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# SOLUTION OF THE 'CUBE' FUNCTIONAL EQUATION IN TERMS OF 'TRILINEAR COEFFICIENTS'[1]

BY
H. T. HEMDAN

1. **Introduction.** We consider the following functional equations

$$\begin{aligned}(1)\qquad & f(x+\nu, y+\nu, z+\nu)+f(x+\nu, y+\nu, z-\nu)+f(x+\nu, y-\nu, z+\nu)\\ &\quad +f(x+\nu, y-\nu, z-\nu)+f(x-\nu, y+\nu, z+\nu)+f(x-\nu, y+\nu, z-\nu)\\ &\quad +f(x-\nu, y-\nu, z+\nu)+f(x-\nu, y-\nu, z-\nu)\\ &=8f(x, y, z),\end{aligned}$$

$$\begin{aligned}(2)\qquad & f(x+\nu, y, z)+f(x-\nu, y, z)+f(x, y+\nu, z)+f(x, y-\nu, z)\\ &\quad +f(x, y, z+\nu)+f(x, y, z-\nu)\\ &=6f(x, y, z),\end{aligned}$$

$$\begin{aligned}(3)\qquad & f(x+\nu, y+\nu, z+\nu)+f(x+\nu, y+\nu, z-\nu)+f(x+\nu, y-\nu, z+\nu)\\ &\quad +f(x+\nu, y-\nu, z-\nu)+f(x-\nu, y+\nu, z+\nu)+f(x-\nu, y+\nu, z-\nu)\\ &\quad +f(x-\nu, y-\nu, z+\nu)+f(x-\nu, y-\nu, z-\nu)\\ &=f(x+\nu, y, z)+f(x-\nu, y, z)+f(x, y+\nu, z)+f(x, y-\nu, z)\\ &\quad +f(x, y, z+\nu)+f(x, y, z-\nu)+2f(x, y, z),\end{aligned}$$

where $f: R^3 \to R$.

Considering their geometric meaning, equations (1) and (2) are known as 'Cube' and 'Octahedron' functional equations, respectively. Under the assumption of continuity, Haruki [2] has proved that (1) and (2) are equivalent. Etigson [3], has proved the equivalence of (1) and (2) under no regularity assumption. We will give here another proof. Also, under the assumption of continuity, Haruki has solved the 'Cube' functional equation. He gave the solution as a certain polynomial of fifth degree in $x$, $y$, $z$ individually whose terms are the partial derivatives of a given polynomial.

In this paper, we first show that each of the 'Cube' and 'Octahedron' functional equations is also equivalent to (3) under the assumption of continuity, then, under this assumption, we give the solution of these functional equations in a form different from that given in [2]. The solution will appear as a certain polynomial of fourth degree in $x$, $y$, $z$ individually with trilinear coefficients, as are defined in the next section.

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2. **Definitions.** In the two dimensional space $xy$, a general bilinear polynomial of the first degree both in $x$ and $y$ will be called a 'Bilinear Coefficient' and will be denoted by $\beta_i$. If $b_i$, $e_i$, $g_i$, $k_i$ are real constants, then

$$\beta_i \equiv b_i xy + e_i x + g_i y + k_i.$$

An ordered 4-tuple of elements may be used to denote $\beta_i$, that is

$$\beta_i \equiv (b_i, e_i, g_i, k_i).$$

In a similar way bilinear coefficients in $xz$ and $yz$ spaces may be defined and denoted by $\gamma_i$ and $\delta_i$ respectively.

