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Gaussian Estimates in Lipschitz Domains

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Abstract. We give upper and lower Gaussian estimates for the diffusion kernel of a divergence and nondivergence form elliptic operator in a Lipschitz domain.

Résumé. On donne des estimations Gaussiennes pour le noyau d'une diffusion, réversible ou pas, dans un domaine Lipschitzien.

0 Introduction

0.1 Notations and Definitions

I shall consider in this paper parabolic second order differential operators of divergence form, as well as time independent non divergence form, in a domain $\Omega \subset \mathbf{R}^d$:

$$M = \frac{\partial}{\partial t} - \partial_i a_{ij}(t,x)\partial_j = \partial_t - \mathcal{A}, \tag{0.1}$$

$$L = \frac{\partial}{\partial t} - a_{ij}(x)\partial_i\partial_j - b_i(x)\partial_i = \partial_t - \mathcal{B}, \tag{0.2}$$

where $\partial_i = \frac{\partial}{\partial x_i}$ $(i = 1, \dots, d)$ and where summation convention is used throughout.

The domain Ω will be a Lipschitz domain. It will be either bounded with the usual definition (*e.g. cf.* [28] [26]) or unbounded, of the upper half space type,

$$x = (x_1, x') \in \mathbf{R}^d = \mathbf{R} \times \mathbf{R}^{d-1}; \quad x_1 > \varphi(x'),\ x \in \mathbf{R}^d, \tag{0.3}$$

where $\varphi \colon \mathbf{R}^{d-1} \to \mathbf{R}$ is a global Lipschitz function that satisfies

$$|\varphi(x') - \varphi(y')| \le A|x' - y'|; \quad x', y' \in \mathbf{R}^{d-1}, \tag{0.4}$$

for some $A > 0$. The domain $\Omega = \mathbf{R}^d$ can also be considered.

We shall assume throughout in (0.1) and (0.2) that

$$|a_{ij}|, |b_i| \le \lambda,\ a_{ij}\xi_i\xi_j \ge \lambda^{-1}\xi_i\xi_i; \quad \xi \in \mathbf{R}^d,\ x \in \Omega,\ t \in \mathbf{R}, \tag{0.5}$$

for some $\lambda > 0$, and we shall also make the qualitative assumption that $a_{ij}, b_i \in C^\infty \cap L^\infty$, *i.e.* that they are smooth and extend in the whole space. But only λ, d, the Lipschitz constants $\mathrm{Lip}(\Omega)$, and sometimes also $\mathrm{diam}(\Omega)$, will be involved in the theorem and in the constants $C, c > 0$ below. As usual the letters C, c, possibly with

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suffixes, will indicate constants that may differ from place to place and only depend on the important parameters. When Ω is the domain (0.3) we set $\mathrm{Lip}(\Omega) = A$ with A as in (0.4). In the case where Ω is bounded we shall scale Ω by dilation to bring it to $\mathrm{diam}(\Omega) = 1$. Then, by definition, there exists some A, $r_0 > 0$ s.t. for all $Q \in \partial\Omega$ we can find local orthonormal coordinates around Q $x = (x_1, x')$ as in (0.3) s.t. $x(Q) = 0$ and such that

$$\Omega \cap [|x_1| \leq r_0, |x'| \leq r_0] = [x_1 > \varphi(x'), |x'| \leq r_0, |x_1| \leq r_0],$$

and some φ as in (0.4). We set then $\mathrm{Lip}(\Omega) = (A, r_0)$.

I shall also adopt throughout the usual notation $X \approx Y$ to indicate that $C^{-1} \leq X/Y \leq C$.

I shall denote by

$$z(t) \in \mathbf{R}^d; \quad t \geq t_0, \tag{0.6}$$

the diffusion generated by (0.1) or (0.2), and I shall consider the first exit time and the probability of life:

$$\tau = \inf[t; z(t) \notin \Omega], \tag{0.7}$$

$$P(s, x; t) = \mathbf{P}_{x,s}[\tau > t] = \mathbf{P}[\tau > t \parallel z(s) = x]; \quad t_0 < s < t, \ x \in \Omega. \tag{0.8}$$

I shall also fix $d\mu(x) (x \in \Omega)$ some measure, as explained below, and consider the corresponding Heat diffusion kernel (parabolic Green function)

$$\int_A p(s, x; t, y)\, d\mu(y) \equiv \mathbf{P}[z(t) \in A; \tau > t]; \quad A \subset \Omega, \ t_0 < s < t, \ x \in \Omega. \tag{0.9}$$

For time-independent operators, and in particular for the operator L in (0.2) the more usual notations

$$p_t(x, y) = p(0, x; t, y), \ P(t, x) = P(0, x; t); \quad t > 0, \ x, y \in \Omega, \tag{0.10}$$

will be used. Very special measures $d\mu$ will be used in (0.9): For the divergence form operator M in (0.1) $d\mu(x) = dx$ the Lebesgue measure will be used throughout. For the operator L in (0.2) the measure $d\mu$ will be chosen as follows: If Ω is bounded and $\Omega \subset B_{r_0}(0)$ lies in the Euclidean ball centered at 0 and radius r_0 we shall fix $m(x) > 0$; $x \in B_{2r_0}(0)$ some adjoint solution:

$$\mathcal{B}^* m = \partial_i \partial_j (a_{ij} m) - \partial_i (b_i m) = 0; \quad x \in B_{2r_0}(0), \tag{0.11}$$

and set

$$d\mu(x) = m(x)\, dx. \tag{0.12}$$

The particular choice of $m(\cdot)$ will not be important and all the estimates will be uniform w.r.t. that choice. We can for instance set (*cf.* [4] [27])

$$m(x) = G(X^*, x); \quad |X^*| = 3r_0, \tag{0.13}$$

with $G(\cdot\,,\cdot)$ the Green function of $L\big(:\ LG(\cdot,x) = -\delta_x(\cdot)\delta_x(\cdot)\big)$ in the ball $B_{4r_0}(0)$ $(:\ G\ |_{\partial B_{4r_0}} \equiv 0)$. If Ω is unbounded no such positive solutions exist in general. We can still set $d\mu$ as in (0.12) with m as in (0.13) where now $G(\cdot\,,\cdot)$ is the global Green function on $\mathbf{R}^d$ provided that we are in the transient case, *i.e.*,

$$G_{\mathbf{R}^d}(x,y) < +\infty. \tag{0.14}$$

$m(\cdot)$ is no longer a solution as in (0.11) but only a supersolution, *i.e.*,

$$\mathcal{B}^* m(x) \le 0; \quad x \in \mathbf{R}^d. \tag{0.15}$$

It will furthermore be necessary that on the actual domain Ω, m is an adjoint solution

$$\mathcal{B}^* m(x) = 0; \quad x \in \Omega, \tag{0.16}$$

e.g. in the transient case (0.14) we can set $m(x) = G(X^*, x)$ with $X^* \notin \Omega$. The reason for this will become apparent in (A.15). It should be observed that positive adjoint solutions exist for any domain if the coefficients are C^∞. In general however these solutions explode when the coefficients "become" L^∞. In the above procedure, on the other hand, we keep control of the L^1-loc norms and this is essential if we want our theorem to make sense for L^∞ coefficients and for the martingale problem (and not merely to hold independently of the smoothness).

If (0.15) holds we can use the measure $d\mu = m\,dx$ to reverse the process (0.6) and construct

$$z^*(t) \in \Omega; \quad t < t_0, \tag{0.17}$$

the process generated by:

$$M^* = \partial_t + \mathcal{A}^*; \quad L^* = \partial_t + \mathcal{B}^*,$$

where $\mathcal{A}^*$, $\mathcal{B}^*$ are the formal adjoints of $\mathcal{A}$ and $\mathcal{B}$ w.r.t. $d\mu$. These operators have 0, or $\mathcal{B}^* m \le 0$, as constant terms so the construction of (0.17) is possible. This process moves backwards in time and I shall use the notations:

$$p^*(s,x;t,y);\ P^*(s,x;t); \quad s_0 > s > t,\ x,y \in \Omega, \tag{0.18}$$

for the Green function and the probability of life of (0.17).

Example (Periodic Coefficients) Let us suppose that the coefficients of L in (0.2) are periodic and that

$$a_{ij}(x+e_k) = a_{ij}(x),\ b_i(x+e_k) = b_i(x); \quad x \in \mathbf{R}^d,$$
$$e_k = (0,\dots,0,1,0,\dots,0) = \text{ coordinate unit vectors}, i,j,k = 1,\dots,d. \tag{0.19}$$

A global adjoint solution $m(\cdot) \underset{\neq}{\geq} 0$, $\mathcal{B}^* m = 0$ on $\mathbf{R}^d$ that is periodic $m(x+e_k) = m(x)$ $(x \in \mathbf{R}^d; k = 1,\dots,d)$ then exists. This is because of obvious reasons of ergodicity (*cf.* [14] V Section 5, [6] Chapter 3, 3.1.)

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0.2 The Doubling Property of the Measure and the Gaussian Kernel

Let now $m(\cdot)$ be defined in some arbitrary domain $P \subset \mathbf{R}^d$ and let us assume that it satisfies $\mathcal{B}^* m \leq 0$ there. Then the measure, $d\mu = m\,dx$ satisfies the doubling property of the volume

$$V_x(r) = \mu\big(B_r(x)\big); \quad B_r(x) \subset P, \tag{0.20}$$

for Euclidean balls (*cf.* [5] [19]). More precisely we have:

$$V_x(2r) \leq CV_x(r); \quad B_{4r}(x) \subset P,\ r < r_o, \tag{0.21}$$

where C depends on r_0 and where we can take $r_0 = +\infty$ if $b_i \equiv 0$ in (0.2). It follows that if Ω, μ, m and r_0 are as in (0.11), (0.12), (0.13) and if we define:

$$\mathrm{Gs} = \mathrm{Gs}_c(t; x, y) = V_x^{-1}(\sqrt{t}) \exp\left(-\frac{|x-y|^2}{ct}\right); \quad x, y \in \Omega, \tag{0.22}$$

then for all $c > 0$ there exists c_1, C s.t.

$$\mathrm{Gs}_c(t; x, y) \leq C\,\mathrm{Gs}_{c_1}(t; y, x); \quad t < r_0^2. \tag{0.23}$$

This means that the definition (0.22) of the above Gaussian kernel is essentially symmetric in x, y and we can even write it in a formally symmetric form:

$$\mathrm{Gs}(t, x, y) \approx V_x^{-\frac{1}{2}}(\sqrt{t}) V_y^{-\frac{1}{2}}(\sqrt{t}) \exp\left(\frac{|x-y|^2}{ct}\right). \tag{0.24}$$

0.3 Statement of the Gaussian Estimates

Theorem *Let Ω be some bounded Lipschitz domain then the Green function of either L of M (cf. (0.8) (0.9)) satisfies*

$$p(s, x; t, y) \approx P(s, x; t) P^*(t, y; s)\,\mathrm{Gs}(t-s; x, y); \tag{0.25}$$
$$x, y \in \Omega,\ s, t \in \mathbf{R},\ 0 < t - s < T,$$

where the constants depend only on d, λ, $\mathrm{Lip}(\Omega)$, $\mathrm{diam}(\Omega)$, T. If we assume that Ω is as in (0.3) and $T = +\infty$, then the same estimate holds for the divergence form operator M in (0.1) and also for the nondivergence form operator L in (0.2) provided that we are in the transient case (0.14) and that the drift term is zero $b_i \equiv 0$ $(1 \leq i \leq d)$.

