Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 December 2023.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

107.E (Stan Dolan)

Let p be a prime of the form $4k + 1$ and split the numbers $1, 2, \ldots, 2k$ into pairs (x, y) such that $x < y$ and $x^2 + y^2 \equiv 0 \pmod p$. Find the sum of all such y.

[For example, when $p = 5$ there is just one pair, (1, 2), and the sum is 2. When $p = 13$ there are three pairs, (1, 5), (2, 3), (4, 6), and the sum is 14.]

107.F (Michael Fox)

The point M lies in the plane of a parallelogram $KLK'L'$ but is not collinear with any pair of its vertices, nor concyclic with any triple. The circles KLM, $K'L'M$ meet again in N; circles $K'LM$, $KL'M$ meet again in N'; circles $K'LN$, $KL'N$ meet again in M'. Prove that K, L, M', N' are concyclic, as are K', L', M', N', and show that the eight named points lie on a conic.

107.G (Mihály Bencze)

Let M be an interior point of a triangle ABC and let R_a, R_b, R_c be the circumradii of triangles MBC, MCA, MAB respectively. Prove that:

(a) $\displaystyle\sum \frac{R_a}{\sqrt{MB.MC}} \geqslant 3,$

(b) $\displaystyle 3 + \sum \frac{R_a}{\sqrt{MB.MC}} \geqslant \sum \frac{1}{\sin\left(\frac{1}{4}B\hat{M}C\right)}.$

107.H (Isaac Sofair)

The probability that a biased coin turns up heads when tossed is p. Derive closed form expressions for the following probabilities:

(a) The probability of getting no consecutive heads in n tosses of the coin.

(b) The probability that, if n people sitting around a circular table each flip the coin, no two adjacent people flip heads.

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Solutions and comments on **106.I**, **106.J**, **106.K**, **106.L** (November 2022).

106.I (Stan Dolan)

Find all non-square positive integers n such that the number of positive integer solutions of $x^2 - 4y^2 = n$ is odd.

Answer: $n \in \{pm^2, 4qm^2, 32m^2\}$ for primes $p \equiv 1 \pmod 4$ and $q \equiv 3 \pmod 4$.

There was an impressive variety of analyses of this problem, including other (equivalent) ways of expressing the answer. The solution that follows is due to the proposer, Stan Dolan, and deals neatly with the various sub-cases that arise.

Let the prime factorisation of n be $n = 2^\alpha \prod_{i=1}^{k} p_i^{\alpha_i}$ where p_i are distinct odd primes. Let $u = x - 2y$, $v = x + 2y$. Then we require the number of solutions of $uv = n$, $u \equiv v \pmod 4$, $u < v$, to be odd. Since n is not a square, the number of solutions without the $u < v$ condition must be singly even (i.e. twice an odd number).

If a solution has $u \equiv v \equiv 1$ *or* $3 \pmod 4$

Then $\alpha = 0$ and $\sum \alpha_i$ is even for primes $p_i \equiv 3 \pmod 4$.

Then all solutions of $uv = n$ satisfy $u \equiv v \pmod 4$ and we require the number of solutions, $\prod(\alpha_i + 1)$, to be singly even. Hence $n = pm^2$ for a prime $p \equiv 1 \pmod 4$ and odd integer m.

If a solution has $u \equiv v \equiv 2 \pmod 4$

Then $\alpha = 2$ and relevant solutions of $uv = n$ have both u and v singly even. Again, $\prod(\alpha_i + 1)$ must be singly even. Hence $n = 4pm^2$ for an odd prime p and odd integer m.

If a solution has $u \equiv v \equiv 0 \pmod 4$

Then $\alpha \geqslant 4$ and relevant solutions of $uv = n$ have u and v both multiples of 4. We then require $(\alpha - 3)\prod(\alpha_i + 1)$ to be singly even. Hence $n = 16pm^2$ for any prime p (including 2) and odd integer m.

The form of the solutions can then be re-expressed as in the given answer.

Correct solutions were received from: M. G. Elliott, I. Hadinata, P. F. Johnson, J. A. Mundie, Z. Retkes, J. Siehler, G. Strickland and the proposer Stan Dolan.

106.J (David Bevan)

Given $n \in \mathbb{N}$ and $\varphi \in \mathbb{R}$, let $S = S(\varphi)$ be the set of n equally distributed points on the unit circle that contain the point $(\cos\varphi, \sin\varphi)$. Let R be the smallest axis-parallel rectangle enclosing S. For each $n > 1$, determine how many values of $\varphi \in [0, 2\pi/n)$ there are for which R is a square.