Also, in the three dimensional space $xyz$, a general trilinear polynomial of the first degree in $x$, $y$, $z$, will be called a 'Trilinear Coefficient' and will be denoted by $\alpha_i$. If $a_i$, $b_i$, $c_i$, $d_i$, $e_i$, $g_i$, $h_i$, $k_i$ are real constants, then

$$\alpha_i \equiv a_i xyz + b_i xy + c_i xz + d_i yz + e_i x + g_i y + h_i z + k_i.$$

An ordered 8-tuple of elements may be used to denote $\alpha_i$, that is

$$\alpha_i \equiv (a_i, b_i, c_i, d_i, e_i, g_i, h_i, k_i).$$

3. **Notations.** For convenience when dealing with functions of three variables, we shall use the symbols $X^\nu$, $Y^\nu$, $Z^\nu$, respectively, for the linear translation operators (which are commutative and distributive)

$$(4) \qquad \begin{cases} X^\nu f(x, y, z) = f(x+\nu, y, z), \\ Y^\nu f(x, y, z) = f(x, y+\nu, z), \\ Z^\nu f(x, y, z) = f(x, y, z+\nu), \end{cases}$$

in place of the customary symbols $E_x^\nu$, $E_y^\nu$, $E_z^\nu$.

Using these symbols, (1) may be written

$$(5) \quad (X^\nu Y^\nu Z^\nu + X^\nu Y^\nu Z^{-\nu} + X^\nu Y^{-\nu} Z^\nu + X^\nu Y^{-\nu} Z^{-\nu} + X^{-\nu} Y^\nu Z^\nu + X^{-\nu} Y^\nu Z^{-\nu} + X^{-\nu} Y^{-\nu} Z^\nu + X^{-\nu} Y^{-\nu} Z^{-\nu}) f(x, y, z) = 8f(x, y, z),$$

or, in factored form,

$$(6) \qquad \{(X^\nu + X^{-\nu})(Y^\nu + Y^{-\nu})(Z^\nu + Z^{-\nu})\} f(x, y, z) = 8f(x, y, z).$$

Similarly, (2) and (3) may be written

$$(7) \qquad (X^\nu + X^{-\nu} + Y^\nu + Y^{-\nu} + Z^\nu + Z^{-\nu}) f(x, y, z) = 6f(x, y, z),$$

$$(8) \quad \{(X^\nu + X^{-\nu})(Y^\nu + Y^{-\nu})(Z^\nu + Z^{-\nu})\} f(x, y, z) = (X^\nu + X^{-\nu} + Y^\nu + Y^{-\nu} + Z^\nu + Z^{-\nu} + 2) f(x, y, z).$$

In addition to the above notations, we introduce the following. Let

$$(9)\quad \begin{cases} X_\nu = X^\nu + X^{-\nu}, & X_{2\nu} = X^{2\nu} + X^{-2\nu}, \dots \\ Y_\nu = Y^\nu + Y^{-\nu}, & Y_{2\nu} = Y^{2\nu} + Y^{-2\nu}, \dots \\ Z_\nu = Z^\nu + Z^{-\nu}, & Z_{2\nu} = Z^{2\nu} + Z^{-2\nu}, \dots \\ L_\nu = X_\nu + Y_\nu + Z_\nu, \dots \\ M_\nu = 2(X_\nu Y_\nu + X_\nu Z_\nu + Y_\nu Z_\nu), \dots \\ N_\nu = X_\nu Y_\nu Z_\nu, \dots \end{cases}$$

According to these notations we will have

$$(10)\quad \begin{cases} X_\nu^2 = X_{2\nu} + 2, \dots \\ Y_\nu^2 = Y_{2\nu} + 2, \dots \\ Z_\nu^2 = Z_{2\nu} + 2, \dots \\ L_\nu^2 = L_{2\nu} + M_\nu + 6, \dots \\ M_\nu^2 = 16L_{2\nu} + 2M_{2\nu} + 8L_\nu N_\nu + 48, \dots \\ N_\nu^2 = 4L_{2\nu} + M_{2\nu} + N_{2\nu} + 8. \end{cases}$$

Equation (6) will become

$$N_\nu f(x, y, z) = 8f(x, y, z),$$

or, more simply, equations (6), (7) and (8) may be written

$$(11)\quad N_\nu = 8$$

$$(12)\quad L_\nu = 6$$

$$(13)\quad N_\nu = L_\nu + 2.$$

4. **Equivalence of** (1), (2), **and** (3). We now state and prove the following theorem.

THEOREM 1. *Under no regularity assumptions, equations* (1), (2) *are equivalent, while under the assumption of continuity each of* (1) *and* (2) *is equivalent to* (3).