When $\Omega = \mathbf{R}^d$ and M is in divergence form (0.25) is the classical Aronson estimate [1]. By scaling, it is furthermore clear that we can multiply $\mathrm{diam}(\Omega)$ by $A > 0$ and T by A^2 and the constants of (0.25) do not change (for the operator L and $A \gg 1$ this is only possible if $b_2 = 0$). The above theorem holds also, and gives significant information, for the operator L as in (0.2) with periodic coefficients (0.19), $b_i \equiv 0$

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and $T = +\infty$. In this case $P(t,x) = P^*(t,x) = 1$ and $V_x(t) \approx t^d (t > 1)$. Not surprisingly therefore we have:

$$\mathbf{P}_x[z(t) \in B_1(y)] \approx t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{ct}\right); \quad x, y \in \mathbf{R}^d,\ t > 1. \tag{0.26}$$

For the operator L, both the kernel and the definition of Gs depend on the choice of the positive adjoint solution, the constants in (0.25) are however uniform w.r.t. that choice. An easy corollary of (0.25) is the following:

Local Gaussian Estimate

(0.27)

$$p(s,x;t,y) \approx \mathrm{Gs}(t-s;x,y); \quad 0 < t-s < T,\ x, y \in \Omega, \delta(x),\ \delta(y) \geq \delta_0 > 0,$$

where here and throughout we denote

$$\delta(x) = d(x, \partial\Omega); \quad x \in \Omega,$$

the Euclidean distance to the boundary, and where in (0.27) the constants depend also on δ_0.

Neumann Conditions The above result is an estimate on the diffusion $z(t) \in \Omega$, $(t > 0)$ with Dirichlet conditions at the boundary. Other boundary conditions can be considered. To fix ideas let us consider the operator M in the symmetric time-independent case $a_{ij}(x) = a_{ji}(x)$. Then the definition of the Neumann conditions at the boundary in terms of Dirichlet forms can be found in [21]. In that case we have $P(t,x) \equiv 1$. Let us also assume that Ω is of the upper half space type. In that case the estimate (0.25) holds again and we have:

$$p_t(x,y) \approx t^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{ct}\right); \quad x, y \in \Omega,\ t > 0.$$

The upper estimate in the above equivalence is already known [39]. Of course there other intermediate boundary conditions (: mixed Dirichlet-Neumann) where $P(t,x) \neq 1$. The question whether (0.25) holds in that generality, and under what boundary conditions, is an interesting problem. But apart from some isolated examples I do not know the answer.

More General Domains It is routine to generalize the theorem for N.T.A. domains [26] [28], and also for noncylindrical $\mathrm{Lip}(1, 1/2)$ domains in time space ([43] 8.3, [28] 31).

0.4 The Discrete Setting

The diffusion considered in (0.6) admits natural discrete Markov chain analogues

$$\mathbf{P}\big(z(n+1) = y \parallel z(n) = x\big) = K(x,y); \quad n = 0, 1, \ldots,\ x, y \in \mathbf{Z}^d, \tag{0.28}$$

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where K is a sub-markovian kernel on $\mathbf{Z}^d$. I chose to discretise here the state space and consider $\mathbf{Z}^d$, but this of course is not necessary, we can consider analogous Markov chains on $\mathbf{R}^d$ with

$$K(x,y) \geq 0; \quad x,y \in \mathbf{R}^d, \int K(x,y)\,dy \leq 1. \tag{0.29}$$

Together with the above submarkovian property, to make the Gaussian estimates (0.25) go through, one has to impose some additional conditions on the kernel:

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(i): There exists $a > 0$ s.t.

$$K(x,y) = 0; \quad |x-y| \geq a \tag{0.30}$$

(ii): There exists $\varepsilon_0 > 0$ s.t.

$$K(x,y) \geq \varepsilon_0; \quad |x-y| \leq 1. \tag{0.31}$$

In addition, and this is in analogy with the divergence form or the non divergence form of the operators M and L, we have to impose one or the other of the following two conditions:

(R): Reversibility The kernel is doubly submarkovian:

$$\sum_y K(y,x) = \int K(y,x)\,dy \leq 1; \quad x \in \mathbf{Z}^d. \tag{0.32}$$

(C): Centered We are dealing with a centered space unhomogeneous random walk (*cf.* [31]) $K(x,y) = e(x,y)(x,y \in \mathbf{Z}^d))$

$$\sum_y e(x,y) = 1; \quad \sum_y e(x,y)(x-y) = 0. \tag{0.33}$$

Time unhomogeneous variants of the above can also be considered.

The two conditions above are verified simultaneously in the most important example of the time and space homogeneous random walk

$$K(x,y) = e(x,y) = \mu(x-y); \quad \int x\,d\mu = 0, \tag{0.34}$$

where $\mu \in \mathbf{P}(\mathbf{Z}^d)$ is a centered probability measure.

The exit time $\tau = 0, 1, 2, \ldots$, and the probability of life and transition kernel

$$P(n,x), p_n(x,y); \quad x,y \in \mathbf{Z}^d, n = 0, 1, \ldots, \tag{0.35}$$

can be defined as in (0.7) (0.8) (0.9) (0.10) with respect to, say, the Lipschitz domain $\Omega \subset \mathbf{R}^d$ of the upper half space type (0.3). Bounded domains Ω can also be considered but in that case, since we cannot let the time t tend to zero, we have to present our results in terms of $\operatorname{diam}(\Omega) \to \infty$.

The exact analogue of our theorem in 0.3 holds in the setting of the random walk (0.34), and also to a certain extent in the above two more general cases (R) and (C).

0.5 The Doubling Property of the Probability of Life

One technical and nontrivial result that will be proved for the operator L and a bounded domain is that

$$(0.36)\qquad 1 \leq \frac{P(t_1,x)}{P(t_2,x)} \leq C\Big(\frac{t_2}{t_1}\Big)^C; \quad 0 < t_1 < t_2 < T,$$

where $C = C(d, \lambda, \mathrm{Lip}(\Omega), \mathrm{diam}(\Omega), T)$. If Ω is as in (0.3) then we can take in (0.36) $T = +\infty$. The analogous result for the operator M is

$$(0.37)\qquad 1 \leq \frac{P(s,x;t_1)}{P(s,x;t_2)} \leq C\Big(\frac{t_2 - s}{t_1 - s}\Big)^C; \quad s < t_1 < t_2,\ x \in \Omega,\ t_2 - s < T.$$

The estimate (0.37) has been proved in [43] and should not be confused with the backwards Harnack estimate of [23] [15] [16] [17] that in the present context says that for a bounded domain we have

$$C^{-1} \leq \frac{P(s_1,x;t)}{P(s_2,x;t)} \leq C; \quad s_1,\ s_2 < t - t_0,\ x \in \Omega,\ |s_1 - s_2| \leq a\delta^2(x),\ t_0 > 0,$$

where $C = C\big(\lambda, d, \mathrm{Lip}(\Omega), \mathrm{diam}(\Omega), a, t_0\big)$. It is of interest however that for the particular coparabolic function $u(s,x) = P(x,s;t)$ the proof of [16] can easily be adapted to the unbounded upper half space domain (0.3). I shall have no use of this fact here and will not give the details. If however we combine this fact with the Gaussian estimate (0.25) we see that the backwards Harnack estimate holds for the unbounded domain Ω and the coparabolic function $u(s,x) = p(s,x;t_0,y_0)$ provided that $|x - y_0| \leq C, C^{-1} \leq t_0 - s \leq C, \delta(y_0) \leq C$. This last point is implicit in [37].

The result (0.37) for the time dependent operator M in divergence form also works for time dependent operators in nondivergence form

$$(0.38)\qquad \frac{\partial}{\partial t} - a_{ij}(t,x)\partial_i\partial_j - b_i(t,x)\partial_i.$$

The proof of this however involves additional complications. This fact will not be needed and the proof will not be given here.

0.6 The Tools of This Paper

The Harnack and the Carleson Principles [35] [29] [13] [4] I shall use throughout the following adapted version of the Euclidean and the parabolic r-Ball (*cf.* [13]):

$$(0.39)\qquad B_r = [|x_1| \leq cr, |x'| \leq r] \subset \mathbf{R}^d; \quad D_r = [-r^2, r^2] \times B_r \subset \dot{\mathbf{R}}^d = \mathbf{R} \times \mathbf{R}^d,$$

where $x = (x_1, x') \in \mathbf{R}^d$ is as in (0.3) and $c > 0$ is appropriate. The standard Harnack principles [35] [29] assert that if $u(t,x) \geq 0$ satisfies $Lu = 0$ or $Mu = 0$ in D_r then

$$(\mathrm{H})\qquad u(0,0) \leq Cu\Big(\frac{r^2}{2}, x\Big); \quad x \in B_{\frac{r}{2}};\ 0 < r.$$

We shall say (following [4] [27]) that a function of the form

$$\widetilde{u}(t,y) = \frac{v(t,y)}{m(y)}; \quad (t,y) \in P \subset D_r, \tag{0.40}$$

is a parabolic normalized adjoint solution (p.n.a.s.) of L in P if $\mathcal{B}^*m = 0$ on P if $(\partial_t - \mathcal{B}^*)v = 0$ in P and if $m > 0$, $\mathcal{B}^*m \leq 0$ is some appropriate larger domain $P_1 \supset \bar{P}$. The following direct analogue of P. Bauman's Harnack principle then holds for positive (p.n.a.s) $u^\sim \geq 0$ in D_r

$$\widetilde{u}(0,0) \leq C\,\widetilde{u}\Big(\frac{r^2}{2}, x\Big); \quad x \in B_{\frac{r}{2}};\ 0 < r. \tag{$\tilde{\mathrm{H}}$}$$

Precise results in that direction will be formulated and proved in the Appendix of this paper. Let now Ω be as in (0.3) and normalize by $0 \in \partial\Omega$, $\varphi(0) = 0$. Let $u(t,x) \geq 0$ satisfy $(\partial_t - L)u$ in $\dot{\Omega} \cap D_{3r}$ and $u^\sim(t,x) \geq 0$ be a (n.p.a.s.) in $\dot{\Omega} \cap D_{3r}$ where $\dot{\Omega} = \mathbf{R} \times \Omega$. And let us assume that $u|_{\partial\dot{\Omega}} = \tilde{u}|_{\partial\dot{\Omega}} = 0$. The following Carleson principles then hold:

$$u(t,x) \leq Cu(\bar{A}_r);\ \widetilde{u}(t,x) \leq C\,\widetilde{u}(\bar{A}_r); \quad (t,x) \in \dot{\Omega} \cap D_r,\ r > 0, \tag{C}$$

where the well codified notations [15] [17]

$$\bar{A}_r(0) = (t = 8r^2, x_1 = cr, x' = 0); \quad \underline{A}_r(0) = (t = -8r^2, x_1 = cr, x' = 0), \tag{0.41}$$

are used. The above estimate for u (in the classical setting) was proved by Carleson in [8]. In the present setting, proofs of (C) for parabolic functions u can found in [23] (for continuous coefficients) and [15] [16] [17] (in full generality). A proof of ($\tilde{\mathrm{H}}$) and a proof of (C) for both u and $u^\sim$ will be outlined in the appendix.