Answer: If $n \equiv 0 \pmod 4$, R is a square for all $\varphi \in \left[0, \frac{2\pi}{n}\right)$.
If $n \equiv 2 \pmod 4$, R is a square for the 2 values $\varphi = \frac{\pi}{2n}, \frac{3\pi}{2n}$.
If n is odd, R is a square for the 4 values $\varphi = \frac{\pi}{4n}, \frac{3\pi}{4n}, \frac{5\pi}{4n}, \frac{7\pi}{4n}$.

All the solutions received for this problem were clear and detailed; the succinct solution which follows is due to the proposer, David Bevan.

Let $\alpha = \frac{2\pi}{n}$.

If $n \equiv 0 \pmod 4$, then for any φ, S exhibits four-fold rotational symmetry about the origin. Thus R is a square for all values of $\varphi \in [0, \alpha)$.

Note that if S contains one of four diagonal points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$, then S exhibits symmetry about one of the two diagonals, and so R is a square.

Suppose that $n \equiv 2 \pmod 4$. Observe first that $S\left(\frac{\alpha}{2}\right)$ is a 90° rotation of $S(0)$, so the set of values of φ for which R is a square has a period of $\frac{\alpha}{2}$. It can be checked that as φ increases from 0 to $\frac{\alpha}{2}$, the width of R decreases monotonically and its height increases monotonically, so there is just one value of $\varphi \in [0, \frac{\alpha}{2})$ for which they are equal and R is square. But $S\left(\frac{\alpha}{4}\right)$ contains a diagonal point, so the required value of φ is $\frac{\alpha}{4}$. By periodicity, there are thus two values of $\varphi \in [0, \alpha)$ for which R is a square: $\frac{\alpha}{4}$ and $\frac{3\alpha}{4}$.

For odd n, the argument is similar. In this case, $S\left(\frac{\alpha}{4}\right)$ is a 90° rotation of $S(0)$, the width of R increases and its height decreases as φ increases from 0 to $\frac{\alpha}{4}$, and $S\left(\frac{\alpha}{8}\right)$ contains a diagonal point. So, by periodicity, there are four values of $\varphi \in [0, \alpha)$ for which R is a square: $\frac{\alpha}{8}, \frac{3\alpha}{8}, \frac{5\alpha}{8}, \frac{7\alpha}{8}$.

Correct solutions were received from: M. G. Elliott, J. A. Mundie, Z. Retkes, and the proposer David Bevan.

106.K (K. S. Bhanu and M. N. Deshpande)

Prove that $$\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n-1}{k}} = 1.$$

What is the corresponding result for $$\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{2n}{k}}\ ?$$

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Answer: The second sum is $\frac{n}{n + 1}$.

This problem attracted a wide variety of solutions with several solvers noting that the following more general identity holds and gives the two required summations as special cases:

$$\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{N}{k}} = \frac{n}{N - n + 1} \text{ for } n \leqslant N. \qquad (*)$$

Samuele Riccarelli, James Mundie, Soham Bhadra and Ulrich Abel proved this by manipulating binomial coefficients.

We have $\frac{\binom{n}{k}}{\binom{N}{k}} = \frac{n!\,(N - k)!}{N!\,(n - k)!} = \frac{\binom{N - k}{n - k}}{\binom{N}{n}}$ and

$$\sum_{k=1}^{n} \binom{N - k}{n - k} = \sum_{r=0}^{n-1} \binom{N - n + r}{r} = \sum_{r=0}^{n-1} \binom{N - n + r + 1}{r} - \binom{N - n + r}{r - 1}$$

$$= \binom{N}{n - 1}, \text{ by telescoping}$$

– a result sometimes known as the "hockey-stick lemma". Thus

$$\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{N}{k}} = \frac{\binom{N}{n - 1}}{\binom{N}{n}} = \frac{n}{N - n + 1}.$$

Henry Ricardo gave a neat probabilistic proof of $(*)$.

Consider an urn with n black balls and $N + 1 - n$ white balls and draw balls randomly and without replacement until a white ball is obtained. The probability this occurs when $k + 1$ balls are drawn is

$$\frac{n}{N + 1} \cdot \frac{n - 1}{N} \cdot \ldots \cdot \frac{n - k + 1}{N - k + 2} \cdot \frac{N - n + 1}{N - k + 1} = \frac{\binom{n}{k}}{\binom{N}{k}} \cdot \frac{N - n + 1}{N + 1}.$$

Since $\sum_{k=0}^{n} \frac{\binom{n}{k}}{\binom{N}{k}} \cdot \frac{N - n + 1}{N + 1} = 1$, we deduce that

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$$\sum_{k=1}^{n} \frac{\binom{n}{k}}{\binom{N}{k}} = \frac{N+1}{N-n+1} - 1 = \frac{n}{N-n+1}.$$

James Mundie located an identity equivalent to (∗) as a problem in [1].