**Proof.** We first prove the equivalence of (1) and (2) (or (11) and (12)). We show that (12) implies (11). Given

$$L_\nu = 6$$
$$\therefore L_\nu^2 = 36.$$

Substituting from (10) and using $L_{2\nu} = 6$, *we get* $M_\nu = 24$

$$(14)\quad \therefore\ M_\nu^2 = (24)^2.$$

Following the same procedure as before, we get $N_\nu = 8$, which is (11).

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Conversely, we show that (11) implies (12). That is, given

$$N_\nu = 8$$
$$\therefore\ N_\nu^2 = 64.$$

Following the same procedure and using $N_{2\nu} = 8$, we get

$$M_\nu = 48 - 4L_\nu. \tag{15}$$

Squaring both sides of (15), using $N_\nu = 8$ and substituting for $M_\nu$, $M_{2\nu}$ from the same equation we get

$$L_{2\nu} = 64L_\nu - 378. \tag{16}$$

Replacing $\nu$ by $2\nu$ and using the same equation we get

$$L_{4\nu} = (64)^2 L_\nu - (65)(378). \tag{17}$$

Squaring (16) and using (10) we get

$$L_{4\nu} + M_{2\nu} + 6 = (64)^2(L_{2\nu} + M_\nu + 6) - (128)(378)L_\nu + (378)^2, \tag{18}$$

using (15), (16), (17) we finally get

$$193{,}536 L_\nu = 1{,}161{,}216,$$

that is

$$L_\nu = 6.$$

This completes the proof of the first part of the theorem. Now we prove the other part of the theorem. Equations (1) and (2) are equivalent without any continuity assumption (as we have just proved) and they obviously imply (3). We show the converse, that is, (3) implies either (1) or (2) under the assumption of continuity. Given is now

$$N_\nu = L_\nu + 2.$$

Squaring both sides of the above equation and using (10) we get

$$M_{2\nu} + 4L_{2\nu} = M_\nu + 4L_\nu.$$

Replacing $\nu$ by $\nu/2$ we get

$$M_\nu + 4L_\nu = M_{\nu/2} + 4L_{\nu/2},$$

and by iteration we get

$$M_\nu + 4L_\nu = M_{\nu/2^n} + 4L_{\nu/2^n}.$$

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Since $f$ is continuous, take the limit as $n \to \infty$, noticing that

$$\lim_{n\to\infty} L_{\nu/2^n} = 6,$$

$$\lim_{n\to\infty} M_{\nu/2^n} = 24,$$

$$\lim_{n\to\infty} N_{\nu/2^n} = 8.$$

Thus

$$M_\nu + 4L_\nu = 48$$
$$4L_\nu = 48 - M_\nu.$$

From this equation, we may proceed in a similar manner as from (15) on, to get

$$L_\nu = 6,$$

as we wanted to show. This completes the proof of the theorem.

5. **Solution of the 'Cube' functional equation.** In terms of the trilinear coefficients $\alpha_i$, we state and prove the following theorem.

THEOREM 2. *If and only if $f: R^3 \to R$ is continuous and satisfies the 'Cube' functional equation* (1) *for all $x$, $y$, $z$, then*

$$f(x, y, z) = \sum_{\substack{0\le i,j,k\le 2\\ i+j+k\le 3}} \alpha_{ijk} x^{2i} y^{2j} z^{2k}, \quad \alpha_{111} \equiv 0$$

*where $\alpha_{ijk} \equiv \alpha_l$, $1 \le l \le 17$, are trilinear coefficients which are not all independent.*

