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Abstract

We show that a genus 2 curve over a number field whose jacobian has complex multiplication will usually have stable bad reduction at some prime. We prove this by computing the Faltings height of the jacobian in two different ways. First, we use a known case of the Colmez conjecture, due to Colmez and Obus, that is valid when the CM field is an abelian extension of the rationals. It links the height and the logarithmic derivatives of an L-function. The second formula involves a decomposition of the height into local terms based on a hyperelliptic model. We use the reduction theory of genus 2 curves as developed by Igusa, Liu, Saito, and Ueno to relate the contribution at the finite places with the stable bad reduction of the curve. The subconvexity bounds by Michel and Venkatesh together with an equidistribution result of Zhang are used to bound the infinite places.

Contents

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1. Introduction

By a curve we mean a smooth, geometrically connected, projective curve C defined over a field k. Its jacobian variety $\mathrm{Jac}(C)$ is a principally polarised abelian variety defined over k. For any abelian variety A defined over k, we write $\mathrm{End}(A)$ for the ring of geometric endomorphisms of A, i.e., the ring of endomorphisms of the base change of A to a given algebraic closure of k. For brevity we say that A has CM if its base change to an algebraic closure of k has complex multiplication and if k has characteristic 0. We also say that C has CM if $\mathrm{Jac}(C)$ does. A curve defined over $\overline{\mathbb{Q}}$ is said to have good reduction everywhere if it has potentially good reduction at all finite places of a number field over which it is defined.

By the work of Serre and Tate [ST68], an abelian variety defined over a number field with CM has potentially good reduction at all finite places. If a curve of positive genus which is defined over a number field has good reduction at a given finite place, then so does its jacobian variety. However, the converse statement is false in the genus 2 case; cf. entry $[I_0\text{-}I_0\text{-}m]$ in Namikawa and Ueno's classification table [NU73] in equicharacteristic 0. The main result of our paper, which we discuss in greater detail below, states that this phenomenon prevails for certain families of CM curves of genus 2.

Theorem 1.1. *Let F be a real quadratic number field. Up to isomorphism there are only finitely many curves C of genus 2 defined over $\overline{\mathbb{Q}}$ with good reduction everywhere and such that $\mathrm{End}(\mathrm{Jac}(C))$ is the maximal order of a quartic, cyclic, totally imaginary number field containing F.*

This finiteness result is of a familiar type for objects in arithmetic geometry. A number field has only finitely many unramified extensions of given degree due to the theorem of Hermite–Minkowski. The Shafarevich conjecture, proved by Faltings [Fal83], ensures that again there are only finitely many curves defined over a fixed number field, of fixed positive genus, with good reduction outside a fixed finite set of places. Fontaine [Fon85, p. 517] proved that there is no non-zero abelian variety of any dimension with good reduction at all finite places if one fixes the field of definition to be $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(i\sqrt{3})$, or $\mathbb{Q}(\sqrt{5})$. In particular, there exists no curve over $\mathbb{Q}$ of positive genus that has good reduction at all primes. Schoof obtained finiteness results along these lines for certain additional cyclotomic fields [Sch03].

Let us stress here that there are infinitely many curves of genus 2 defined over $\overline{\mathbb{Q}}$ with good reduction everywhere. One can deduce this fact from Moret-Bailly [MB01, Exemple 0.9].

Our result does not seem to be a direct consequence of the theorems mentioned above. Instead of working over a fixed number field our finiteness result concerns curves over the algebraically closed field $\overline{\mathbb{Q}}$. Indeed, it is not possible to uniformly bound the degree over $\mathbb{Q}$ of a curve of genus 2 whose jacobian variety has complex multiplication.

Example 1.2. Let us exhibit an infinite family of genus 2 curves with CM such that the endomorphism ring is the ring of algebraic integers in a cyclic extension of $\mathbb{Q}$ that contains $\mathbb{Q}(\sqrt{5})$.

Suppose that $p \equiv 1 \mod 12$ is a prime; then

$$f = x^4 + 10px^2 + 5p^2$$

has roots

$$\pm\sqrt{-p(5 \pm 2\sqrt{5})}. \tag{1.1}$$

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So, the splitting field $K = K_p$ of f over $\mathbb{Q}$ is a CM field with maximal totally real subfield $\mathbb{Q}(\sqrt{5})$. The product of two roots lies in $\mathbb{Q}(\sqrt{5})$, so $K/\mathbb{Q}$ is a Galois extension. But such a product does not lie in $\mathbb{Q}$, so the Galois group of $K/\mathbb{Q}$ is not isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. This means that $K/\mathbb{Q}$ is a cyclic extension.

By (1.1), K is ramified above p, so we obtain infinitely many fields K as infinitely many primes p satisfy $p \equiv 1 \mod 12$.

There exists a principally polarised abelian surface whose endomorphism ring is the ring of algebraic integers in K; see for example the paragraph after the proof of Theorem 4 of van Wamelen [vWa99a].

In our situation this abelian surface is necessarily simple; cf. Lemma 3.10 below.

A principally polarised abelian surface that is not a product of elliptic curves with the product polarisation is the jacobian of a curve of genus 2 by [BL04, Corollary 11.8.2]. Therefore, there is a curve $C = C_p$ defined over a number field such that $\mathrm{Jac}(C)$ has complex multiplication by the ring of algebraic integers in K. According to Theorem 1.1, the curve C has potentially good reduction everywhere for at most finitely many p.

We set $m = (p-1)/12 \in \mathbb{Z}$ and observe that

$$2^{-4} f(2x+1) = x^4 + 2x^3 + (30m+4)x^2 + (30m+3)x + 45m^2 + 15m + 1$$

is irreducible modulo 2 and modulo 3. This implies that $K/\mathbb{Q}$ is unramified above these primes and even that they are inert in K. We may apply Goren [Gor97, Theorem 1] to see that the semi-stable reduction of $\mathrm{Jac}(C)$ at all places above 2 and 3 is isogenous but not isomorphic to a product of supersingular elliptic curves. By the paragraph before Proposition 2 of Liu [Liu93], the curve C has potentially good reduction at places above 2 and 3. So, bad reduction is not a consequence of the obstruction described by Ibukiyama *et al.* [IKO86, Theorem 3.3(III)]; cf. Goren and Lauter's comment of [GL07, p. 477].

