A PROPERTY OF CONVEX PSEUDOPOLYHEDRA

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In this note we prove one theorem and make a few conjectures, all of which are connected with the following problem raised by S. Mazur [1]: does there exist a closed convex surface whose plane sections give all plane closed convex curves, up to affinities? For our purposes we define a convex pseudopolyhedron to be the closed convex hull of a countable bounded nonplanar sequence of points in E_3 with exactly one limit point.

THEOREM 1. There exists a convex pseudopolyhedron U the set of whose plane sections contains a triangle similar to any preassigned triangle. We start with the following

LEMMA. Let $A(\alpha)$ be the trihedral angle whose vertex angles are $\pi/2$, $\pi/2$ and α, $0 < \alpha < \pi$. Let T be a triangle with angles a, b, and c, and let $\max(a, b, c) < \alpha$. Then T is similar to a plane section of $A(\alpha)$.

The proof, which is easily obtainable by inspection or by computation, is elementary and will be omitted.

To construct U consider first an increasing sequence $\{\alpha_n\}$, $n = 1, 2, \ldots$, of positive numbers with $\lim \alpha_n = \pi$. Let $\{A(\alpha_n)\}$ be the associated sequence of trihedral angles as in the lemma. Let S be the sphere of radius 1. On S we draw a sequence $\{C_n\}$, $n = 1, 2, \ldots$, of circles, subject to these conditions: the distance between the centres of any two circles is greater than the sum of their radii, the radii are steadily decreasing and tend to 0, and the sequence of the centres has exactly one limit point s.

Let W_n be the circular cone touching S along C_n. Take now $A(\alpha_1)$ and place it so that the following conditions are satisfied: S is tangent to the three edges of $A(\alpha_1)$ the vertex v_1 of $A(\alpha_1)$ is within W_1, and C_1 is within $A(\alpha_1)$. It is clear that these conditions can be satisfied if α_1 is sufficiently close to π. Let p_1, q_1, r_1 be the points of tangency of S to the edges of $A(\alpha_1)$ and

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p_1, q_1, r_1 be the points of tangency of S to the edges of $A(\alpha_1)$ and let v_1 be the position of the vertex of $A(\alpha_1)$. Repeat the same operation with $A(\alpha_2)$ and C_2, then with $A(\alpha_3)$ and C_3, and so on. In this way we obtain four sequences of points: $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ and $\{v_n\}$. Their union K is a countable bounded non-planar set with exactly one limit point, namely s. Therefore the closed convex hull U of K is a pseudopolyhedron. If follows from this construction that U has, among others, vertices $v_1, v_2, \ldots$, and within a sufficiently small neighbourhood of v_n U coincides with $A(\alpha_n)$.

Let T be an arbitrary triangle. Let a, b, c be its angles and let a = max (a, b, c). There is a member of $\{\alpha_n\}$, say α_m, such that $a < \alpha_m$. Then by the lemma some plane section of $A(\alpha_m)$ is similar to T. Therefore some section of U by a plane sufficiently close to its vertex v_m is also similar to T. This completes the proof.

We conclude with several related problems, stated here as conjectures.

(1) Theorem 1 does not hold for any convex polyhedron.
(2) There exists a convex pseudopolyhedron the set of whose plane sections contains a quadrilateral affinely equivalent to any preassigned convex quadrilateral.
(3) The above is not true if 'affinely equivalent' is replaced by 'similar', of if 'quadrilateral' is replaced by 'pentagon', even if one allows general convex solids instead of convex pseudopolyhedra.

REFERENCES

1. S.M. Ulam, Mathematical Problems, mimeographed notes, (Los Alamos, 1954).

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