If

$$\begin{array}{llll} \alpha_{210} \equiv \alpha_1, & \alpha_{120} \equiv \alpha_2, & \alpha_{110} \equiv \alpha_3, & \alpha_{201} \equiv \alpha_4, \\ \alpha_{102} \equiv \alpha_5, & \alpha_{101} \equiv \alpha_6, & \alpha_{021} \equiv \alpha_7, & \alpha_{012} \equiv \alpha_8, \\ \alpha_{011} \equiv \alpha_9, & \alpha_{200} \equiv \alpha_{10}, & \alpha_{100} \equiv \alpha_{11}, & \alpha_{020} \equiv \alpha_{12}, \\ \alpha_{010} \equiv \alpha_{13}, & \alpha_{002} \equiv \alpha_{14}, & \alpha_{001} \equiv \alpha_{15}, & \alpha_{000} \equiv \alpha_{16}, \\ \alpha_{111} \equiv \alpha_{17} \equiv 0, & & & \end{array}$$

then, the 'dependence relations' between the coefficients are as follows:

$$\alpha_2 = (-a_1, -b_1, -5c_1/3, -3d_1/5, -5e_1/3, -3g_1/5, -h_1, -k_1),$$
$$\alpha_4 = (-a_1, -3b_1, -c_1/3, -d_1, -e_1, -3g_1, -h_1/3, -k_1),$$
$$\alpha_5 = (a_1, 5b_1, c_1/3, 3d_1/5, 5e_1/3, 3g_1, h_1/5, k_1),$$
$$\alpha_7 = (a_1, 3b_1, 5c_1/3, d_1/5, 5e_1, 3g_1/5, h_1/3, k_1),$$
$$\alpha_8 = (-a_1, -5b_1, -c_1, -d_1/5, -5e_1, -g_1, -h_1/5, -k_1),$$

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and $(\alpha_3, \alpha_9, \alpha_{12})$, $(\alpha_6, \alpha_9, \alpha_{14})$, $(\alpha_3, \alpha_6, \alpha_{10})$, $(\alpha_{11}, \alpha_{13}, \alpha_{15})$ are related by the following four sets of conditions

$$\begin{aligned}
3a_3+3a_9+10a_{12}&=0 & 3e_3+e_9+6e_{12}&=0\\
3b_3+b_9+10b_{12}&=0 & g_3+g_9+10g_{12}&=0\\
c_3+c_9+2c_{12}&=0 & h_3+3h_9+6h_{12}&=0\\
d_3+3d_9+10d_{12}&=0 & k_3+k_9+6k_{12}&=0
\end{aligned}$$

$$\begin{aligned}
3a_6+3a_9+10a_{14}&=0 & 3e_6+e_9+6e_{14}&=0\\
b_6+b_9+2b_{14}&=0 & g_6+3g_9+6g_{14}&=0\\
3c_6+c_9+10c_{14}&=0 & h_6+h_9+10h_{14}&=0\\
d_6+3d_9+10d_{14}&=0 & k_6+k_9+6k_{14}&=0
\end{aligned}$$

$$\begin{aligned}
3a_3+3a_6+10a_{10}&=0 & e_3+e_6+10e_{10}&=0\\
3b_3+b_6+10b_{10}&=0 & 3g_3+g_6+6g_{10}&=0\\
c_3+3c_6+10c_{10}&=0 & h_3+3h_6+6h_{10}&=0\\
d_3+d_6+2d_{10}&=0 & k_3+k_6+6k_{10}&=0
\end{aligned}$$

$$\begin{aligned}
a_{11}+a_{13}+a_{15}&=0 & 3e_{11}+e_{13}+e_{15}&=0\\
3b_{11}+3b_{13}+b_{15}&=0 & g_{11}+3g_{13}+g_{15}&=0\\
3c_{11}+c_{13}+3c_{15}&=0 & h_{11}+h_{13}+3h_{15}&=0\\
d_{11}+3d_{13}+3d_{15}&=0 & k_{11}+k_{13}+k_{15}&=0.
\end{aligned}$$

Except for these relations between the coefficients $\alpha_l$, they are arbitrary. This shows that the solution contains 128 real constants but only 56 of them can be chosen arbitrary. One way of choosing the arbitrary constants is to take all the constants of each linear coefficient as either independent (arbitrary) or dependent (and in that sense we may consider the linear coefficient as independent or dependent). Thus only 7 linear coefficients are independent. These are: one of $(\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_7, \alpha_8)$, five of $(\alpha_3, \alpha_6, \alpha_9, \alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{15})$ and $\alpha_{16}$. For example, we can consider $\alpha_1, \alpha_3, \alpha_6, \alpha_9, \alpha_{11}, \alpha_{13}, \alpha_{16}$ as independent, the others as dependent.