The proof of Theorem 1.1 relies heavily on various aspects of the stable Faltings height $h(A)$ of an abelian variety A defined over a number field. Indeed, it follows by computing the said height of $\mathrm{Jac}(C)$ in two different ways if C is a genus 2 curve defined over $\overline{\mathbb{Q}}$. We will be able to bound one of these expressions from below and the other one from above. The resulting inequality will yield Theorem 1.4 below, a more precise version of our result above.

The first expression of the Faltings height of $\mathrm{Jac}(C)$ uses the additional hypothesis that C has CM as in Theorem 1.1. We will use Colmez's conjecture, a theorem in our case due to Colmez [Col93] and Obus [Obu13], as the CM field K is an abelian extension of $\mathbb{Q}$. Yang [Yan10] was the first to prove the Colmez conjecture in some non-abelian quartic cases. Colmez's conjecture enables us to express $h(\mathrm{Jac}(C))$ in terms of the logarithmic derivative of an L-function. Using this presentation, Colmez [Col98] found a lower bound for the Faltings height of an elliptic curve with CM when the endomorphism ring is a maximal order. The bound grows logarithmically in the discriminant of the CM field. We recall that the discriminant Δ_K of K is a positive integer as K is a quartic CM field. In our case we obtain a lower bound which is linear in $\log \Delta_K$. Let B be a real number. So, by the theorem of Hermite–Minkowski, there are only finitely many possibilities for K up to isomorphism if $h(\mathrm{Jac}(C)) \leqslant B$. In the situation of Theorem 1.1, the endomorphism ring of $\mathrm{Jac}(C)$ is the maximal order of K. So, there are only finitely many possibilities for $\mathrm{Jac}(C)$ up to isomorphism for fixed B. Torelli's theorem will imply that there are at most finitely many possibilities for C up to isomorphism.

Our theorem would follow if we could establish a uniform height upper bound B as before. We were not able to do this directly. Instead, we will show that for any $\epsilon > 0$ there is a constant $c(\epsilon, F)$

with

$$h(\mathrm{Jac}(C)) \leqslant \epsilon \log \Delta_K + c(\epsilon, F), \tag{1.2}$$

with F the maximal totally real subfield of K. For small ϵ this upper bound is strong enough to compete with the logarithmic lower bound coming from Colmez's conjecture because F is fixed in our Theorem 1.1. It would be interesting to know if this inequality could be made uniform in F.

The upper bound requires the second expression for the Faltings height of $\mathrm{Jac}(C)$ alluded to above. We still work with a curve C of genus 2 defined over $\overline{\mathbb{Q}}$, but now do not require that $\mathrm{Jac}(C)$ has CM. Suppose that C is the base change to $\overline{\mathbb{Q}}$ of a curve C_k defined over a number field $k \subseteq \overline{\mathbb{Q}}$. If C_k has good reduction at all places above 2, then Ueno [Uen88] decomposed $h(\mathrm{Jac}(C))$ into a sum over all places of k.[1] We present another expression for the Faltings height in Theorem 4.5 by decomposing it into local terms. In contrast to Ueno's formula and with our application in mind, we require that $\mathrm{Jac}(C_k)$ has good reduction at all finite places but in turn allow C_k to have bad reduction above 2. Our proof of Theorem 4.5 makes use of the reduction theory of genus 2 curves as developed by Igusa [Igu60] and later by Liu [Liu93, Liu94] as well as Saito's generalisation [Sai88] of Ogg's formula for the conductor of an elliptic curve. In our decomposition of $h(\mathrm{Jac}(C))$ into local terms a non-zero contribution at a finite place indicates that the *curve* C_k has bad stable reduction at the said place. In other words, if C has good reduction everywhere, as in Theorem 1.1, then the finite places do not contribute to $h(\mathrm{Jac}(C))$. We will also express the local contribution in $h(\mathrm{Jac}(C))$ at the finite places in terms of the classical Igusa invariants attached to C.

The terms at the archimedean places in Theorem 4.5 are expressed using a Siegel modular cusp form of degree 2 and weight 10. We must bound these infinite places from above in order to arrive at (1.2). One issue is that the archimedean local term has a logarithmic singularity along the divisor where the cusp form vanishes. This vanishing locus corresponds to the principally polarised abelian surfaces that are isomorphic to a product of elliptic curves with the product polarisation. The jacobian variety of a genus 2 curve defined over $\mathbb{C}$ is never such a product. So, in our application, we are never on the logarithmic singularity.

To obtain the upper bound for $h(\mathrm{Jac}(C))$ we must ensure first that not too many period matrices coming from the conjugates of $\mathrm{Jac}(C)$ are close to the logarithmic singularity. Second, we must show that no period matrix is excessively close to the said singularity.

To achieve the first goal we require Zhang's equidistribution theorem [Zha05] for Galois orbits of CM points on Hilbert modular surfaces. Zhang's result relies on the powerful subconvexity estimate due to Michel and Venkatesh [MV10]; Cohen [Coh05] and Clozel and Ullmo [CU05] have related equidistribution results. Roughly speaking, equidistribution guarantees that only a small proportion of period matrices coming from the Galois orbit of $\mathrm{Jac}(C)$ lie close to the problematic divisor.

However, equidistribution does not rule out the possibility that some period matrix is excessively close to the singular locus. To handle this contingency we use the following simple but crucial observation. Inside Siegel's fundamental domain, the divisor consists of diagonal period matrices

$$\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}.$$

[1] For other explicit formulas, the reader may consult Autissier [Aut06, Theorem 5.1, p. 1457] or the second-named author [Paz12b, Theorems 1.3 and 1.4].

A period matrix lying close to this divisor has small off-diagonal entries. It is a classical fact that the period matrix of a CM abelian variety is algebraic. Moreover, the degree over $\mathbb{Q}$ of each entry is bounded from above in terms of the dimension of the abelian variety. We will use Liouville's inequality to bound the modulus of the off-diagonal entries from below. This enables us to handle the contribution coming from the vanishing locus of the cusp form.

The archimedean contribution to the Faltings height of $\mathrm{Jac}(C)$ is also unbounded near the cusp in the Siegel upper half-space. We will again use the subconvexity estimates to control this contribution on average.

Our efforts at the archimedean places can be reduced to obtaining an upper bound for the archimedean contribution to the Faltings height of $\mathrm{Jac}(C)$. We state this bound separately.