The** A_∞ **Weights [19] [32] [28] [27] The function m that is either a positive solution or supersolution, $\mathcal{B}^*m = 0$ or $\mathcal{B}^*m \leq 0$, in B_{2r}, satisfies the A_∞ condition in B_r (it gives rise to an A_∞-weight). This means that for any ball $B \subset B_r$ and any subset $E \subset B$ the measure μ in (0.12) satisfies

$$\frac{\mu(E)}{\mu(B)} \leq C\Big(\frac{|E|}{|B|}\Big)^c, \tag{0.42}$$

for appropriate constants, and $|\cdot|$ indicating the Lebesgue measure. Proofs and background material can be found in the above references. Together with the doubling property (0.21) we shall also make explicit use of the following reverse Hölder inequality that is verified by m (*cf.* [19])

$$|B|^{-1}\int_B m^{\frac{d}{d-1}}(x)\,dx \leq C\Big(|B|^{-1}\int_B m(x)\,dx\Big)^{\frac{d}{d-1}}, \tag{0.43}$$

where $B \subset B_r$ is an arbitrary Euclidean ball. As usual the above two estimates (0.42) (0.43) hold for $r < r_0$ and the constants depend on r_0 but we can take $r_0 = +\infty$ if $b_i \equiv 0$ in (0.2).

Remark An analogue of (0.43) for time dependent adjoint solutions $m(t,x)$ of (0.38) also holds, and this fact is essential if we wish to give a proof of (0.37) for the operator (0.38). In that analogue however B has to be replaced by D_r, a ball in space time as in (0.39). And furthermore the ball on the right hand side of (0.43) has to be larger than that on the left.

0.7 General Plan of the Paper

The logical order in which the various components of the proof are presented is the following:

A_1, A_2, A_3: The proof of $(\tilde{H})$
Section 1: The interior Gaussian estimate
B_3: The proof of (C) for $u^{\sim}$ (and for u)
Section 2: The doubling property for $P(t,x)$
Section 3: Proof of the Theorem: The upper estimate
Section 4: Proof of the Theorem: The lower estimate.

The reason that made me push to the appendix the proofs of $(\tilde{H})$ and (C) is that these results stand apart and because the proofs are straightforward adaptations of the work of L. Carleson [8] and P. Bauman [4]. In that appendix I will be brief and the reader who is not an expert in the subject may very well have to consult the original references.

In the methods of this paper we can distinguish two separate directions: The potential theoretic arguments that center around the appendix and Section 4, and the Gaussian estimate techniques that are used in Section 0.3. In Section 0.3 a special device is borrowed from Usakov (*cf.* [39]). These two directions can be studied independently.

1 The Local Gaussian Estimate

1.1 The Gaussian Mass Escape

The following estimates are basic for the diffusion $z(t) \in \mathbf{R}^d$ $(t > 0)$ that is generated by L or M.

$$\mathbf{P}_0[|z(t)| > R] \le C \exp\Big(-\frac{R^2}{ct}\Big); \quad R, t > 0, \tag{1.1}$$

$$\mathbf{P}_0[\sup_{0<s<t} |z(s)| > R] \le C \exp\Big(-\frac{R^2}{ct}\Big); R, t > 0. \tag{1.2}$$

The maximal estimate (1.2) is an easy consequence of (1.1) and of the Markov property (in [42] the reader will find a general; discussion of (1.1) and of the Markov property). The estimates (1.1) and (1.2) for the operator M follow from the Aronson estimates ([1] *cf.* also [20]). For the operator L, which here could even be time dependent, the estimates are a consequence of Martingale theory. We have (*cf.* [33]), in

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terms of Ito calculus and vector notations in the coordinates of $\mathbf{R}^d$,

$$z(t) = \int_0^t \sqrt{A}\, d\beta + \int_0^t B\, dt, \tag{1.3}$$

where $\beta(t) = \big(\beta_1(t), \dots\big) \in \mathbf{R}^d$ is standard brownian motion, and $A = (a_{ij}), B = (b_1, \dots, b_d)$ are the coefficients of L. The estimates (1.1) (1.2) then follow by representing the coordinates of $z(t)$ as time changed brownian motion plus a drift term. The upshot is that (1.1) and (1.2) hold for all $t > 0$ if the drift of $B \equiv 0$, otherwise they only hold for $0 < t < t_0$ and the constants depend only on d, λ and t_0.

1.2 The Upper Gaussian Estimate For the Operator L

The function $u^{\sim}(t, y) = p_t(x, y)$ $(t > 0, y \in \Omega)$ is a (n.p.a.s.) and by (1.1) it satisfies:

$$\int_{B_R(y)} \widetilde{u}(t, \xi)\, d\mu(\xi) \leq C \exp\Big(-\frac{R^2}{ct}\Big); \quad 0 < t < t_0,\ x, y \in \Omega, \tag{1.4}$$
$$\delta(x), \delta(y) \geq \delta_0,\ |x - y| \geq 2R,\ B_R(y) \subset \Omega.$$

This together with the scaled Harnack estimate ($\tilde{\mathrm{H}}$) for the function $u^{\sim}(t, \xi)$ in the ball $B_R(y)$ gives the upper estimate of (0.27).

1.3 The Mass Escape For the Adjoint Diffusion

One should observe that from the local upper Gaussian estimate (0.27) (applied to a larger domain if necessary) we can recover back the Gaussian mass escape property simply by integrating:

$$\mathbf{P}_0[|z(t)| \geq R] = \int_{|y|>R} p_t(0, y)\, d\mu(y). \tag{1.5}$$

The doubling property (0.21) is essential here. Given that the Gaussian estimates are symmetric with respect to the two diffusions $z(t)$ and $z^*(t)$ (*cf.* (0.6) (0.17)) we conclude from the about remark that the Gaussian mass escape estimates (1.1) and (1.2) hold also for the adjoint diffusion $z^*(t)$ $(t < 0)$.

1.4 The Lower Gaussian Estimate

The procedure that allows us to obtain the lower Gaussian estimate in (0.27) as a simple consequence of the Gaussian mass escape and of Harnack (H) is standard (*e.g.* [44]). The only novelty here, when $d\mu$ is not the Lebesgue measure, is that we have to use the doubling property (0.21). The first step is to use (1.1) to deduce that for $0 < t < t_0$ and $c > 0$ large enough we have

$$\mathbf{P}_x[z(t) \in B_x(c\sqrt{t})] = 1 - \int_{|x-y|>c\sqrt{t}} p_t(x, y)\, d\mu(y) \geq \frac{1}{2}. \tag{1.6}$$

Harnack [*i.e.*, ($\tilde{\text{H}}$) for the n.p.a.s $u^{\sim}(t,y) = p_t(x,y)$] and (1.6) implies therefore that

$$p_t(x,x) \geq CV_x^{-1}(\sqrt{t}); \quad x \in \Omega,\ \delta(x) \geq \delta_0,\ 0 < t < t_0. \tag{1.7}$$

This is the lower estimate (0.27) for $x = y$. The off diagonal estimate (0.27) is easily deduced from (1.7) by the following argument. In time-space we "join" $\dot{x} = (t, x)$ to $\dot{y} = (2t, y) \in \dot{\Omega}$ by a sequence of points $\dot{x} = \dot{a}_0, \dots, \dot{a}_{N+1} = \dot{y}$

$$\dot{a}_j = \left(a_j, \left(1 + \frac{j}{N+1}\right) t = t_j \right), \quad x = a_0, \dots, a_{N+1} \in \Omega, \tag{1.8}$$

$$|a_{j+1} - a_j| \approx \frac{|x-y|}{N}; \quad 0 \leq j \leq N, \tag{1.9}$$

and we apply scaled ($\tilde{\text{H}}$) successively. For this to be possible we must have

$$|a_{j+1} - a_j|^2 \sim \frac{t}{N}; \quad 0 \leq j \leq N+1. \tag{1.10}$$

(1.9) and (1.10) put together give

$$N \approx \frac{|x-y|^2}{t}. \tag{1.11}$$

The errors, on the other hand, that pile up from the use of these Harnack estimates put together amount to $C^N \approx e^{cN}$. Hence the Gaussian factor $\exp(-\frac{|x-y|^2}{ct})$ (where we have the minus sign in the Gaussian because we are dealing with a lower estimate).

