \left\{ {\cos \atop \sin} \right\}$ $p = \left\{ {2m \atop 2m+1} \right\}$ $d = \left\{ {0 \atop 1} \right\}$, $m \in \mathbb{N}_0$. Replacing these values in (4.3), we obtain a particular case of the formula (4.6) for $p = 2m$, $m \in \mathbb{N}_0$:

$$S_\alpha^{J_{2m,0}} = \sum_{n=1}^{\infty} \frac{(s)^{n-1} J_{2m}((an-b)x)}{(an-b)^\alpha} = \frac{c\pi(\frac{x}{2})^{\alpha-1}}{2\Gamma(\frac{\alpha+1}{2}-m)\Gamma(\frac{\alpha+1}{2}+m)\cos(\frac{\alpha\pi}{2})} + \sum_{i=0}^{\infty} \frac{(-1)^i (\frac{x}{2})^{2i+2m} F(\alpha-2i-2m)}{i!\,(i+2m)!}.$$

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Actually, this formula is a particular case of our previous formula (2.2) for $\nu = 2m$, which we have expected anyway, since for $q = 0$ Bourget functions reduce to Bessel functions of integer order.

However, for $p = 2m+1$, $m \in \mathbb{N}_0$, from (4.6) we obtain a new summation formula for the series over the Weber functions

$$\sum_{n=1}^{\infty} \frac{(s)^{n-1}\mathbf{E}_{2m+1}((an-b)x)}{(an-b)^{\alpha}} = \frac{c\pi(-1)^m(\frac{x}{2})^{\alpha-1}}{2\Gamma(\frac{\alpha}{2}-m)\Gamma(\frac{\alpha}{2}+m+1)\sin(\frac{\alpha\pi}{2})} + \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{x}{2})^{2k+2m+1}F(\alpha-2k-2m-1)}{k!\,(k+2m+1)!}.$$

4.2 Limiting Value Cases

When $h = \sin$ and $\alpha = 2k$ or $h = \cos$ and $\alpha = 2k-1$, $k \in \mathbb{N}$, the limiting value of the right-hand side of (4.6) should be taken. For instance, if $a = 2$, $b = 1$, $s = 1$, then $c = \frac{1}{2}$ and $F = \lambda$ (from Table (1.1)). If $\varphi_{p,q} = J_{p,q}$, we see that there follows $h = f$, $\tau = 1$, and taking $f = \cos$, from the first line after (4.6) we read $\delta = 0$. That means $h = \cos$, and we must have $d = 0$. So for $p = 3$ and $q = 1$, the formula (4.6) becomes

$$\sum_{n=1}^{\infty} \frac{J_{3,1}((2n-1)x)}{(2n-1)^{\alpha}} = \frac{x^{\alpha-1}I_{\alpha-1}}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} + \frac{2}{\pi}\sum_{i=0}^{\infty} \frac{(-1)^i\lambda(\alpha-2i)}{(2i)!}x^{2i}I_{2i}.$$

We now evaluate

$$\begin{aligned}\Phi_{2k+1}(x) &= \lim_{\alpha\to 2k+1}\Big[\frac{x^{\alpha-1}I_{\alpha-1}}{2\Gamma(\alpha)\cos(\frac{\pi\alpha}{2})} + \frac{2}{\pi}\sum_{i=0}^{k}\frac{(-1)^i\lambda(\alpha-2i)}{(2i)!}x^{2i}I_{2i}\Big]\\ &= \frac{(-1)^k x^{2k}I_{2k}}{\pi(2k)!}\Big(\psi(2k+1) + \tfrac{1}{2}\psi(k+3) + \gamma - \log\tfrac{x}{2}\\ &\quad + \tfrac{1}{4k}\psi(k+\tfrac{1}{2}) - \tfrac{2k+1}{4k}\psi(k+\tfrac{3}{2})\Big)\\ &\quad + \frac{2}{\pi}\sum_{i=0}^{k-1}\frac{(-1)^i\lambda(2k+1-2i)}{(2i)!}x^{2i}I_{2i},\end{aligned}$$

where γ is Euler's constant and ψ is the digamma function, $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$, whose relation to the harmonic numbers $H_n = \sum_{j=1}^{n}\frac{1}{j}$ is $\psi(n) = H_{n-1} - \gamma$, with $\psi(1) = -\gamma = \Gamma'(1)$, and $\psi(n+\frac{1}{2}) = -\gamma - 2\log 2 + 2\sum_{k=0}^{n-1}\frac{1}{2k+1}$. The above series now becomes