Theorem 1.3. *For any $\epsilon > 0$ and any real quadratic field F, there exists a constant $c = c(\epsilon, F) > 0$ with the following property. Let C be a curve of genus 2 defined over a number field k such that its jacobian $\mathrm{Jac}(C)$ has complex multiplication by the ring of integers of a CM field K containing F that is quartic and cyclic over $\mathbb{Q}$. For an embedding $\sigma : k \to \mathbb{C}$, let Z_σ denote a period matrix of $\mathrm{Jac}(C) \otimes_\sigma \mathbb{C}$. Then $\chi_{10}(Z_\sigma) \neq 0$ and*

$$-\frac{1}{[k:\mathbb{Q}]}\sum_{\sigma:k\to\mathbb{C}} \log(|\chi_{10}(Z_\sigma)| \det \mathrm{Im}(Z_\sigma)^5) \leqslant \epsilon(\log \Delta_K) + c, \tag{1.3}$$

where $\Delta_K > 0$ is the discriminant of K and χ_{10} is the Siegel cusp mentioned above. (The value of $|\chi_{10}(Z_\sigma)| \det \mathrm{Im}(Z_\sigma)^5$ does not depend on the choice of period matrix.)

These various estimates combine to give (1.2). The quantitative nature of our approach allows for the following quantitative estimate, which implies Theorem 1.1, as we will see. We will measure the amount of bad stable reduction of a curve C_k of genus 2 defined over a number field k using the minimal discriminant $\Delta^0_{\min}(C)$ in the sense of Definition 4.4. It is a non-zero ideal in the ring of integers of k and $\mathrm{N}(\Delta^0_{\min}(C))$ denotes its norm below.

Theorem 1.4. *Let F be a real quadratic number field. There exists a constant $c(F) > 0$ with the following property. Let C be a curve of genus 2 defined over $\overline{\mathbb{Q}}$ such that $\mathrm{End}(\mathrm{Jac}(C))$ is the maximal order of an imaginary quadratic extension K of F with $K/\mathbb{Q}$ cyclic. Then C is the base change to $\overline{\mathbb{Q}}$ of a curve C_k defined over a number field $k \subseteq \overline{\mathbb{Q}}$ with*

$$\log \Delta_K \leqslant c(F)\left(1 + \frac{1}{[k:\mathbb{Q}]} \log \mathrm{N}(\Delta^0_{\min}(C_k))\right), \tag{1.4}$$

where the normalised norm on the right is invariant under finite field extensions of k.

The choice of k will be made during the proof.

In Theorem 4.5(ii), we will be able to express the normalised norm in terms of the Igusa invariants of the curve C.

Theorem 1.4 implies finiteness results for more general families than curves with potentially good reduction everywhere. Indeed, an analog of Theorem 1.1 is obtained for any collection where the normalised norm of $\Delta^0_{\min}(C_k)$ is uniformly bounded from above.

Let K be a quartic CM field that is not bi-quadratic. Goren and Lauter [GL06] called a rational prime p *evil* for K if there is a principally polarised abelian variety with CM by the maximal order of K whose reduction over a place above p is a product of two supersingular elliptic curves with the product polarisation. This corresponds to a genus 2 curve whose semi-stable

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reduction is bad at a place above p and whose jacobian variety has CM by the maximal order of K. Goren and Lauter proved that evilness prevails by showing that a given prime is evil for infinitely many K containing a fixed real quadratic field with trivial narrow-class group. In our Theorem 1.1, the prime p varies; using Goren and Lauter's terminology we can restate our result as follows. For all but finitely many quartic and cyclic CM number fields containing a given real quadratic field, there is an evil prime.

Let us now recall the fundamental result of Deligne and Mumford of [DM69, Theorem 2.4, p. 89].

THEOREM 1.5 (Deligne–Mumford). *Let k be a field with a discrete valuation and with algebraically closed residue field. Let C be a curve over k of genus at least 2. Then the jacobian variety $\mathrm{Jac}(C)$ has semi-stable reduction if and only if C has semi-stable reduction.*

The reader should keep in mind that even though a curve and its jacobian variety have semi-stable reduction simultaneously, it does not mean that the type of reduction (good or bad) is the same.

We conclude this introduction by posing some questions related to our results and to $\mathcal{A}_g$, the coarse moduli space of principally polarised abelian varieties of dimension $g \geqslant 1$.

The authors conjecture that there are only finitely many curves C of genus 2 defined over $\overline{\mathbb{Q}}$ which have good reduction everywhere and for which $\mathrm{Jac}(C)$ has complex multiplication by an order containing the ring of integers of F.

Our restriction in Theorem 1.1 that $K/\mathbb{Q}$ is abelian reflects the current status of Colmez's conjecture. This conjecture is open for general quartic extensions of $\mathbb{Q}$. However, Yang [Yan10] has proved some non-abelian cases for quartic CM fields. Recently, Andreatta–Goren–Howard–Madapusi Pera and independently Yuan–Zhang announced a version of Colmez's conjecture when averaging over all CM types of a CM field. An averaged version is not strong enough for our purposes. It is, however, if all members in the average have the same Faltings height (which is the case in our paper). We hope that this will allow us to extend our theorems above to quartic CM fields that are not normal over $\mathbb{Q}$.

Nakkajima and Taguchi [NT91] computed the Faltings height of an elliptic curve with complex multiplication by a general order. They reduced the computation to the case of a maximal order which is covered by the Chowla–Selberg formula. Very recently, Mocz has announced a variant of Nakkajima–Taguchi's result for abelian varieties in higher dimension.

Our approach relies heavily on equidistribution of Galois orbits on Hilbert modular surfaces. For this reason we must fix the maximal total real subfield in our theorem. However, it is natural to ask if the finiteness statement in Theorem 1.1 holds without fixing F. For example, is the set of points in $\mathcal{A}_2$ consisting of jacobians of curves defined over $\overline{\mathbb{Q}}$ with CM and with good reduction everywhere Zariski non-dense in $\mathcal{A}_2$? One could even speculate whether this set is finite.

In genus $g = 3$ the image of the Torelli morphism again dominates $\mathcal{A}_3$. Here too this image contains infinitely many jacobian varieties with CM. So, we ask whether the set of CM points that come from genus 3 curves with good reduction everywhere is Zariski non-dense in $\mathcal{A}_3$ or perhaps even finite. A simplified variant of this question would ask for non-denseness or finiteness under the restriction that the CM field contains a fixed totally real cubic subfield. Hyperelliptic curves of genus 3 do not lie Zariski dense in the moduli space of genus 3 curves. Thus, a statement like Theorem 4.5 for non-hyperelliptic curves would be necessary. This would be interesting in its own right.