**Proof.** To prove this theorem, we need the following three lemmas:

LEMMA 1. *If $f:R^3 \to R$ is of class $C^\infty$ and satisfies the 'Cube' functional equation, then $(\partial^{i+j+k}/\partial x^i\, \partial y^j\, \partial z^k)f$, $i, j, k = 1, 2, 3, \ldots$ also satisfy the 'Cube' functional equation.*

(See [2]). The proof of this lemma is easy.

LEMMA 2. *If and only if $f:R^2 \to R$ is continuous and satisfies the following functional equation*

$$f(x+\nu, y+\nu)+f(x+\nu, y-\nu)+f(x-\nu, y+\nu)+f(x-\nu, y-\nu)=4f(x, y),$$

*which is known as 'Square' functional equation, then*

(19) $$f(x, y) = axy(x^2 - y^2) + b(3x^2y - y^3) + c(x^3 - 3xy^2) + dxy + e(x^2 - y^2) + gx + hy + k,$$

*where a, b, c, d, e, g, h, k are real constants.*

The proof of this lemma is given in [1] and will not be repeated here.

LEMMA 3. *If* $f: R^3 \to R$ *is of class* $C^\infty$ *and satisfies the 'Cube' functional equation for all x, y, z, then*

$$\partial^6 f/\partial x^6 = 0, \quad \partial^6 f/\partial y^6 = 0, \quad \partial^6 f/\partial z^6 = 0, \quad \partial^6 f/\partial x^2\, \partial y^2\, \partial z^2 = 0.$$

**Proof.** In Theorem 1, we have proved the equivalence of 'Cube' and 'Octahedron' functional equations. The latter may be written as

(20) $$X_\nu + Y_\nu + Z_\nu = 6.$$

Multiply both sides of (20) by $X_\nu$, add $Y_\nu Z_\nu$ to both sides and use (14) to get

(21) $$X_\nu^2 + 12 = 6X_\nu + Y_\nu Z_\nu.$$

Multiply both sides of (21) by $X_\nu$ and use (11) to get

$$X_\nu^3 - 6X_\nu^2 + 12X_\nu - 8 = 0,$$

which is

(22) $$(X^{3\nu} + X^{-3\nu} - 6X^{2\nu} - 6X^{-2\nu} + 15X^{\nu} + 15X^{-\nu} - 20)f(x, y, z) = 0,$$

that is

(23) $$f(x+3\nu, y, z) + f(x-3\nu, y, z) - 6f(x+2\nu, y, z) - 6f(x-2\nu, y, z) + 15f(x+\nu, y, z) + 15f(x-\nu, y, z) - 20f(x, y, z) = 0.$$

Differentiate (23) six times with respect to $\nu$ and put $\nu = 0$, to get for all $x, y, z$

(24) $$\partial^6 f/\partial x^6 = 0,$$

as we wanted to show. In a similar manner we can show that

(25) $$\partial^6 f/\partial y^6 = 0 \quad \text{and} \quad \partial^6 f/\partial z^6 = 0.$$

To prove that $\partial^6 f/\partial x^2\, \partial y^2\, \partial z^2 = 0$, we make use of the above results, as follows:

Differentiate (1) twice with respect to $\nu$ and put $\nu = 0$ to get for all $x, y, z$

(26) $$\frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial z^2} f = 0.$$

Differentiate (1) four times with respect to $\nu$ and put $\nu = 0$ to get for all $x, y, z$

(27) $$\frac{\partial^4}{\partial x^4} f + \frac{\partial^4}{\partial y^4} f + \frac{\partial^4}{\partial z^4} f + 6\left(\frac{\partial^4}{\partial x^2\, \partial y^2} f + \frac{\partial^4}{\partial x^2\, \partial z^2} f + \frac{\partial^4}{\partial y^2\, \partial z^2} f\right) = 0.$$