1.5 The Discrete Setting

The Gaussian mass escape estimates hold in both the discrete settings (R) and (C) of Section 0.4. Indeed the setting (R) in its global form with $\Omega = \mathbf{R}^d$ has been exhaustively studied by many authors: Gaussian estimates, upper and lower, Harnack estimates, *etc.* The paper [25] contains the upper Gaussian estimate in $\mathbf{R}^d$ and therefore the mass escape (1.1). Observe however that in the particular case of the random walk (0.34) the local Gaussian estimate (upper and lower) is an easy consequence of the Edgeworth, expansion for the central limit theorem (*cf.* [22]) and a coarse Gaussian estimate of the form $\mu^{*n}(x) \leq cn^A \exp(-\frac{|x|^2}{cn})$ $(n \geq 1; x \in \mathbf{Z}^d)$ (*cf.* [40] [9]). The Gaussian mass escape in the setting (C) also holds and is a consequence of martingale theory (*cf.* [10]). This is because under the setting (C) the coordinate functions

$$x_i\big(z(n)\big); \quad i = 1, 2, \dots, d,\ x = (x_1, \dots, x_d) \in \mathbf{R}^d,\ n \geq 1, \tag{1.12}$$

are Martingales and it is only a matter of estimating the maximal function in terms of the uniform norm of the square function. An alternative proof can be found in [43] Section 8.1 where I show that the transition Markov operator T attached to the chain (C) of Section 0.4 satisfies the estimate

$$\|e^{-s\varphi} T e^{s\varphi}\|_{\infty \to \infty} \leq \exp(cs^2); \quad s \in \mathbf{R}, \tag{1.13}$$

where φ is any function on $\mathbf{Z}^d$ that satisfies the Lipschitz condition

$$|\varphi(x) - \varphi(y)| \leq |x - y|; \quad x, y \in \mathbf{Z}^d. \tag{1.14}$$

2 The Doubling Property of the Probability of Life

I shall assume the Ω is bounded and I shall prove the doubling property for the probability of life for the operator L (*cf.* Section 0.5); the corresponding results for M are contained in [43]. Scaling will be used and I shall consider the scaled domains:

$$r\Omega = \Omega_r; \quad r > 0, \tag{2.1}$$

I shall then show that

$$P(1, x) - P(1 + h, x) \le ch^{\alpha} P(1, x); \quad x \in \Omega_r,\ r \ge R > 0, \tag{2.2}$$

where we may as well assume that $0 < h < 1$, and where C and α only depend on d, λ, $\mathrm{Lip}(\Omega)$, $\mathrm{diam}(\Omega)$ and R but not on r. The point here is that Ω_r as $r \to \infty$ looks more and more like the domain (0.3) and $\mathrm{diam}(\Omega_r) \to \infty$ but the Lipschitz constants of the functions that define the boundary stay the same. In the course of the proof I shall have to introduce the measure $d\mu = m\,dx$. A subtle point occurs in the use of this reference measure, because we certainly cannot use the same m for all these domains simultaneously. This however will cause no problem. The argument that I will give below adapts to the upper half space domain (0.3) and $T = +\infty$ provided that we are in the transient case (0.14). The estimate (2.2) then holds for all $r > 0$ (in this case we can even use the same m in all scales). Alternatively we can assume that the coefficients are C^∞ and consider a global reference measure, since that reference measure does not appear in (2.2) or (0.36) the fact that this reference measure might explode as the coefficients become L^∞ makes no difference.

The proof of (2.2) that I shall give below is an adaptation of the corresponding proof in [43] for the operator M, and, just as in [43], once we have (2.2) the estimate (0.36) follows. The only difference being that $0 < t < T$ because now $r \ge R$.

Proof of (2.2) We shall assume that r is large enough that $0 < h < 1$ and to simplify notations drop the r from the notations and set $\Omega_r = \Omega$. We have then

$$P(1, x) - P(1 + h, x) = \int_\Omega p_1(x, y) M(h, y) m(y)\, dy = \int_{\Omega_1} + \int_{\Omega_2} = J + J', \tag{2.3}$$

where

$$\Omega_1 = [x \in \Omega; \delta(x) \le 1]; \quad \Omega_2 = [x \in \Omega; \delta(x) > 1], \tag{2.4}$$

where $m(\cdot) \ge 0$ satisfies $\mathcal{B}^* m = 0$ as in Section 0.1 in some ball B with $\frac{1}{2}B \supset \Omega$ and where

$$M(h, y) = \mathbf{P}_y[z(s) \notin \Omega \text{ for some } 0 < s < h]; \quad y \in \Omega,\ h > 0. \tag{2.5}$$

By the estimate (1.2) we have

$$M(h, y) \le C \exp\left(-\frac{\delta^2(y)}{ch}\right); \quad y \in \Omega,\ h > 0, \tag{2.6}$$

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and therefore the contribution of J' satisfies the estimate (2.2). It is therefore J that we must estimate. Toward that we shall fix $Q_1, Q_2, \ldots, \in \partial\Omega$ a finite number of points and local Euclidean coordinates near each Q_j, as in (0.3), *i.e.*, $x = (x_1, x')$ and $x(Q_j) = 0$, to define:

$$T_j = T(Q_j) = [\varphi(x') < x_1 < C; |x'| \leq C]; \quad j \geq 1. \tag{2.7}$$

This can be done in such a way that

$$\Omega_1 \leq \bigcup_j T_j. \tag{2.8}$$

For every $j = 1, \ldots,$ we have (we drop the j here)

$$\int_{T_j} \leq \int_T p_1(x, y) \exp\Big(-\frac{\delta^2(y)}{ch}\Big) m(y)\, dy \leq ABC,$$

where

$$A = \Big(\int_T p_1^\alpha(x, y)\, dy\Big)^{\frac{1}{\alpha}}; \quad B = \bigg(\int_T \exp\Big(-\frac{\delta^2(y)}{ch}\Big)\, dy\bigg)^{\frac{1}{\beta}} \leq ch^{\frac{1}{2\beta}}; \tag{2.9}$$

$$C = \Big(\int_T m^\gamma(y)\, dy\Big)^{\frac{1}{\gamma}},$$

The Hölder indices $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 1$ will be chosen so that (*cf.* (0.43))

$$C_j \leq C \int_{B_j} m(y)\, dy, \tag{2.10}$$

where $B_j = B_a\big(Q_j + (1, 0, \ldots, 0)\big)$. These are balls with centers at Q_j pushed inside Ω by a unit vector. Clearly if r is large enough the above balls are disjoint ($r \gg 0$ can always be achieved by scaling) and if their radius a is small enough we have, by the Carleson principle

$$A_j \leq C \inf_{y \in B_j} p_2(x, y). \tag{2.11}$$

If we put all this together we conclude that

$$\int_{T_j} \leq ch^{\frac{1}{2\beta}} \int_{B_j} p_2(x, y) m(y)\, dy. \tag{2.12}$$

We can sum over j and we obtain

$$\begin{aligned} J &\leq \sum \int_{T_j} \leq ch^{\frac{1}{2\beta}} \sum \int_{B_j} p_2(x, y) m(y)\, dy \\ &\leq ch^{\frac{1}{2\beta}} \int_\Omega p_2(x, y) m(y)\, dy = Ch^{\frac{1}{2\beta}} P(2, x). \end{aligned} \tag{2.13}$$

This is the required estimate (2.2) because $P(t, x)$ is decreasing in t.

Remark The above argument adapts to the time dependent operator (0.38) and even for a domain (0.3) in a nontransient case. To do that we have to use the remark in Section 0.7 and the "doubling property" of Lemma 4.1(a) of [37]. The details will be left to the interested reader.

The Doubling Property for the Reversed Diffusion It will be essential in what follows to prove the same doubling property for the probability of life of the reversed diffusion $z^*(t)$ $(t > 0)$ [*cf.* (0.17) (0.1)]. In other words that we have:

$$1 \leq \frac{P^*(t_1, x)}{P^*(t_2, x)} \leq C\left(\frac{t_2}{t_1}\right)^C; \quad 0 < t_1 < t_2 < T. \tag{2.14}$$

where to avoid confusion we replace here t by $-t$ and make the process move forward in time. This however is not an additional problem. Indeed the ingredients that have been used in the above proof are the Gaussian (maximal) mass escape (1.2), the interior Harnack principle and the Carleson principle for (n.p.a.s.), *i.e.*, for the y-variable in the diffusion kernel $p_t(x, y)$. The diffusion kernel of $z^*(t)$ with respect to $d\mu = m\,dx$ is

$$p_t^*(x, y) = p_t(y, x),$$

and the Harnack and Carleson principles hold again for the y-variable. It follows that the same proof can be repeated for $z^*(t)$ and we obtain (2.14).

3 The Upper Gaussian Estimate

I shall treat simultaneously here the case of the operators M and L in (0.1) and (0.2) and use the notation of Section 0.1 where the Heat diffusion kernel is taken w.r.t. $d\mu$, but to simplify notations I shall denote throughout $d\mu$ by dx, which therefore, in the case of the operator L, will not in general be Lebesgue measure. Once more the key step is the Gaussian mass escape.

3.1 The Gaussian Mass Escape

We shall prove the following estimate:

$$\begin{aligned} I(s, x; R, t) &= \int_{|x-y| \geq R} p(s, x; t, y)\,dy \\ &\leq CP(s, x; t) \exp\left(-\frac{R^2}{c(t-s)}\right); \quad s < t,\ x \in \Omega,\ t - s < T \end{aligned} \tag{3.1}$$

To fix ideas I shall assume that Ω is the "upper half space" (0.3) that $T = +\infty$ and that $s = 0$. We set then

$$I(0, x; R, t) = I_t(R); \quad P(0, x; t) = p(t), \tag{3.2}$$

and by the doubling property Section 0.5, Section 2 we have:

$$1 \le \frac{p(t_1)}{p(t_2)} \le C\left(\frac{t_2}{t_1}\right)^C; \quad 0 < t_1 < t_2. \tag{3.3}$$

We shall also denote

$$T_{t,s}f(x) = \int p(s,x;t,y)f(y)\,dy, \quad T_t = T_{t,0};\ f \in L^\infty,\ s < t, \tag{3.4}$$

the (time dependent) semigroup attached to the process (0.6). The proof of (3.1) follows closely an argument due to Usakov [39]. The first step consists in showing that

(3.5)

$$I_{t_2}(R_2) \le C\exp\left(-c\frac{(R_2-R_1)^2}{t_2-t_1}\right)p(t_1) + I_{t_1}(R_1); \quad 0 < t_1 < t_2;\ 0 < R_1 < R_2.$$

I shall set

$$t_1 = \tau,\ t_2 = t,\ R_1 = \rho,\ R_2 = R; \quad u = u(y) = p(0,x;\tau,y), \tag{3.6}$$

$$u = u\chi_\rho + u(1-\chi_\rho) = u_1 + u_2, \tag{3.7}$$

where χ_ρ is the characteristic function of the ball $B_\rho(x)$. We have

$$I_t(R) \le \int_{|y-x|\ge R} T_{t-\tau}u_1\,dy + I_\tau(\rho). \tag{3.8}$$

To see this we use the fact that T^*_{t-s} is sub-markovian and therefore that:

$$\|T_{t-\tau}u_2\|_1 \le \|u_2\|_1 = I_\tau(\rho). \tag{3.9}$$

To conclude the proof of (3.5) it suffices to estimate the first term of the right hand side of (3.8) by

$$\exp\left(-c\frac{(R-\rho)^2}{t-\tau}\right)p(\tau). \tag{3.10}$$

This follows from the estimates

$$\|u_1\|_1 = p(\tau); \quad \operatorname{supp} u_1 \subset B_\rho(x), \tag{3.11}$$

together with the Gaussian mass escape (1.1).