Reference

1. R. L. Graham, D. E. Knuth and O. Patshniz, *Concrete mathematics: a foundation for computer science* (2nd ed.), Addison-Wesley (1994) pp. 173-175.

Correct solutions were received from: U. Abel, S. Bhadra, M. G. Elliott, I. Hadinata, P. F. Johnson, J. A. Mundie, Z. Retkes, H. Ricardo (2 solutions), S. Riccarelli, J. Siehler, S. Stewart and the proposers K. S. Bhanu and M. N. Deshpande.

106.L (Ankush Kumar Parcha and Toyesh Prakash Sharma)

Find the values of the following infinite nested radicals

(a) $\sqrt[3]{2 + 2\sqrt[3]{3 + 6\sqrt[3]{4 + 12\sqrt[3]{5 + 20\sqrt[3]{6 + 30\sqrt[3]{7 + \ldots}}}}}}$,

(b) $\sqrt[3]{-1 + 7\sqrt[3]{-1 + 13\sqrt[3]{-1 + 21\sqrt[3]{-1 + 31\sqrt[3]{-1 + 43\sqrt[3]{-1 + \ldots}}}}}}$.

Answer: (a) 2, (b) –3.08462… .

As a number of respondents noted, we have to be careful about the definition and convergence of the sequences which define these infinite nested radicals. Stan Dolan established the following general result.

Let a_i, b_i, c_i be polynomials in i, which are positive for $i = 1, 2, \ldots$ and satisfy $c_i^3 = a_i + b_i c_{i+1}$.

Then $T_i(x) = \sqrt[3]{a_i + b_i x}$ is a monotone increasing function with $T_i(c_{i+1}) = c_i$. Hence, for all $n \geqslant m$, $T_m \ldots T_n(0) < T_m \ldots T_n(c_{n+1}) = c_m$, so the increasing sequence $T_m \ldots T_n(0)$ converges to a limit L_m satisfying $L_m \leqslant c_m$ and the recurrence relation $L_i^3 = a_i + b_i L_{i+1}$ (∗). This limit assigns a meaning to the infinite nested radical

$$L_m = \sqrt[3]{a_m + b_m\sqrt[3]{a_{m+1} + b_{m+1}\sqrt[3]{\ldots}}}.$$

We claim that $L_m = c_m$ for all m. For if $\frac{L_i}{c_i} < 1$, then $\frac{L_{i+1}}{c_{i+1}} = \frac{L_i^3 - a_i}{c_i^3 - a_i} < \left(\frac{L_i}{c_i}\right)^3$ and so, as $n \to \infty$, $\frac{L_n}{C_n}$ would tend to 0 exponentially. Since a_i, b_i, c_i are positive polynomials, L_n would then tend to 0 exponentially as well, contradicting (∗),

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(a) Here, we take $a_i = i + 1$, $c_i = i + 1$, $b_i = i(i + 1)$ corresponding to the identity $(i + 1)^3 = i + 1 + i(i + 1)(i + 2)$. Then the required limit is $L_1 = c_1 = 2$.

(b) Here, we take $a_i = -1$, $c_i = i + 2$, $b_i = i^2 + 3i + 3$ corresponding to the identity $(i + 2)^3 = -1 + \left(i^2 + 3i + 3\right)(i + 3)$. But here we must be careful: a_i is *not* positive.

The first part of the proof above still works to show that $T_m \ldots T_n(0)$ converges to a limit, but the second part does not go through to establish that $L_m = c_m$. Numerical evaluation gives the required limit $L_1 \approx -3.08462$.

For (b), a number of solvers chose a different truncation procedure corresponding to

$$3 = \sqrt[3]{-1 + 7 \times 4} = \sqrt[3]{-1 + 7\sqrt[3]{-1 + 13 \times 5}} = \sqrt[3]{-1 + 7\sqrt[3]{-1 + 13\sqrt[3]{-1 + 21 \times 6}}} = \ldots$$

which gives 3 as its limit.

Correct solutions were received from: S. Bhadra, S. Dolan, M. G. Elliott, I. Huseynov, J. A. Mundie, P. Pandey, D. Pinchon, Z. Retkes, C. Starr, K. Tummebaze, L. Wimmer and the proposers Ankush Kumar Parcha and Toyesh Prakash Sharma.

N.J.L.

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