$$\sum_{n=1}^{\infty}\frac{J_{3,1}((2n-1)x)}{(2n-1)^{2k+1}} = \Phi_{2k+1}(x) + \frac{2}{\pi}\sum_{i=k+1}^{\infty}\frac{(-1)^i\lambda(2k+1-2i)}{(2i)!}x^{2i}I_{2i},$$

where $0 < x < \pi$ (see Table (1.1)). For instance,

$$\sum_{n=1}^{\infty} \frac{J_{3,1}((2n-1)x)}{(2n-1)^7} = \frac{x^6}{10240}\Big(\frac{227}{120} - \log\frac{x}{4}\Big) - \frac{7x^4}{1024}\zeta(3) + \frac{31x^2}{256}\zeta(5) + \frac{2}{\pi}\sum_{i=8}^{\infty}\frac{(-1)^i\lambda(7-2i)}{(2i)!}x^{2i}I_{2i}.$$

4.3 Closed Form Cases

If $\alpha - d = 2m$ and $F = \zeta, \eta, \lambda$ or $\alpha - d = 2m+1$ and $F = \beta$ ($m \in \mathbb{N}_0$), the formula (4.6) can be brought into closed form, which means that the sum in (4.6) then becomes finite, because the functions ζ, η, and λ vanish at negative even numbers and the function β vanishes at negative odd numbers. So we write $\alpha = 2m + d + \varepsilon$, where

$$\varepsilon = \begin{cases} 0 & F = \zeta, \eta, \lambda, \\ 1 & F = \beta, \end{cases}$$

and we have

$$\begin{aligned} S_{2m+d+\varepsilon}^{\varphi_{p,q}} &= \sum_{n=1}^{\infty} \frac{(s)^{n-1}\varphi_{p,q}((an-b)x)}{(an-b)^{2m+d+\varepsilon}} \\ &= \frac{c\tau 2^{q-1}x^{2m+d+\varepsilon-1}I_{2m+d+\varepsilon-1}}{\Gamma(2m+d+\varepsilon)h(m\pi + \frac{d+\varepsilon}{2})} + \frac{\tau 2^q}{\pi}\sum_{i=0}^{m}\frac{(-1)^i F(2m-2i+\varepsilon)}{(2i+d)!}x^{2i+d}I_{2i+d}, \end{aligned}$$

where $\varphi_{p,q} = \left\{\begin{smallmatrix} J_{p,q} \\ I_{p,q} \end{smallmatrix}\right\}$ $h = \left\{\begin{smallmatrix} f \\ \overline{f} \end{smallmatrix}\right\}$ $\tau = \left\{\begin{smallmatrix} 1 \\ (-1)^{\delta+1} \end{smallmatrix}\right\}$; $f = \left\{\begin{smallmatrix} \cos \\ \sin \end{smallmatrix}\right\}$ $\delta = \left\{\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right\}$; $h = \left\{\begin{smallmatrix} \cos \\ \sin \end{smallmatrix}\right\}$ $d = \left\{\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right\}$. The rest of the parameters we find in Table (1.1).

For example, if $a = 1$, $b = 0$, $s = -1$, then $c = 0$ and $F = \eta$ (see Table (1.1)), meaning that $\varepsilon = 0$. If $\varphi_{p,q} = I_{p,q}$, there follows $h = \overline{f}$, and if we take $f = \sin$, then $\delta = 1$, which implies $\tau = 1$. So $h = \cos$, and according to the choice of h, we have $d = 0$. So for $p = 2$ and $q = 3$, $m = 2$, the above formula becomes

$$\sum_{n=1}^{\infty}\frac{(-1)^n I_{2,3}(nx)}{n^4} = \frac{8}{\pi}\sum_{i=0}^{2}\frac{(-1)^i\eta(4-2i)}{(2i)!}x^{2i}I_{2i} = \frac{14\pi^3}{225} - \frac{8\pi x^2}{105} + \frac{16x^4}{945\pi},$$

where $-\pi < x < \pi$ (see Table (1.1)). We have made use of the property $\eta(z) = (1 - 2^{1-z})\zeta(z)$.

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