Remark The time-dependent semigroup (3.4) can be defined for noncylindrical domains in time space and the above argument goes through for divergence form operators and Lip$(1,1/2)$ domains.

3.2 A Digression: The Discrete Setting

The proof of the estimate (3.5) only used the Gaussian mass escape of the diffusion. Because of Section 1.5 this proof extends therefore at once to the two discrete settings (R) and (C) of Section 0.4.

3.3 The Iteration and the Proof of (3.1)

We go back to the continuous time setting and we complete the proof of (3.1). To do this we follow [39] and iterate (3.5). We obtain:

$$I_t(R) \leq C \sum_{j=1}^{m} p(t_j) \exp\left(-c\frac{(R_j - R_{j-1})^2}{t_{j-1} - t_j}\right) + I_{t_m}(R_m), \tag{3.12}$$

where $m = 1, 2, \ldots; R, t > 0$ and
(3.13)
$$t = t_0 > t_1 > \cdots > t_j = 2^{-j}t \cdots; \quad R = R_0 > R_1 > \cdots > R_j = R/j > \cdots .$$

The Gaussian mass escape (2.1) implies that

$$I_{t_m}(R_m) \to 0. \tag{3.14}$$

This allows us to eliminate the second term on the right hand side of (3.12) and replace the first by the infinite series. We can then use (3.3) to estimate the sum of the infinite series. (This elementary computations, where we can assume that $R^2 \gg t$, has been carried out in excruciating detail in [24]). The estimate (3.1) follows.

3.4 The Discrete Time Process

The first part of the argument goes as before. Indeed the estimate (3.1) needs proof only for $1 \leq t \ll R^2$. When R is large enough the second term in the right hand side of (3.12) is 0 for some m because of the finite span condition (0.30). It follows that, as long as we have at our disposal the (discrete) analogue of (3.3), we can finish the proof as before. The discrete analogue of (3.3) is in general a real issue, but there is one context at least where this holds, this is the context of the:

Homogeneous Random Walk (0.34) Indeed the results of [43] imply that in that case (Ω is as in (0.3)) we have:

$$1 \leq \frac{P(n_1, x)}{P(n_2, x)} \leq C\left(\frac{n_2}{n_1}\right)^C; \quad C < n_1 < n_2,\ x \in \mathbf{Z}^d \cap \Omega,\ \delta(x) \geq C. \tag{3.15}$$

The behaviour for $n_1, n_2 = 0, 1, \ldots, C$ is trivial to control for then $P(\cdot, x) \approx 1$. The behaviour for $0 < \delta(x) \leq C$ is equally trivial for the following (ellipticity and the geometry of Ω is used however) Harnack estimate holds: There exists $C > 0$ s.t.

$$CP(n + C; x_1) \geq P(n, x_2); \quad n = 0, 1, \ldots,\ x_1, x_2 \in \Omega,\ |x_1 - x_2| \leq 1, \tag{3.16}$$

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and (3.16) allows us to move away from the boundary simply by shifting the time. It follows that we have in general

$$(3.17)\qquad 1 \le \frac{P(n_1,x)}{P(n_2,x)} \le C\Big(\frac{n_2}{n_1+1}\Big)^C; \quad 0 \le n_1 < n_2,\ x \in \mathbf{Z}^d \cap \Omega.$$

This means that we can finish the proof of (3.1) in the discrete setting as before.

The Non Homogeneous Case The same estimate (3.1) holds in the general nonhomogeneous cases (0.32) and (0.33) but this needs proving. I shall not give the proof of this fact in this paper. The details will be written elsewhere.

3.5 The Gaussian Mass Escape of the Reversed Process

Together with the estimate (3.1) we also have:

(3.18)

$$\int_{|x-y|\ge R} p(s,x;t,y)\,dx \le CP^*(t,y;s)\exp\Big(-\frac{R^2}{c(t-s)}\Big); \quad s<t;\ x\in\Omega,\ t-s<T.$$

It is only a matter of repeating the same proof for the adjoined process $z^*(t)$, $t<0$ (*cf.* (0.17)) for indeed all the necessary ingredients of the proof (: Gaussian mass escape, doubling property of the probability of life *etc.*) are shared by the process $z^*(t)$ (0.17).

3.6 Proof of the Upper Gaussian Estimate (0.25)

There are several ways of playing this "end game". The first that comes to mind (but perhaps not the most intelligent) is the following: We use interior Harnack or Carleson and (3.1) to deduce that

$$(3.19)\qquad p(s,x;t,y) \le CP(s,x;t)\,\mathrm{Gs}(t-s;x,y); \quad x,y\in\Omega,\ 0<t-s<T,$$

which, by the doubling property of the measure, gives

(3.20)

$$J(s,x;t) = \int_\Omega p^2(s,x;t,y)\exp\Big(\varepsilon\frac{|x-y|^2}{t-s}\Big)\,dy \le CP^2(s,x;t)V_x^{-1}\big((t-s)^{\frac12}\big);$$
$$x\in\Omega,\ s<t,$$

for $\varepsilon>0$ small enough. The corresponding adjoint estimate

(3.21)

$$J_*(t,y;s) = \int_\Omega p^2(s,x;t,y)\exp\Big(\varepsilon\frac{|x-y|^2}{t-s}\Big)\,dx \le CP^{*2}(t,y;s)V_y^{-1}\big((t-s)^{\frac12}\big),$$

also holds.

We can then use Hölder, and some intermediate time point, say $s < u = \frac{t+s}{2} < t$, (and the triangle inequality) to deduce that

$$(3.22)\qquad p(s,x;t,y) = \int_\Omega p(s,x;u,z)p(u,z;t,y)\,dz \leq C\exp\Big(-\frac{|x-y|^2}{c(t-s)}\Big)\,J^{\frac12}(s,x;u)J_*^{\frac12}(t,y;u).$$

This together with the doubling property of P and V completes the proof of the upper Gaussian estimate (0.25) and also the corresponding upper Gaussian estimate in the setting of the Homogeneous random walk (0.34). These random walks are thus seen to satisfy:

$$(3.23)\qquad p_n(x,y) \leq CP(n,x)P^*(n,y)\,\mathrm{Gs}(n;x,y);\quad x,y\in \mathbf{Z}^d\cap\Omega,\ n=0,1,\dots,$$

where $P^* \approx P$ corresponds to the random walk induced by the measure $\mu^*(x) = \mu(-x)$.

3.7 The Time Dependent Non-Divergence Form Operator (0.38)

Our starting estimate (3.1) holds also for the time dependent operator (0.38) because, as we already pointed out at the end of Section 0.5, the estimate (3.3) holds for these operators also.

Unfortunately however it is not easy to exploit this estimate, as we did in this section, and obtain the corresponding Gaussian estimate for the kernel. The reason is simply that we no longer have at our disposal the reference measure (0.12) and the adjoint solutions of (0.38) are time-dependent. We shall therefore leave matters at that.

4 The Lower Gaussian Estimate

To simplify notations I shall assume that the operators L and M are time independent and I shall give a proof of the lower estimate of (0.25). We shall fix $x, y \in \Omega$ (Ω, say, is as in (0.3)) and join them as we did in Section 1.4 by a chain of points similar to what we had in (1.8) (1.9) that satisfy (1.10) and (1.11). We shall show that we can chose the points $a_1,\dots,a_N$ so that

$$(4.1)\qquad p_t(x,a_1) \geq CV_x^{-1}(\sqrt{t})P(t,x);\quad p_t(a_N,y)\geq CV_y^{-1}(\sqrt{t})P^*(t,y).$$

$$(4.2)\qquad \delta(a_j)\geq C\sqrt{t};\quad j=1,2,\dots,N,\ a_0=x,\ a_{N+1}=y.$$

From (4.1) and (4.2) we can easily conclude the lower Gaussian estimate (0.25). Indeed we can use the scaled interior adjoint parabolic Harnack to compare $p_{t_j}(x,a_j)$ with $p_{t_{j+1}}(x,a_{j+1})$ for $j=1,\dots,N-1$. This is possible because of (4.2) and will take us, as in Section 5.1, from $p_t(x,a_1)$ to $p_{t_N}(x,a_N)$ with a factor C^N. This together with

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the first condition (4.1) gives

$$(4.3)\qquad p_{2t}(x,\xi) \geq C p_{t_N}(x,a_N) \geq C V_x^{-1}(\sqrt{t}) P(t,x) \exp\left(-c\frac{|x-y|^2}{t}\right); \quad \xi \in B_{a_N}(c\sqrt{t}) = B_N.$$

By the semigroup property, we have on the other hand:

$$(4.4)\qquad p_{3t}(x,y) \geq \int_{B_N} p_{2t}(x,u) p_t(u,y)\,du.$$

The second condition of (4.1) and the doubling property of the volume together with (4.3) (4.4) implies therefore the lower estimate (0.25). It remains therefore to construct a sequence $a_1, \dots, a_N$ that satisfies (4.1) and (4.2). But, by the geometry of Ω, only a_1 and a_N present a problem, because they have to satisfy (4.1) *and*

$$(4.5)\qquad \delta(a_1), \delta(a_N) \geq c\sqrt{t}.$$

The other points can then be picked up more or less at will on some curve that joins a_1 to a_N that stays away from the boundary $\partial\Omega$. I shall describe below two different constructions that allow us to have (4.1) and (4.5) simultaneously. Observe however that we only need to construct $a_1 = a$ with the following two properties:

$$(4.6)\qquad p_t(x,a) \geq C V_x^{-1}(\sqrt{t}) P(t,x); \quad \delta(a) \geq c\sqrt{t}.$$

For then the same construction, applied to the reversed diffusion $z^*(t)$ (*cf.* (0.17)), will give the last point a_N.