Starting from genus $g = 4$ it is no longer true that the Torelli morphism dominates $\mathcal{A}_4$. The André–Oort conjecture, which is known unconditionally in this case by work of Pila and

Tsimerman [PT14], yields an additional obstruction for a curve of genus 4 to have CM. Coleman conjectured that there are only finitely many curves of fixed genus $g \geqslant 4$ with CM. However, this conjecture is known to be false for $g = 4$ and $g = 6$ by work of de Jong and Noot [dJN91]. In any case, a version of Theorem 1.1 for higher genus curves is entangled with other problems in arithmetic geometry.

In genus $g = 1$ no finiteness result such as Theorem 1.1 can hold true, as an elliptic curve with complex multiplication has potentially good reduction at all finite places. However, the first-named author proved [Hab15] the following finiteness result, which is reminiscent of the current work. Up to $\overline{\mathbb{Q}}$-isomorphism there are only finitely many elliptic curves with complex multiplication whose j-invariants are algebraic units. This connection reinforces the heuristics that CM points behave similarly to integral points on a curve in the context of Siegel's theorem. Indeed, the jacobian variety of a curve of genus 2 defined over $\overline{\mathbb{Q}}$ and with good reduction everywhere corresponds to an algebraic point on $\mathcal{A}_2$ that is integral with respect to the divisor given by products of elliptic curves with their product polarisation. Theorem 1.1 is a finiteness result on the set of certain CM points of $\mathcal{A}_2$ that are integral with respect to the said divisor. It would be interesting to know if, e.g., Vojta's Theorem 0.4 on integral points on semi-abelian varieties [Voj99] has an analog for $\mathcal{A}_g$ and other Shimura varieties.

Finally, one can ask if the questions posed above remain valid in an S-integer setting. In other words, are there only finitely many curves C of genus 2 or 3 which have good reduction above the complement of a finite set of primes, where $\mathrm{Jac}(C)$ has CM, and where possibly further conditions are met?

The paper is structured as follows. In the next section we introduce some basic notation. In § 3 we cover some properties of abelian varieties with complex multiplication, and recall Shimura's theorem on the Galois orbit for the cases we are interested in. In § 4 we recall first the Faltings height of an abelian variety. Then in § 4.2 we use a case of Colmez's conjecture to express the Faltings height of certain abelian varieties with CM. Section 4.4 contains the local decomposition of the Faltings height of a jacobian surface with good reduction at all finite places. The archimedean places in this decomposition are bounded from above in § 5. Finally, the proof of both our theorems is completed in § 6. In the appendix we both express, using Colmez's conjecture, and approximate numerically, using the result in § 4.4, the Faltings height of three jacobian varieties of genus 2 curves. Each pair of heights are equal up to the prescribed precision. The computations and statements made in the appendix are not necessary for the proof of our theorems.

2. Notation

In this paper it will be convenient to take $\overline{\mathbb{Q}}$ as the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and all number fields to be subfields of $\overline{\mathbb{Q}}$.

The letter i stands for an element of $\overline{\mathbb{Q}}$ such that $i^2 = -1$.

We let $K^\times$ denote the multiplicative group of any field K. If K is a number field, then Δ_K is its discriminant and Cl_K is the class group of K. We use the symbol $\mathcal{O}_K$ for the ring of integers of K and ${\mathcal{O}_K}^\times$ is the group of units of $\mathcal{O}_K$. If $\mathfrak{A}$ is a fractional ideal of K, then $[\mathfrak{A}]$ denotes its class in Cl_K. If K/F is an extension of number fields, then $\mathscr{D}_{K/F}$ is its different and $\mathfrak{d}_{K/F}$ is its relative discriminant. The norm of $\mathfrak{A}$ is $\mathrm{N}(\mathfrak{A})$, so $\mathrm{N}(\mathfrak{A}) = [\mathcal{O}_K : \mathfrak{A}]$ if $\mathfrak{A} \subseteq \mathcal{O}_K$. For the norm of $\mathfrak{A}$ relative to K/F, a fractional ideal of F, we use the symbol $\mathrm{N}_{K/F}(\mathfrak{A})$. If $\alpha \in K$, then $\mathrm{N}_{K/F}(\alpha) \in F$ and $\mathrm{Tr}_{K/F}(\alpha) \in F$ are norm and trace, respectively, of α relative to K/F.

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A place ν of K is an absolute value on K whose restriction to $\mathbb{Q}$ is the standard absolute value on $\mathbb{Q}$ or a p-adic absolute value for some prime number p. The former places are called infinite or archimedean and we write $\nu \mid \infty$ whereas the latter are called finite or non-archimedean and we write $\nu \nmid \infty$ or $\nu \mid p$. The set of finite places is M_K^0. Any $\nu \in M_K^0$ corresponds to a maximal ideal of $\mathcal{O}_K$ and we write $\operatorname{ord}_\nu(\mathfrak{A}) \in \mathbb{Z}$ for the power with which this ideal appears in the factorisation of $\mathfrak{A}$. If $\alpha \in K^\times$, then $\operatorname{ord}_\nu(\alpha) = \operatorname{ord}_\nu(\alpha\mathcal{O}_K)$. We write K_ν for the completion of K with respect to ν and $d_\nu = [K_\nu : \mathbb{Q}_{\nu'}]$, where ν' is the restriction of ν to $\mathbb{Q}$.

We will often use K to denote a CM field and F its totally real subfield. Complex conjugation on K will be denoted by $\alpha \mapsto \overline{\alpha}$. If a CM type Φ of K is given, then we write K^* for the associated reflex field.

For the field of definition of an algebraic variety we use lower case letters, k for instance.