Differentiate (26) twice with respect to $x$, $y$, $z$ respectively, add these last three equations and use (27) to get for all $x$, $y$, $z$

(28) $$\frac{\partial^4}{\partial x^4} f + \frac{\partial^4}{\partial y^4} f + \frac{\partial^4}{\partial z^4} f = 0.$$

Differentiate (26): (a) twice with respect to $x$ then twice with respect to $y$, (b) twice with respect to $x$ then twice with respect to $z$ and (c) twice with respect to $y$ then twice with respect to $z$, add these last three equations to get

(29) $$3\frac{\partial^6}{\partial x^2 \partial y^2 \partial z^2} f + \frac{\partial^6}{\partial x^4 \partial y^2} f + \frac{\partial^6}{\partial x^4 \partial z^2} f + \frac{\partial^6}{\partial y^4 \partial x^2} f + \frac{\partial^6}{\partial y^4 \partial z^2} f + \frac{\partial^6}{\partial z^4 \partial x^2} f + \frac{\partial^6}{\partial z^4 \partial y^2} f = 0.$$

Differentiate (28) twice with respect to $x$, $y$, $z$ respectively, add these last three equations, use (24) and (25) and compare with (29) to get for all $x$, $y$, $z$

(30) $$\frac{\partial^6}{\partial x^2 \partial y^2 \partial z^2} f = 0,$$

as we wanted to show. This completes the proof of the lemma.

Using distributions, the assumption that $f$ is of class $C^\infty$ can be reduced to $f$ is continuous since the distribution for $f$ satisfies Laplace equation as we have seen before.

Now we will start proving the theorem. In terms of the bilinear coefficients $\beta_i$, the solution (19) of the 'Square' functional equation, may be written in the following form

$$f(x, y) = \beta_1 x^2 + \beta_2 y^2 + \beta_3,$$

where the coefficients $\beta_1$ and $\beta_3$ are independent, while $\beta_2$ depends on $\beta_1$ and is given by $\beta_2 = (-e_1, -3b_1, -\frac{1}{3}g_1, -k_1)$.

Equation (30) implies

(31) $$(\partial^4/\partial x^2 \partial y^2) f(x, y, z) = z\phi_1(x, y) + \psi_1(x, y).$$

Using lemma 1 and substituting in (1), we find that each of $\phi_1$ and $\psi_1$ satisfies the 'Square' functional equation, thus we may write

$$\phi_1(x, y) = \beta_1 x^2 + \beta_2 y^2 + \beta_3,$$
$$\psi_1(x, y) = \beta_4 x^2 + \beta_5 y^2 + \beta_6,$$

and equation (31) may be written in the following form

(32) $$(\partial^4/\partial x^2 \partial y^2) f(x, y, z) = \alpha_1^1 x^2 + \alpha_2^1 y^2 + \alpha_3^1,$$

where $\alpha_1^1$, $\alpha_2^1$, $\alpha_3^1$ are trilinear coefficients as defined before. Equation (32) implies

$$\frac{\partial^2}{\partial x^2} f(x, y, z) = \alpha_1 x^2 y^2 + \alpha_2 y^4 + \alpha_3 y^2 + (y\phi_2(x, z) + \psi_2(x, z)). \tag{33}$$

Differentiating (33) twice with respect to $z$, we get

$$\frac{\partial^4}{\partial x^2 \partial z^2} f(x, y, z) = y \frac{\partial^2}{\partial z^2} \phi_2(x, z) + \frac{\partial^2}{\partial z^2} \psi_2(x, z).$$

Following the same procedure as before, we will find that each of $(\partial^2/\partial z^2)\phi_2$ and $(\partial^2/\partial z^2)\psi_2$ satisfies the 'Square' functional equation (in the $xz$ space), thus we may write

$$\frac{\partial^2}{\partial z^2} \phi_2(x, z) = \gamma_1 x^2 + \gamma_2 z^2 + \gamma_3, \tag{34}$$

$$\frac{\partial^2}{\partial z^2} \psi_2(x, z) = \gamma_4 x^2 + \gamma_5 z^2 + \gamma_6. \tag{35}$$