First Method. Based on the Parabolic Boundary Harnack Principle We shall start from the fact that:

$$(4.7)\qquad p_t(z,z) \geq C V_z^{-1}(\sqrt{t}); \quad z \in \Omega,\ \delta(z) \geq c\sqrt{t}.$$

This is a consequence of the local Gaussian. It follows that if $\delta(x) \geq c\sqrt{t}$ we can take $a = x$. If $\delta(x) \ll \sqrt{t}$ we shall take $a = x + (\sqrt{t}, 0, \dots, 0)$ (notations of (0.3). Let us consider the two parabolic functions.

$$(4.8)\qquad u(s,z) = p_s(z,a),\ v(s,z) = P(s,z); \quad s > 0,\ z \in \Omega.$$

These functions satisfy:

$$(4.9)\qquad u(t,a) \approx V_a^{-1}(\sqrt{t}),\ v(t,a) \approx 1; \quad 0 < t < T.$$

The first estimate (4.9) comes from (4.7), the second follows from the Gaussian mass escape applied to a ball of radius $\approx \sqrt{t}$ around a. The parabolic Harnack boundary principle (*cf.* [23] [17]; this principle is stated in (A.5) for the special case $\Omega =$

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Euclidean ball) will then be used and will allow us to "shift" the a in (4.9) to x close to the boundary. If this argument is done with care, it can easily absorb the time lag in the Harnack principle (because of the doubling property of the volume in (4.9)). The argument works therefore without having to resort to any kind of doubling property (or backwards estimates [17]) for the probability of life.

The only drawback that the above argument has is that it uses the boundary Harnack principle not only for parabolic functions (where the result can be found in the literature (*cf.* [23] [15] [16] [17]) but also n.p.a.s. For the operator M (even for the time dependent case *cf.* [43]) this is no problem because the adjoint operator M^* is of the same form as M. But for the operator L this has to be proved.

The Harnack boundary principle for p.n.a.s., as pointed out in the appendix B_4, holds good. But unfortunately the details of the proof are to be found nowhere. It follows therefore that unless I am prepared here to write down the proof of the Harnack principle for p.n.a.s. the above construction of the point a_N is incomplete.

Second Method. Based on the Upper Estimate Let Ω again be as in (0.3). Then the upper Gaussian estimate (0.25) allows us to assert that there exist $A > 0$ large enough s.t.

$$\begin{aligned} \mathbf{P}_x\big[\,|x - z(t)| \le A\sqrt{t}; \delta\big(z(t)\big) &\ge A^{-1}\sqrt{t}; \tau > t\big] \\ &= \int p_t(x, y)[|x - y| \le A\sqrt{t}; \delta(y) \ge A^{-1}\sqrt{t}]\, dy \\ &\ge \frac{1}{2}P(t, x); \quad x \in \Omega,\ 0 < t < T, \end{aligned} \tag{4.10}$$

where dy denotes as before the measure $d\mu(y)$, and $[\cdots]$ inside the integral denotes the indicator function. To see this we can scale and assume that $t = 1$. We have already seen in (3.1) that

$$P^{-1}(1, x) \int_{|x-y|>A} p_1(x, y)\, dy \to 0, \tag{4.11}$$

uniformly in x as $A \to \infty$. It remains to verify that for fixed $A > 0$ we can find $\varepsilon = \varepsilon(A) > 0$ s.t.

$$\int_{|x-y|\le A} p_1(x, y)[\delta(y) < \varepsilon]\, dy < \frac{1}{10}P(1, x). \tag{4.12}$$

If we use the upper estimate (0.25) we see that the left hand side of (4.12) can be estimated from above by

$$V_x^{-1}(1)P(1, x) \int_{|x-y|<A} [\delta(y) < \varepsilon]\, dy. \tag{4.13}$$

It remains therefore to show that for fixed $A > 0$

$$\frac{1}{V_x(1)} \int_{|x-y|<A} [\delta(y) < \varepsilon]\, dy \to 0, \tag{4.14}$$

uniformly in x, as $\varepsilon \to 0$. This is a consequence of (0.21) and (0.42).

From (4.10) it follows in particular that they exists $a \in \Omega$ s.t.

$$|x-a| \leq A\sqrt{t}, \quad \delta(a) \geq A^{-1}\sqrt{t}, \quad p_t(x,a) \geq CP(t,x)V_x^{-1}(\sqrt{t}), \tag{4.15}$$

where the doubling property is used once more. This concludes the construction for (4.6).

5 The Discrete Results

5.1 Centered Random Walks: (C) of Section 0.4

We have already seen how in the setting of the homogeneous random walk (0.34) in Ω (Ω is, say, as in (0.3)) we can prove the upper Gaussian estimate (3.23). We can also prove the following lower Gaussian estimate: There exists $c > 0$, that depends on $\mathrm{Lip}(\Omega)$, d and the parameters a and ε_0 (0.30), (0.31), of the measure (0.34) μ, s.t. the diffusion kernel and the probability of life (0.35) satisfy

$$p_n(x,y) \geq cn^{-\frac{d}{2}} P(n,x)P^*(n,y) \exp\left(-\frac{|x-y|^2}{cn}\right); \qquad x,y \in \Omega,\ |x-y| \leq cn,\ n \geq 0. \tag{5.1}$$

c should be thought as a sufficiently small constant. The restriction to the range:

$$|x-y| < ct, \tag{5.2}$$

is essential here. This is because, by (0.30), $p_n(x,y) = 0$ for $|x-y| \gg Cn$. Observe also that (5.1) holds trivially if $n < C^{-1}$ (for then $x = y$), so it is only a matter of proving it for n large enough. This result has important applications. I shall therefore explain below how one adapts the proof of Section 4 to make it work in this discrete setting (C) of (0.34).

The proof of (5.1) in this discrete setting consists of two parts:

Step 1: The Construction of the Points a_1 and a_N that Satisfy (4.1) and (4.5) From these the other intermediate points $a_2, \dots, a_{N-1}$ as in (1.8) (1.9) (1.10) (1.11) (4.2) can constructed.

Step 2: The Use of the Interior Parabolic Harnack in the Successive Steps Between a_j and a_{j+1}, $j = 1, 2, \dots$ Here we only need the parabolic Harnack and we do not need to worry about n.p.a.s. because the reversed process $z^*(n)$ $n = 1, \dots$ (*i.e.*, the analogue of (0.17)) is just the random walk generated by μ^*. It is here however that the condition (5.2) becomes essential.

Indeed there is no way that we can use that discrete parabolic Harnack (in any form whatsoever) unless the time step $t_{j+1} - t_j$, ($t_k = 0, 1, 2, \dots$) between $\dot{a}_{j+1}$ and $\dot{a}_j$ in (1.8) satisfies

$$\frac{t}{N} \approx t_{j+1} - t_j \geq 1. \tag{5.3}$$

This together with (1.11) forces us to restrict ourselves to the range (5.2). If the condition (5.2) is verified however the Step 2 can be carried out in the above discrete setting without any difficulty. We can for instance use the discrete parabolic Harnack that is proved in [30]. The results of [30] work in the setting of space unhomogeneous random walks (0.33). In Section 5.3 below I will outline a simple proof of this parabolic Harnack in the homogeneous case (0.34). To carry out the Step 1 on the other hand we only need the upper Gaussian estimate and (implicitly) the doubling property of $P(n, x)$. These facts have already been seen to hold for homogeneous random walks (*cf.* Section 3.4). This completes the proof of (5.1).

The above proof and (5.1) easily extends to the general unhomogeneous random walks (0.33) provided that we can carry out the following steps:

(1) Prove the parabolic Carleson principle.

(2) Define the analogue of the adjoined solutions $m(x)$ $x \in \mathbf{Z}^d$ (*cf.* (0.11)) and prove that they have the "correct" A_∞ properties.

(3) Use the adjoined solution $m(\cdot)$ to reverse the process and prove the Harnack principle for the corresponding n.p.a.s.

(4) Prove the Carleson principle for n.p.a.s.

(5) If we use the same strategy toward (3) we have to prove "en route" the parabolic Harnack boundary principle (*cf.* (A.5)) and also some form of a backwards estimate (*cf.* (A.6)). [5′) While "we are at it" we might as well prove the Harnack boundary principle for n.p.a.s. (*cf.* B_4) but this is not essential].

The above program is carried out in a forthcoming joint paper with S. Mustapha.

5.2 Reversible Chains: (R) of Section 0.4

The Gaussian estimate both upper and lower (with the additional range restriction (5.2) for the lower estimate) hold for the process in (R) of Section 0.4.

The situation here is considerably simpler because the local Gaussian and the parabolic Harnack are known to hold. As I already pointed out the local upper Gaussian, and the Gaussian mass escape where proved for the first time in [25]. This paper contains new ideas and is significant. Alternatively one can use the argument that we gave in Section 3 (which essentially is due to Usakov) to prove these facts.

The interior parabolic Harnack (in all scales) also holds in the context of (R). This fact in the classical setting (divergence form symmetric diffusion) is the celebrated Moser Harnack theorem *cf.* [35] [34] [36]. Moser's ideas have been adapted by various authors (including myself [41]) and have been made to work in various settings. In [11] one finds a "manifold" version that brings out the essential geometric features of Moser's proof. Another variant of Moser's proofs that puts [11] in a discrete setting and works for graphs can be found in [12]. For us this is what is needed because $\mathbf{Z}^d$ is a graph. All is well therefore for a symmetric (*i.e.*, $K(x, y) = K(y, x); x, y \in \mathbf{Z}^d$) Markov chain as in (R) Section 0.4.

Having this, the program 1) -5) described in the previous section can be carried out quite easily and the Gaussian estimate (0.25) follows.

In fact the only "real problem" in carrying the program 1) -5) out in the case of a centered random walk (0.33) was the analysis and the study of the p.n.a.s. in the discrete setting. This difficulty disappears in the case of the reversible Markov chains (0.32).

5.3 The Discrete Parabolic Harnack For Homogeneous Random Walks

In the proof of the lower Gaussian estimate that I gave in Section 5.1. I used the discrete parabolic Harnack of [30]. This is unfortunate because the paper [30] (which gives the discrete analogue of the Krylov-Safonov parabolic Harnack [29]) is long and technical. However in the case of a homogeneous random walks (0.34) a much easier proof of the parabolic Harnack estimate can be given. Here is how it goes:

Start from the local Gaussian estimate, which as we already pointed out is a consequence of standard [22] and relatively easy [40] [9] facts. Then use the procedure of [20] Section 3 to deduce Harnack. That procedure has its origin in one of the constructions of [29] and [8], but the way that it is presented in [20] it is easy and self contained. For homogeneous random walks this is therefore a much more satisfactory way to prove the parabolic Harnack estimate.