Let $g \geqslant 1$ be an integer and $\mathbb{H}_g$ the Siegel upper half-space, i.e., $g \times g$ symmetric matrices with entries in $\mathbb{C}$ and positive-definite imaginary parts. For brevity, $\mathbb{H} = \mathbb{H}_1$ denotes the upper half-plane. The symplectic group $\operatorname{Sp}_{2g}(\mathbb{Z})$ acts on $\mathbb{H}_g$ by

$$\gamma Z = (\alpha Z + \beta)(\lambda Z + \mu)^{-1} \quad \text{if } \gamma = \begin{pmatrix} \alpha & \beta \\ \lambda & \mu \end{pmatrix} \in \operatorname{Sp}_{2g}(\mathbb{Z}).$$

We recall that $Z = (z_{lm})_{1 \leqslant l,m \leqslant g} \in \mathbb{H}_g$ is called Siegel reduced and lies in Siegel's fundamental domain $\mathcal{F}_g$ if and only if the following properties are met; cf. [Kli90, §§ 2 and 3].

(i) For every $\gamma \in \operatorname{Sp}_{2g}(\mathbb{Z})$, one has $\det \operatorname{Im}(\gamma Z) \leqslant \det \operatorname{Im}(Z)$, where $\operatorname{Im}(\cdot)$ denotes the imaginary part.

(ii) The real part is bounded by

$$|\operatorname{Re}(z_{lm})| \leqslant \tfrac{1}{2} \quad \text{for all } (l, m) \in \{1, \ldots, g\}^2.$$

(iii)$_a$ For all $l \in \{1, \ldots, g\}$ and all $\xi = (\xi_1, \ldots, \xi_g) \in \mathbb{Z}^g$ with $\gcd(\xi_l, \ldots, \xi_g) = 1$, we have ${}^t\xi \operatorname{Im}(Z)\xi \geqslant \operatorname{Im}(z_{ll})$.

(iii)$_b$ For all $l \in \{1, \ldots, g-1\}$, we have $\operatorname{Im}(z_{l,l+1}) \geqslant 0$.

The properties (iii)$_a$ and (iii)$_b$ state that $\operatorname{Im}(Z)$ is Minkowski reduced.

We write $\operatorname{diag}(\alpha_1, \ldots, \alpha_g)$ for the diagonal matrix with diagonal elements $\alpha_1, \ldots, \alpha_g$ which are contained in some field.

3. Abelian varieties

In the next sections we collect some statements on Hilbert modular varieties and abelian varieties that we require later on.

3.1 Hilbert modular varieties

Theorem 1.1 concerns jacobian varieties whose endomorphism algebras contain a fixed real quadratic number field. So, Hilbert modular surfaces arise naturally. In this section we discuss some properties of a fundamental set of the action of Hilbert modular groups on $\mathbb{H}^g = \mathbb{H} \times \cdots \times \mathbb{H}$, the g-fold product of the complex upper half-plane $\mathbb{H} \subseteq \mathbb{C}$. Our main reference for this section is Chapter I of van der Geer's book [vdG88]. However, we will work in a slightly modified setting and therefore provide some additional details.

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Let F be a totally real number field of degree g with distinct real embeddings $\varphi_1,\ldots,\varphi_g : F \to \mathbb{R}$. Throughout this section $\mathfrak{a}$ is a fractional ideal of $\mathcal{O}_F$. Later on we will be mainly interested in the case $\mathfrak{a} = \mathscr{D}_{F/\mathbb{Q}}^{-1}$, the inverse of the different of $F/\mathbb{Q}$.

Let $\mathcal{O}_F^{\times,+}$ be the group of totally positive units in $\mathcal{O}_F$ and

$$\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) : a, d \in \mathcal{O}_F,\, b \in \mathfrak{a}^{-1},\, c \in \mathfrak{a}, \text{ and } ad - bc \in \mathcal{O}_F^{\times,+} \right\}. \tag{3.1}$$

The group $\mathrm{GL}_2(F)$ acts on $\mathbb{P}^1(F)$. Its subgroup $\mathrm{GL}_2^+(F)$ of matrices with totally positive determinant acts on $\mathbb{H}^g$ by fractional linear transformations through the g embeddings $\varphi_1,\ldots,\varphi_g$.

We are interested in the restriction of this action to the subgroup $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{a})$. As this group's center acts trivially on $\mathbb{H}^g$, let us consider also

$$\widehat{\Gamma}(\mathfrak{a}) = \mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{a}) \Big/ \left\{ \begin{pmatrix} u & \\ & u \end{pmatrix} : u \in \mathcal{O}_F^\times \right\}. \tag{3.2}$$

The group $\widehat{\Gamma}(\mathfrak{a})$ also acts on $\mathbb{P}^1(F)$.

The $\widehat{\Gamma}(\mathfrak{a})$-action on $\mathbb{P}^1(F)$ consists of $h = \#\,\mathrm{Cl}_F < +\infty$ orbits which represent the cusps of $\widehat{\Gamma}(\mathfrak{a})\backslash\mathbb{H}^g$. For $\eta = [\alpha : \beta] \in \mathbb{P}^1(F)$ with $\alpha, \beta \in F$ and $\tau = (\tau_1,\ldots,\tau_g) \in \mathbb{H}^g$, we define

$$\mu(\eta,\tau) = \mathrm{N}(\alpha\mathcal{O}_F + \beta\mathfrak{a}^{-1})^2 \prod_{l=1}^{g} \frac{\mathrm{Im}(\tau_l)}{|\varphi_l(\alpha) - \varphi_l(\beta)\tau_l|^2} > 0.$$

The quantity $\mu(\eta,\tau)^{-1/2}$ measures the distance of the point in $\widehat{\Gamma}(\mathfrak{a})\backslash\mathbb{H}^g$ represented by τ to the cusp represented by η.

If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(F)$, then

$$\mu(\gamma\eta,\gamma\tau) = \frac{\mu(\eta,\tau)}{\mathrm{N}_{F/\mathbb{Q}}(\det\gamma)^2} \frac{\mathrm{N}(\alpha'\mathcal{O}_F + \beta'\mathfrak{a}^{-1})^2}{\mathrm{N}(\alpha\mathcal{O}_F + \beta\mathfrak{a}^{-1})^2}, \tag{3.3}$$

where $\alpha' = a\alpha + b\beta$ and $\beta' = c\alpha + d\beta$.