The last two equations imply

$$\begin{aligned} y\phi_2(x, z) + \psi_2(x, z) = \alpha_4 x^2 z^2 + \alpha_5 z^4 + \alpha_6 z^2 \\ + (yzf_1(x) + yf_2(x) + zf_3(x) + f_4(x)). \end{aligned} \tag{36}$$

Substitute from (36) into (33) and differentiate this last equation four times with respect to $x$ and use lemma 3 to get

$$yzf_1^{(4)}(x) + yf_2^{(4)}(x) + zf_3^{(4)}(x) + f_4^{(4)}(x) = 0, \tag{37}$$

which implies, by differentiation with respect to $y$ and $z$, that each of $f_1$, $f_2$, $f_3$, $f_4$ is a polynomial of the third degree in $x$. Equation (33) implies

$$\begin{aligned} f(x, y, z) = \alpha_1 x^4 y^2 + \alpha_2 x^2 y^4 + \alpha_3 x^2 y^2 + \alpha_4 x^4 z^2 + \alpha_5 x^2 z^4 \\ + \alpha_6 x^2 z^2 + x^2 yzf_1(x) + x^2 yf_2(x) + x^2 zf_3(x) \\ + x^2 f_4(x) + (x\phi_3(y, z) + \psi_3(y, z)). \end{aligned} \tag{38}$$

Differentiating (38) twice with respect to $y$ and twice with respect to $z$ we get

$$\frac{\partial^4}{\partial y^2 \partial z^2} f(x, y, z) = x \frac{\partial^4}{\partial y^2 \partial z^2} \phi_3(y, z) + \frac{\partial^4}{\partial y^2 \partial z^2} \psi_3(y, z). \tag{39}$$

Following the same procedure as before, we find that each of $(\partial^4/\partial y^2 \partial z^2)\phi_3$ and $(\partial^4/\partial y^2 \partial z^2)\psi_3$ satisfies the 'Square' functional equation (in the $yz$ space), thus we may write

$$\frac{\partial^4}{\partial y^2 \partial z^2} \phi_3(y, z) = \delta_1 y^2 + \delta_2 z^2 + \delta_3, \tag{40}$$

$$\frac{\partial^4}{\partial y^2 \partial z^2} \psi_3(y, z) = \delta_4 y^2 + \delta_5 z^2 + \delta_6, \tag{41}$$

and, as before, these last two equations imply

$$x\phi_3(y, z)+\psi_3(y, z)=\alpha_7y^4z^2+\alpha_8y^2z^4+\alpha_9y^2z^2 + xzf_5(y)+xf_6(y)+zf_7(y)+f_8(y) + \iint (xyf_9(z)+xf_{10}(z)+yf_{11}(z)+f_{12}(z))\,dz\,dz,$$

and equation (38) may be rewritten as

$$\begin{aligned} f(x, y, z) &= \alpha_1x^4y^2+\alpha_2x^2y^4+\alpha_3x^2y^2+\alpha_4x^4z^2+\alpha_5x^2z^4 \\ &+\alpha_6x^2z^2+\alpha_7y^4z^2+\alpha_8y^2z^4+\alpha_9y^2z^2 \\ &+x^2yzf_1(x)+x^2yf_2(x)+x^2zf_3(x)+x^2f_4(x) \\ &+xzf_5(y)+xf_6(y)+zf_7(y)+f_8(y) \\ &+\iint (xyf_9(z)+xf_{10}(z)+yf_{11}(z)+f_{12}(z))\,dz\,dz. \end{aligned} \tag{42}$$