6 The Neumann Boundary Conditions (an Outline)

Only an outline of the Neumann estimate in Section 0.3 will be given here because the upper estimate is already known *cf.* [39] but I shall propose here an alternative approach. The Dirichlet form analysis that I developed in the decade of the 80's (*cf.* [41] [44]) applies here to the semigroup generated by the Dirichlet form:

$$D^2(\varphi) = \int_\Omega a_{ij}(x)\partial_i\varphi(x)\partial_j\varphi(x)\,dx; \quad \varphi \in C_0^\infty(\Omega)$$

closed with Neumann conditions. To be able to prove the basic estimate (*cf.* [41] [44]):

$$\|f\|_{\frac{2n}{n-2}} \leq CD(f); \quad f \in \mathrm{Dom}(\text{Neumann in } \Omega),\ n \geq 3,$$

(the case $n = 1, 2$, as usual, has to be treated *a posteriori*) one can use the Calderon extension theorems [7]. A strong form of that extension can be found in [38] where the author constructs a linear operator that extends $H_1(\bar{\Omega})$, the Sobolev space $(: f \in L^2, |\nabla f| \in L^2)$, and the $L_p(\bar{\Omega})$ spaces simultaneously, from $\bar{\Omega}$ to $\mathbf{R}^d$. Once this is done the diagonal estimate

$$p_t(x,x) \leq ct^{-\frac{d}{2}}; \quad x \in \Omega,\ t > 0,$$

follows by the general theory [44] [41]. The upper Gaussian (global on the whole of $\bar{\Omega}$) follows then by [25] or [39]. It should be noted that the original proof in [39] is more flexible (but more involved) and it applies to more general domains (*i.e.*, $\partial\Omega$ is not necessarily Lipschitz).

The lower Gaussian Neumann estimate all the way up to the boundary can now be deduced from the upper estimate as in Section 4. Once more it is not essential that $\partial\Omega$ should be Lipschitz for this argument to work. The fact that

$$|B|^{-1}\,\mathrm{Vol}_n[(\varepsilon - \text{Nhd of } \partial\Omega) \cap B] \to 0; \quad \varepsilon \to 0,$$

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uniformly for all Euclidean balls B suffices.

An Example (Without Proof!) Let $\Omega = (x_1 > 0)$ be the (genuine) upper half space in $\mathbf{R}^d$ and let $E \subset \partial\Omega = \mathbf{R}^{d-1}$ be the $(d-1)$-dimension Cantor set (*i.e.*, the product of the Cantor middle-$\frac{1}{3}$ set with itself $(d-1)$-times). Let T_t be the Heat diffusion semigroup (*i.e.*, generated by $\triangle = \sum \frac{\partial^2}{\partial x_i^2}$, or more generally by a symmetric $M(0.1)$, in Ω) with mixed boundary conditions:

$$\text{Dirichet on } E; \quad \text{Neumann on } \partial\Omega \setminus E.$$

Then the estimate (0.25) holds. The proof is not hard and generalizes to the more general subsets E that have the property that for any ball $B \subset \partial\Omega = \mathbf{R}^{d-1}$ there exists $B_1 \subset B \cap (\partial\Omega \setminus E)$ such that $|B_1| \geq \varepsilon_0 |B|$ for some fixed $\varepsilon_0 > 0$. All these sets E are of $(d-1)$-dimensional Lebesgue measure zero. For this reason I do not consider these examples as particularly significant, and I shall leave matters at that. The reader who wishes to write down the proof for himself should note that the above mixed boundary conditions correspond, by reflexion, to Dirichlet boundary conditions in $\Omega' = \mathbf{R}^d \setminus E$.

Appendix

A. Interior Estimates

***A_1. The Parabolic Hopf Principle for** L* It will be convenient here to define the parabolic ball not as in (0.39) but

$$D_r = [-8r^2, 8r^2] \times [|x| < r] = [-8r^2, 8r^2] \times B_r \subset \dot{\mathbf{R}}^d; \quad r > 0. \tag{A.1}$$

For any point $\xi = (t, Q) \in \partial_l D_r = [-8r^2, 8r^2] \times \partial B_r$ the lateral boundary of the cylinder D_r I shall denote by $\frac{\partial}{\partial \nu_Q}$ the derivative in the inward normal direction ν_Q. (In the notations and the letters used here I try in what follows, to stay close to P. Bauman's choices whose proof, in [4], I shall adapt). Let $u(\xi) \geq 0$ $(\xi = (t, Q) \in D_r)$ be a parabolic function (*i.e.*, $Lu = 0$ with L in (0.2). If we want the considerations that follow to hold for large as well as for small r we must impose the zero drift condition $b_i \equiv 0$ in (0.2)). I shall make the additional assumption that $u|_{\partial_l D_r} \equiv 0$, *i.e.*, that u vanishes on the lateral boundary of D_r. In this section we shall show that

$$\frac{\partial}{\partial \nu_Q} u(\xi) \approx \frac{c}{r} u(\zeta); \; \xi \in [-r^2, r^2] \times \partial B_r, \quad \zeta \in D_{\frac{r}{2}}, \; 0 < r < r_0. \tag{A.2}$$

The above estimate should be compared with the Lemma 4.3 in [4] where u is replaced by $\nu(\cdot)$ the time-independent function that vanishes on the lateral boundary

$$\nu(t, P) = \nu(P) = G(P, Y_0); \quad P \in B_r, \tag{A.3}$$

where G is the Green function of $\mathcal{B}$ in B_r with a pole at Y_0, with $|Y_0| = \frac{r}{4}$, say. The estimate (A.2) holds then (*cf.* [4]) for v:

$$\frac{\partial}{\partial \nu_Q} v(\xi) \approx \frac{c}{r} v(\zeta); \quad \xi \in [-4r^2, 4r^2] \times \partial B_r, \quad \zeta = (t, Q),\ |Q| = \frac{r}{2}. \tag{A.4}$$

We shall deduce (A.2) from (A.4) and from the parabolic Harnack principle at the boundary [23] [17]. With the notations as above and as in (0.41), this principle can be formulated as follows:

Let us fix $\xi = (t, Q) \in \partial_l D_r$ $|t| \leq r^2$ and let φ, ψ be two positive parabolic functions that are defined in some Nhd of ξ in D_r and vanish on the lateral boundary $\varphi|_{\partial_l D_r} = \psi|_{\partial_l D_r} = 0$. We then have

$$\frac{\varphi(\zeta)}{\psi(\zeta)} \leq C \frac{\varphi\big(\bar{A}_\rho(\xi)\big)}{\psi\big(\underline{A}_\rho(\xi)\big)}; \quad 0 < \rho < \rho_0 \text{ small enough},\ \zeta \in D_r,\ \xi - \zeta \in D_{\frac{\rho}{8}}. \tag{A.5}$$

Furthermore if we assume that φ is as above and is globally defined in D_r and vanish on the lateral boundary we can improve this estimate because we then have [15]:

$$\varphi(t_1, P) \approx \varphi(t_2, P); \quad P \in B_r,\ -r^2 < t_1,\ t_2 < r^2. \tag{A.6}$$

The proof of these Harnack type of principles can be found in [23] [17]. We can now conclude the proof of (A.2). Indeed, with the notations that we have already established, we have

$$\frac{\frac{\partial}{\partial \nu_Q} u(t, Q)}{\frac{\partial}{\partial \nu_Q} v(Q)} = \lim_{\varepsilon \to 0} \frac{u(t, Q + \varepsilon \nu_Q)}{v(Q + \varepsilon \nu_Q)}; \quad -2r^2 < t < 2r^2,\ Q \in \partial B_r. \tag{A.7}$$

If we combine (A.4) (A.5) (A.6) (A.7) the estimate (A.2) follows.

A_2. The Heat Diffusion Kernel Let $B \subset \mathbf{R}^d$ denote the Euclidean unit ball and let $p_t(Q, Y)$ $(Q, Y \in B)$ be the diffusion kernel of L in B. In out context here we shall think of

$$u(t, Q) = p_t(Q, Y) \quad \text{for } t > 0,\ u(t, \cdot) \equiv 0 \quad \text{for } t \leq 0,$$

which is a parabolic function for $(t, Q) \neq (0, Y)$, *i.e.*, the Green function of $(\frac{\partial}{\partial t} - L)$ in $\dot{B} = \mathbf{R} \times B$ (*cf.* [13]). If we use these notations, and put together the estimates of the previous section, with $r = 1$ we obtain

$$\frac{\partial}{\partial \nu_Q} p_t(Q, Y) \leq C\Pi(Y) = C p_1(0, Y); \quad Q \in \partial B,\ t \leq t_0,\ |Y| \leq \frac{1}{2}. \tag{A.8}$$

Where C depends on t_0 here, we also have

$$\frac{\partial}{\partial \nu_Q} p_s(Q, Y) \approx \Pi(Y); \tag{A.9}$$

$$\frac{p_{s_1}(Q, Y_1)}{p_{s_2}(Q, Y_2)} \approx \frac{\Pi(Y_1)}{\Pi(Y_2)}; \tag{A.10}$$

$$Y, Y_1, Y_2 \in B_{\frac{1}{2}},\ Q \in B,\ 0 < \delta_0 \leq s,\ s_1, s_2 \leq \delta_0^{-1},$$

where the constants depend on δ_0. To prove the estimate (A.10) we perform the shift $s_1 \to s_2 = s_1 + \alpha$ and apply the comparison results (A.5) (A.6) to the two functions

$$\varphi(s, Q) = p_s(Q, Y_1); \quad \psi(s, Q) = p_{s+\alpha}(Q, Y_2),$$

the time independence of L is used in an essential way here (*cf.* [15]).

A*$_3$ *The Interior Parabolic Adjoint Harnack Principle I shall preserve the notations of Section A_1, Section A_2 (with $r = 1$) and I shall consider $u^{\sim}(t, Y) \geq 0$ $(0 < t < 4$, $Y \in B)$ some p.n.a.s. as in (0.40). By a simple use of Green's formula (*cf.* [13] for the classical case and [4] for the elliptic case) we have:

$$\text{(A.11)} \qquad \begin{aligned} \widetilde{u}(\xi) = \widetilde{u}(t, Y) &\approx \int_{\partial_l} \widetilde{u}(t - \tau, Q) \frac{m(Q)}{m(Y)} \frac{\partial}{\partial \nu_Q} p_\tau(Q, Y)\, ds(\tau, Q)) \\ &\quad + \int_{\partial_0 = [Q \in B]} \widetilde{u}(0, Q) \frac{m(Q)}{m(Y)} p_t(Q, Y)\, dQ \\ &= I_l(t, Y; \widetilde{u}) + I_0(t, Y; \widetilde{u}); \quad t > 0,\ Y \in B. \end{aligned}$$

Here $p_t(Q, Y)$ is the diffusion kernel of L in the unit ball B, $\partial_l = [t, 0] \times \partial B$ is the lateral boundary of the cylinder, so that $0 < \tau < t$ on ∂_l, and $s(\tau, Q)$ denotes the n-dimensional surface measure of the lateral boundary. ∂_0 is the bottom boundary $[\tau = t]$ of the cylinder. The $\approx$ indicates once more positive constants that only depend on d and λ.