Let us study two important special cases. First, if $\gamma \in \mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{a})$, then $\det\gamma \in \mathcal{O}_F^{\times,+}$ and the ideals appearing on the right of (3.3) coincide. So, the equality simplifies to $\mu(\gamma\eta,\gamma\tau) = \mu(\eta,\tau)$. Second, let us suppose that $\gamma \in \mathrm{SL}_2(F)$ and fix a positive integer λ with $\lambda a, \lambda d \in \mathcal{O}_F$, $\lambda b \in \mathfrak{a}^{-1}$, and $\lambda c \in \mathfrak{a}$. Then $\lambda\alpha'\mathcal{O}_F + \lambda\beta'\mathfrak{a}^{-1} \subseteq \alpha\mathcal{O}_F + \beta\mathfrak{a}^{-1}$ and so the norm of the ideal on the left is at least the norm of the ideal on the right. Equality (3.3) implies that $\mu(\gamma\eta,\gamma\tau) \geqslant \lambda^{-2g}\mu(\eta,\tau)$.

On applying the same argument to γ^{-1}, we find

$$c^{-1} \leqslant \frac{\mu(\gamma\eta,\gamma\tau)}{\mu(\eta,\tau)} \leqslant c, \tag{3.4}$$

where $c > 0$ depends only on γ and not on $\eta \in \mathbb{P}^1(F)$ or on $\tau \in \mathbb{H}^g$.

A *fundamental set* for the action of $\widehat{\Gamma}(\mathfrak{a})$ on $\mathbb{H}^g$ is a subset of $\mathbb{H}^g$ that meets all $\widehat{\Gamma}(\mathfrak{a})$-orbits. We do not require a fundamental set to be connected and we do not exclude that two distinct points are in the same orbit. In the following we will describe a fundamental set much as van der Geer's construction of a fundamental domain for the action of $\mathrm{SL}_2(\mathcal{O}_F)$ on $\mathbb{H}^g$ in [vdG88, ch. I.3].

First, let us fix a set of representatives $\eta_1 = [\alpha_1 : \beta_1],\ldots,\eta_h = [\alpha_h : \beta_h] \in \mathbb{P}^1(F)$ of the cusps. We may assume that $\alpha_1 = 1$ and $\beta_1 = 0$, i.e., $[\alpha_1 : \beta_1] = \infty$. Only a slight variation in the argument of [vdG88, Lemma I.2.2] is required to obtain

$$\max\{\mu(\eta_1,\tau),\ldots,\mu(\eta_h,\tau)\} \gg 1; \tag{3.5}$$

the constants implicit in $\ll$ and $\gg$ here and below depend only on $\mathfrak{a}$ and the α_m, β_m.

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Proposition 3.1. *There is a closed fundamental set $\mathcal{F}(\mathfrak{a})$ for the action of $\widehat{\Gamma}(\mathfrak{a})$ on $\mathbb{H}^g$ with the following property. If $\tau = (\tau_1, \ldots, \tau_g) \in \mathcal{F}(\mathfrak{a})$, then $|\mathrm{Re}(\tau_l)| \ll 1$ and*

$$\Big(\max_{1\leqslant m\leqslant h} \mu(\eta_m, \tau)\Big)^{-2/g} \ll \mathrm{Im}(\tau_l) \ll \Big(\max_{1\leqslant m\leqslant h} \mu(\eta_m, \tau)\Big)^{1/g} \tag{3.6}$$

for all $1 \leqslant l \leqslant g$.

Proof. A given $\tau \in \mathbb{H}^g$ is in

$$S = \{\tau' \in \mathbb{H}^g : \mu(\eta_m, \tau') = \max\{\mu(\eta_1, \tau'), \ldots, \mu(\eta_h, \tau')\}\} \quad \text{for some } m,$$

the sphere of influence of the cusp η_m. We abbreviate $\eta = \eta_m, \alpha = \alpha_m$, and $\beta = \beta_m$. Thus,

$$\mu(\eta, \tau) \gg 1 \tag{3.7}$$

by (3.5). Let us define the fractional ideal $\mathfrak{b} = \alpha\mathcal{O}_F + \beta\mathfrak{a}^{-1}$ of $\mathcal{O}_F$. Next we choose $\gamma \in \mathrm{SL}_2(F)$ with

$$\gamma^{-1} = \begin{pmatrix} \alpha & \alpha^* \\ \beta & \beta^* \end{pmatrix},$$

where $\alpha^* \in (\mathfrak{a}\mathfrak{b})^{-1}$ and $\beta^* \in \mathfrak{b}^{-1}$. So, $\gamma\eta = \infty$ and we observe that an application of (3.4) and (3.7) yields

$$\mu(\infty, \gamma\tau) = \mu(\gamma\eta, \gamma\tau) \gg \mu(\eta, \tau) \gg 1.$$

The left-hand side is $\mathrm{Im}(\tau'_1) \cdots \mathrm{Im}(\tau'_g) \gg 1$, where $\gamma\tau = (\tau'_1, \ldots, \tau'_g)$.

We observe that

$$\gamma\widehat{\Gamma}(\mathfrak{a})\gamma^{-1} = \gamma\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{a})\gamma^{-1} = \mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{a}\mathfrak{b}^2) \tag{3.8}$$

and use $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{a}\mathfrak{b}^2)$ to act on $\gamma\tau$. In fact, we will use only elements in the stabiliser of ∞, i.e., the subgroup of upper triangular matrices in $\mathrm{GL}^+(\mathcal{O}_F \oplus \mathfrak{a}\mathfrak{b}^2)$. As in [vdG88, ch. I.3], we find γ' in the said group such that if $\gamma'\gamma\tau = (\tau''_1, \ldots, \tau''_g) = \tau''$, then

$$|\mathrm{Re}(\tau''_l)| \ll 1 \quad \text{and} \quad \mathrm{Im}(\tau''_l) \ll \mathrm{Im}(\tau''_{l'}) \quad \text{for all } 1 \leqslant l, l' \leqslant g. \tag{3.9}$$

We note that $\mathrm{Im}(\tau''_1) \cdots \mathrm{Im}(\tau''_g) = \mathrm{Im}(\tau'_1) \cdots \mathrm{Im}(\tau'_g) \gg 1$ and thus

$$\mathrm{Im}(\tau''_l) \gg 1 \quad \text{for all } 1 \leqslant l \leqslant g. \tag{3.10}$$

The point $\gamma^{-1}\gamma'\gamma\tau = \gamma^{-1}\tau''$ lies in the $\widehat{\Gamma}(\mathfrak{a})$-orbit of τ by (3.8). We define D as the set of τ'' that satisfy (3.9) and (3.10). We take $\gamma^{-1}D$ as a part of the fundamental set whose entirety $\mathcal{F}(\mathfrak{a})$ is obtained by taking the union of the sets coming from all h cusps. Observe that $\gamma^{-1}D$ is closed in $\mathbb{H}^g$ and so $\mathcal{F}(\mathfrak{a})$ is closed too.