Differentiating this equation six times with respect to $y$ and using lemma 3, we get

$$xzf_5^{(6)}(y)+xf_6^{(6)}(y)+zf_7^{(6)}(y)+f_8^{(6)}(y)=0. \tag{43}$$

As before, the above equation implies that each of $f_5$, $f_6$, $f_7$, $f_8$ is a polynomial of the fifth degree in $y$. Also by differentiating (42) six times with respect to $z$ and using lemma 3, we get

$$xyf_9^{(4)}(z)+xf_{10}^{(4)}(z)+yf_{11}^{(4)}(z)+f_{12}^{(4)}(z)=0, \tag{44}$$

which implies that each of $f_9$, $f_{10}$, $f_{11}$, $f_{12}$ is a polynomial of the third degree in $z$. Finally equation (42) may be written in the following form

$$\begin{aligned} f(x, y, z) &= \alpha_1x^4y^2+\alpha_2x^2y^4+\alpha_3x^2y^2+\alpha_4x^4z^2+\alpha_5x^2z^4 \\ &+\alpha_6x^2z^2+\alpha_7x^4z^2+\alpha_8y^2z^4+\alpha_9y^2z^2+\alpha_{10}x^4 \\ &+\alpha_{11}x^2+\alpha_{12}y^4+\alpha_{13}y^2+\alpha_{14}z^4+\alpha_{15}z^2+\alpha_{16}. \end{aligned}$$

and when the summation sign is used, the above solution may be written in the following final form

$$f(x, y, z)=\sum_{\substack{0\le i,j,k\le 2\\ i+j+k\le 3}} \alpha_{ijk}x^{2i}y^{2j}z^{2k},\quad \alpha_{111}\equiv 0, \tag{45}$$

where $\alpha_{ijk}\equiv\alpha_l$ are trilinear coefficients as defined before. This completes the proof of the 'if' part of the theorem.

To prove the converse, that is, the function given by (45) and satisfying the 'dependence relations' (given in the statement of the theorem), satisfies the 'Cube' functional equation (1), it is enough to show that (45) satisfies (2). This can be shown easily if we expand the left side of (2) using Taylor's formula

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around the point $(x, y, z)$. We notice that the 'dependence relations' are consequences of (26) and (28). This completes the proof of the theorem.

6. **Particular solutions of the 'Cube' functional equation.** As mentioned before we will take the coefficients $\alpha_1$, $\alpha_3$, $\alpha_6$, $\alpha_9$, $\alpha_{11}$, $\alpha_{13}$, $\alpha_{16}$ as independent, the others as dependant. A special solution arises from the choice

$$\alpha_1 = (0, 0, 0, 0, 0, 0, 0, 0),$$
$$\alpha_1 = \alpha_3 = \alpha_6 = \alpha_9 = \alpha_{11} = \alpha_{13},$$
$$\alpha_{16} = (1, 1, 1, 1, 1, 1, 1, 1).$$

From the 'dependence relations' it follows that all other coefficients are zeros and equation (45) gives

$$f(x, y, z) = xyz + xy + xz + yz + x + y + z + 1,$$

which is a polynomial of the first degree in each of $x$, $y$, $z$. Another particular solution results from the choice

$$\alpha_1 = \alpha_3 = \alpha_6 = \alpha_9 = (0, 0, 0, 0, 0, 0, 0, 0),$$
$$\alpha_{11} = \alpha_{13} = \alpha_{16} = (1, 1, 1, 1, 1, 1, 1, 1).$$

The 'dependence relations' show that all of $\alpha_2$, $\alpha_4$, $\alpha_5$, $\alpha_7$, $\alpha_8$, $\alpha_{10}$, $\alpha_{12}$, $\alpha_{14}$ are zeros, while $\alpha_{15}$ is given by

$$\alpha_{15} = (-2, -6, -\tfrac{4}{3}, -\tfrac{4}{3}, -4, -4, -\tfrac{2}{3}, -2),$$

and we get the following solution

$$f(x, y, z) = (x^2 + y^2 + 1)(xyz + xy + xz + yz + x + y + z + 1)$$
$$-2z^2(xyz + 3xy + 2xz/3 + 2yz/3 + 2x + 2y + z/3 + 1),$$

which is a polynomial of the third degree in each of $x$, $y$, $z$. It can be easily shown that these particular solutions satisfy the 'Cube' functional equation by following the proof of the second half of Theorem 2.

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FACULTY OF MATHEMATICS,
UNIVERSITY OF WATERLOO,
WATERLOO, ONTARIO, CANADA