A simple use of the Section A_2 estimates and of (A.14) that will be proved below implies:

$$\text{(A.12)} \qquad I_0(\xi_1; \widetilde{u}) \approx I_0(\xi_2; \widetilde{u}); \quad \xi_1, \xi_2 \in [1, 2] \times B_{\frac{1}{2}},$$

uniformly in $u^{\sim}$. Let us denote by

$$\text{(A.13)} \qquad F(Y) = \frac{\Pi(Y)}{m(Y)}; \quad Y \in B,$$

We then have:

$$\text{(A.14)} \qquad F(Y_1) \approx F(Y_2); \quad Y_1, Y_2 \in B_{\frac{1}{2}}.$$

To see this we fix $\varepsilon_0 > 0$ and apply the formula (A.11) to $u^{\sim} \equiv 1$, $t = 1$. We have

$$\text{(A.15)} \qquad 1 \approx F(Y) \int_B m(Q) \frac{P_1(Q, 0)}{\Pi(0)}\, dQ + \int_{\partial_l; \tau > \frac{1}{2}} + \int_{\partial_l; \varepsilon_0 < \tau < \frac{1}{2}} + \int_{\partial_l; 0 < \tau < \varepsilon_0}$$

where (A.10) is used in I_0. The estimates (A.8), (A.9) imply then that, for ε_0 small enough, and $Y \in B_{\frac{1}{2}}$, the second term in the right hand side of (A.15) will absorb the fourth (no matter what the additional constants that are involved in the comparisons of A_2 might be). The fourth term can thus be ignored and we are left with

$$1 \approx F(Y)\Big[\int_B m(Q)\frac{P_1(Q,0)}{\Pi(0)}\,dQ + \int_{\partial_l;\tau>\varepsilon_0} m(Q)\,ds(\tau,Q)\Big]; \quad |Y| \le \frac{1}{2}, \tag{A.16}$$

by (A.9) (with constants that depend on the choice of ε_0). The estimate (A.14) follows.

Let us now consider $\xi_i = (t_i, y_i)$, $i = 1, 2$ two points such that $Y_1, Y_2 \in B_{\frac{1}{2}}$ and with $1 < t_1 < t_2 < 2$, $\varepsilon_0 < t_2 - t_1 < 2\varepsilon_0$, for some $\varepsilon_0 > 0$ to be chosen later. We have

$$I_l(\xi_i, \widetilde{u}) = \int_{\tau<t_2-t_1} + \int_{\tau>t_2-t_1} = I_1 + I_2; \quad i = 1, 2. \tag{A.17}$$

Using the decomposition (A.17) and the estimates of A_2 as before we see that if ε_0 is small enough we have (with constants that depend on ε_0)

$$I_l(\xi_2) \approx F(Y_2)\int_{\partial_l;t_2-t_1<\tau<t_2} \widetilde{u}(t_2-\tau;Q)m(Q)\,ds, \tag{A.18}$$

$$I_l(\xi_1) \le C_{\varepsilon_0}F(Y_1)\int_{\partial_l;0<\tau<t_1} \widetilde{u}(t_1-\tau;Q)m(Q)\,ds. \tag{A.19}$$

Since the integrals on the right hand sides of (A.18) and (A.19) are identical, if we combine (A.12) (A.14) (A.17) we obtain the required Harnack estimate (for $r = 1$):

$$\widetilde{u}(\xi_1) \le C\,\widetilde{u}(\xi_2). \tag{A.20}$$

This can be scaled to any $0 < r < r_0$, and if the drift term in L vanishes (: $b_1 \equiv 0$) it can be scaled to any $r > 0$.

B. Boundary Estimates

B_1 Lower Bound of the Caloric Measure Let $0 \in \partial\Omega$ with Ω as in (0.3) and let $B_r = [|x_1| \le cr, |x'| < r]$ and D_r be as in (0.39). We shall denote by $\dot{\Omega} = \mathbf{R} \times \Omega$ and by $h_r(E)$, $E \subset \partial(D_r \cap \dot{\Omega})$ the caloric measure of L in $D_r \cap \dot{\Omega}$ at the point $\bar{A}_r = (\theta_1 r^2; c\theta_2 r, 0) \in D_r$ (with the same c as in the definition of B_r) with $\theta_1, \theta_2 \in (0, \frac{1}{2})$. The caloric measure is the hitting probability of the process $\big(-t, z(t)\big) = \dot{z}(t)$, $t > 0$ (*cf.* (0.6), [13] [15]). We have then:

$$h_r(D_r \cap \partial\dot{\Omega}) > C; \quad 0 < r < r_0, \tag{A.21}$$

where $C = C\big(\lambda, d, \mathrm{Lip}(\Omega), r_0\big)$ but is independent of r and of θ_1, θ_2.

The argument to prove (A.21) is easy. Let $P(r)$ be the probability of the paths that have the following properties:

$$\dot{z}(0) = \bar{A}_r,\ \dot{z}(s) \subset D_r; \quad 0 < s < t,$$

and such that the distance of $z(t)$ from the point $x_1 = -r$, $x' = 0$ is $\ll r$, and where we assume that $t \approx r^2$. Such paths clearly exit $D_r \cap \dot{\Omega}$ at $D_r \cap \partial\dot{\Omega}$ and we have $h_r(D_r \cap \partial\dot{\Omega}) \geq P(r)$. We have $P(r) \geq c > 0$ and (A.21) follows. To see this last point the reader could observe that the local Gaussian (0.27) implies, by scaling, a lower estimates:

$$p_t^R(x, y) \geq C\,\mathrm{Gs}(t; x, y);\ x, y \in B_{(1-\delta)R}(0) \quad 0 < t < AR^2,$$

where p^R indicates the diffusion killed at ∂B_R, and the constant C only depends on d, λ, δ and A but is uniform in R.

We can easily improve and obtain similar lower bound for the caloric and the adjoint caloric measure of the set $D_r \cap \partial\dot{\Omega} \cap (t < 0)$. To see this we just have to condition on the position of $\dot{z}$ as it crosses the level $t = 0$.

B*$_2$. *The Hölder Continuity at the Boundary Let the notations be as before with $0 \in \partial\Omega$ and for any function φ let as denote by:

$$M(r) = M(\varphi; r) = \sup[\varphi(t, x); -r^2 < t < 0; x \in B_r \cap \Omega].$$

The representation of a parabolic function u or of a n.p.a.s. $u^\sim$ in terms of caloric or adjoint caloric measure together with B_1 implies that for every $0 < \theta < 1$ there exists $0 < \rho < 1$ s.t. the above indicator $M(r)$ for u or $u^\sim$ satisfies

$$M(\theta r) \leq \rho M(r); \quad 0 < r < r_0. \tag{A.22}$$

provided that $u\big|_{\partial\dot{\Omega}} \equiv u^\sim\,|_{\partial\dot{\Omega}} \equiv 0$. An immediate consequence of this is that:

$$u(\xi), \widetilde{u}(\xi) = O\big(d^\alpha(\xi, \partial\dot{\Omega})\big) \tag{A.23}$$

where $\alpha > 0$ depends on d, λ, $\mathrm{Lip}\,\Omega$. The proof of (A.23) that one finds in the literature (*cf.* [18] [27]) for solutions $\mathcal{B}u = 0$, or for parabolic functions, depend on the construction of appropriate barrier functions and not on Gaussian estimates. The advantage of our approach is that it is symmetric and works equally well for parabolic functions as well as for n.p.a.s.

B*$_3$ *The Carleson Principle for the Operator L This principle asserts that

$$M(\varphi; r) \leq C\varphi(\bar{A}_r); \quad 0 < r < r_0, \tag{A.24}$$

where $M(\cdot)$ is as in the previous section $\bar{A}_r$ is as in (0.41) and $\varphi \geq 0$ is either a parabolic function or a n.p.a.s. in $D_{10r} \cap \dot{\Omega}$ (Section 0.6(C)) that satisfies $\varphi|_{\partial\dot{\Omega}=0}$. This follows from the interior Harnack estimates (H) or ($\tilde{\mathrm{H}}$) that hold for φ and from (A.22). The argument goes as follows: We normalize $r = 1$, $\varphi(\bar{A}_1) = 1$. Let then $m > 0$ be large enough and let us assume $\xi_1 = (t_1, x_1)$, $-1 < t_1 < 0, x_1 \in B_1 \cap \Omega$ is such that $\varphi(\xi_1) > m$. The parabolic Harnack on the other hand implies the "chain condition" (*cf.* [28] Section 1.3 adapted to the parabolic setting). This together with the normalization $\varphi(A_1) = 1$ forces $\dot{d}(\xi_1, \partial\dot{\Omega}) < Cm^{-c}$. One then uses (A.22) to find $\xi_2 \in D_{10} \cap \dot{\Omega}$ with an even higher value $\varphi(\xi_2) \geq \rho^{-a}\varphi(\xi_1)$ and a parabolic distance $\dot{d}(\xi_1, \xi_2) \leq Cm^{-c}\theta^{-a}$, where a can be chosen at will. The normalization $\varphi(A_1) = 1$ is used again and so on. The sequences so constructed $\xi_1, \xi_2, \dots$, if $m > 0$ is large enough, will stay in D_2 (*cf.* (0.39)) and has therefore a limit point. This gives a contradiction. This argument goes back to Carleson [8] and can be found in the literature in a large number of places (*e.g.* [26] Lemma 4.4). The reader should check this and fill in the details for himself.

It is worth noting that the same type of argument can be used to show that the interior Harnack estimate (H), ($\tilde{\text{H}}$) will follow if we already know that the local Gaussian estimate holds. This is explained in details in [20] (Section 3).

B_4 The Boundary Harnack Principle For (N.P.A.S.) The result described here is not essential for the paper but it is related. The claim is that the (local) parabolic boundary Harnack principle holds for the comparison of two nonnegative n.p.a.s. defined in $D_r \cap \dot{\Omega}$ that vanish on $D_r \cap \partial\dot{\Omega}$. One way that this can be seen is by adapting the proof in [3] [2]. The proofs in [3], [2] are probabilistic and therefore they easily adapt to the time space process $\big(-t, z(t)\big) \in \dot{\mathbf{R}}^d$ (*cf.* (0.6) [13]). Furthermore we have at our disposal the necessary tools (($\tilde{\text{H}}$), *etc.*) to adapt these proofs equally well to the process $\big(-t, z^*(t)\big) \in \dot{\mathbf{R}}^d$.

I do not claim of course that the above is a proof. But since no essential use will be made of this boundary Harnack principle for (n.p.a.s.) I will leave matters at that.

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