It remains to prove that the various bounds in the assertion hold for $\gamma^{-1}\tau'' \in \gamma^{-1}D$. To simplify notation we write $\tau = \gamma^{-1}\tau''$ and recall that $\gamma\eta = \infty$ still holds. We use the second set of inequalities in (3.9) to bound $\mathrm{Im}(\tau''_l) \ll (\mathrm{Im}(\tau''_1) \cdots \mathrm{Im}(\tau''_g))^{1/g} = \mu(\infty, \tau'')^{1/g} = \mu(\gamma\eta, \tau'')^{1/g} \ll \mu(\eta, \gamma^{-1}\tau'')^{1/g}$. So, $\mathrm{Im}(\tau''_l) \ll \mu(\eta, \tau)^{1/g}$ and in particular $\mu(\eta, \tau) \gg 1$ by (3.10). We find that $|\tau''_l| \ll \mu(\eta, \tau)^{1/g}$ as the real part of τ''_l is bounded by (3.9). Now

$$\mathrm{Im}(\tau_l) = \mathrm{Im}(\gamma_l^{-1}\tau''_l) = \frac{\mathrm{Im}(\tau''_l)}{|\beta_l\tau''_l + \beta^*_l|^2} \geqslant \frac{\mathrm{Im}(\tau''_l)}{(|\beta_l\tau''_l| + |\beta^*_l|)^2} \gg \frac{1}{\mu(\eta, \tau)^{2/g}},$$

where the subscript l in β_l, β^*_l, and γ_l indicates that φ_l was applied. This yields the lower bound in (3.6).

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To deduce the upper bound, we split up into two cases. If $\beta_l \neq 0$, then $\mathrm{Im}(\gamma_l^{-1}\tau_l'') \leqslant \mathrm{Im}(\tau_l'')/(|\beta_l|^2 \mathrm{Im}(\tau_l'')^2) \ll 1$ and in particular $\mathrm{Im}(\gamma^{-1}\tau_l'') \ll \mu(\eta,\tau)^{1/g}$. So, the upper bound holds in this case. What if $\beta_l = 0$? Then $\mathrm{Im}(\gamma_l^{-1}\tau_l'') = \mathrm{Im}(\tau_l'')/|\beta_l^*|^2$. Further up we have seen that $\mathrm{Im}(\tau_l'') \ll \mu(\eta,\tau)^{1/g}$ and the upper bound follows from this.

To bound the real part, we use

$$|\mathrm{Re}(\tau_l)| = |\mathrm{Re}(\gamma_l^{-1}\tau_l'')| = \frac{|\alpha_l\beta_l|\tau_l''|^2 + \alpha_l^*\beta_l^* + (\alpha_l\beta_l^* + \alpha_l^*\beta_l)\,\mathrm{Re}(\tau_l'')|}{|\beta_l\tau_l'' + \beta_l^*|^2}.$$

The denominator is at least $|\beta_l|^2\,\mathrm{Im}(\tau_l'')^2 \gg 1$ if $\beta_l \neq 0$ and it equals $|\beta_l^*|^2 \gg 1$ if $\beta_l = 0$. Using elementary estimates we conclude that $|\mathrm{Re}(\tau_l)| \ll 1$ by treating separately the cases $|\beta_l\tau_l''| > 2|\beta_l^*|$ and $|\beta_l\tau_l''| \leqslant 2|\beta_l^*|$. $\square$

3.2 Abelian varieties with complex multiplication

In this section we recall some basic facts on a certain class of abelian varieties with CM. Furthermore, we prove several estimates that will play important roles in sections to come.

Let K be a CM field with $[K:\mathbb{Q}] = 2g$ and F the maximal, totally real subfield of K.

We suppose that A is an abelian variety of dimension g defined over $\mathbb{C}$ such that there is a ring homomorphism from an order $\mathcal{O}$ of K into $\mathrm{End}(A)$ which maps 1 to the identity map on A. In addition, we suppose that A is principally polarised.

As $[K:\mathbb{Q}] = 2\dim A$, the natural action of K on the tangent space of A at $0 \in A(\mathbb{C})$ is equivalent to a direct sum of embeddings $\varphi_1,\ldots,\varphi_g : K \to \mathbb{C}$, which are distinct modulo complex conjugation. In this way, A gives rise to a CM type $\Phi = \{\varphi_1,\ldots,\varphi_g\}$ of K. To keep notation elementary we fix a basis of the said tangent space and identify it with $\mathbb{C}^g$ such that the action of K is given by

$$\alpha(z_1,\ldots,z_g) = (\varphi_1(\alpha)z_1,\ldots,\varphi_g(\alpha)z_g)$$

for $(z_1,\ldots,z_g) \in \mathbb{C}^g$.

By abuse of notation, we write $\Phi(\alpha) = (\varphi_1(\alpha),\ldots,\varphi_g(\alpha))$ if $\alpha \in K$.

The period lattice of A is a discrete subgroup $\Pi \subseteq \mathbb{C}^g$ of rank $2g$. After scaling coordinates we may suppose that $(1,\ldots,1) \in \Pi$.

The set

$$\mathfrak{M} = \{\alpha \in K : \Phi(\alpha) \in \Pi\}$$

is an $\mathcal{O}_F$-module since $\mathcal{O}_F$ acts on the period lattice via Φ. It is finitely generated as such and it contains an order of K. Moreover, $\mathfrak{M}$ is torsion-free and $\mathcal{O}_F$ is a Dedekind ring; thus, $\mathfrak{M}$ is a projective $\mathcal{O}_F$-module. It is of rank 2 making it isomorphic to $\mathcal{O}_F \oplus \mathfrak{a}$, where $\mathfrak{a}$ is a fractional ideal of $\mathcal{O}_F$. Now $\mathfrak{a}$ is uniquely determined by its ideal class and later on we will show that $\mathfrak{a}$ lies in the class of $\mathscr{D}_{F/\mathbb{Q}}^{-1}$. Let us fix $\omega_1,\omega_2 \in K \smallsetminus \{0\}$ with

$$\mathfrak{M} = \omega_1\mathcal{O}_F + \omega_2\mathfrak{a}. \tag{3.11}$$

We note that $\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2 \neq 0$, where as usual $\bar{\cdot}$ denotes complex conjugation on K, and define

$$t_0 = (\omega_1\overline{\omega}_2 - \overline{\omega}_1\omega_2)^{-1}. \tag{3.12}$$

Observe that if the order $\mathcal{O}$ equals $\mathcal{O}_K$, then $\mathfrak{M}$ is a fractional ideal of $\mathcal{O}_K$. In this case we will use the symbol $\mathfrak{A}$ to denote $\mathfrak{M}$.

As A is principally polarised, it comes with an $\mathbb{R}$-bilinear form $E : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{R}$ which restricts to an integral symplectic form of determinant 1 on $\Pi \times \Pi$. We note that

$$H(z, w) = E(iz, w) + iE(z, w)$$

is a positive-definite hermitian form whose imaginary part is integral on $\Pi \times \Pi$.

Our form E satisfies the condition of Theorem 4, ch. II in Shimura's book [Shi97]. So, there is $t \in K$ with $\overline{t} = -t$ and $\mathrm{Im}(\varphi_m(t)) > 0$ for all m such that

$$E(z, w) = \sum_{j=1}^{g} \varphi_j(t)(\overline{z_j} w_j - z_j \overline{w_j}) \tag{3.13}$$

for all $z = (z_1, \ldots, z_g)$ and $w = (w_1, \ldots, w_g)$ in $\mathbb{C}^g$. Then

$$E(\Phi(\alpha), \Phi(\beta)) = \mathrm{Tr}_{K/\mathbb{Q}}(t\overline{\alpha}\beta) \tag{3.14}$$

for all $\alpha, \beta \in K$.

Lemma 3.2. *Let us keep the notation from above and also set $u = t/t_0$.*

(i) *We have $u \in F$ and*

$$E(\Phi(\mu\omega_1 + \lambda\omega_2), \Phi(\mu'\omega_1 + \lambda'\omega_2)) = \mathrm{Tr}_{F/\mathbb{Q}}(u(\mu'\lambda - \mu\lambda'))$$

for all $\mu, \mu, \lambda, \lambda' \in F$.

(ii) *We have $u\mathfrak{a} = \mathscr{D}_{F/\mathbb{Q}}^{-1}$.*

Proof. As $\overline{t_0} = -t_0$, we find that $\overline{u} = u$ and thus $u \in F$. We find that $\mathrm{Tr}_{K/\mathbb{Q}}(t\mu\mu'\omega_1\overline{\omega}_1) = 0$ as $\mu\mu'\omega_1\overline{\omega}_1 \in F$ and similarly $\mathrm{Tr}_{K/\mathbb{Q}}(t\lambda\lambda'\omega_2\overline{\omega}_2) = 0$. Therefore, by (3.14),

$$\begin{aligned}E(\Phi(\mu\omega_1 + \lambda\omega_2), \Phi(\mu'\omega_1 + \lambda'\omega_2)) &= \mathrm{Tr}_{K/\mathbb{Q}}(t(\mu\overline{\omega}_1 + \lambda\overline{\omega}_2)(\mu'\omega_1 + \lambda'\omega_2)) \\ &= \mathrm{Tr}_{K/\mathbb{Q}}(t(\lambda\mu'\omega_1\overline{\omega}_2 + \mu\lambda'\overline{\omega}_1\omega_2)) \\ &= \sum_{j=1}^{g} \varphi_j(t(\lambda\mu'\omega_1\overline{\omega}_2 + \mu\lambda'\overline{\omega}_1\omega_2 - \lambda\mu'\overline{\omega}_1\omega_2 - \mu\lambda'\omega_1\overline{\omega}_2)) \\ &= \mathrm{Tr}_{F/\mathbb{Q}}(u(\lambda\mu' - \mu\lambda')),\end{aligned}$$

where the final equality used $t = ut_0$ and (3.12). Part (i) follows.

The symplectic form E has determinant 1 as it corresponds to a principal polarisation of A. So, there exist a $\mathbb{Z}$-basis $(\mu_1, \ldots, \mu_g)$ of $\mathcal{O}_F$ and a $\mathbb{Z}$-basis of $(\lambda_1, \ldots, \lambda_g)$ of $\mathfrak{a}$ such that $E(\Phi(\lambda_l\omega_2), \Phi(\mu_m\omega_1)) = 0$ except if $l = m$ when the value is 1. Part (i) yields

$$E(\Phi(\lambda_l\omega_2), \Phi(\mu_m\omega_1)) = \mathrm{Tr}_{F/\mathbb{Q}}(u\mu_m\lambda_l).$$

So, if we arrange the g column vectors $\Phi(\mu_m)$ to a square matrix U and do the same with $\Phi(\lambda_l)$ to obtain Λ, then ${}^tU\,\mathrm{diag}(\varphi_1(u), \ldots, \varphi_g(u))\Lambda$ is the $g \times g$ unit matrix. Thus, $\det(U)\mathrm{N}_{F/\mathbb{Q}}(u)\det(\Lambda) = 1$. Now $|\det\Lambda| = \mathrm{N}(\mathfrak{a})|\Delta_F|^{1/2}$ and $|\det U| = |\Delta_F|^{1/2}$ and thus $|\mathrm{N}_{F/\mathbb{Q}}(u)|\mathrm{N}(\mathfrak{a}) = |\Delta_F|^{-1}$. We conclude that $\mathrm{N}(u\mathfrak{a}) = \mathrm{N}(\mathscr{D}_{F/\mathbb{Q}}^{-1})$.

If $\lambda \in \mathfrak{a}$ is arbitrary, then $\mathrm{Tr}_{F/\mathbb{Q}}(u\lambda) = E(\Phi(\lambda\omega_2, \omega_1))$ by part (i). This is an integer and so $u\mathfrak{a} \subseteq \mathscr{D}_{F/\mathbb{Q}}^{-1}$. But we proved above that these two fractional ideals have equal norm; thus, part (ii) follows. $\square$

Part (ii) of the lemma above establishes our claim that $\mathfrak{a}$ and $\mathscr{D}_{F/\mathbb{Q}}^{-1}$ are in the same ideal class. So, we can take $\mathfrak{a} = \mathscr{D}_{F/\mathbb{Q}}^{-1}$ to start out with. Part (ii) of the previous lemma implies that $u \in \mathcal{O}_F^\times$. We now replace ω_1 and ω_2 with ω_1 and $u^{-1}\omega_2$, respectively. With these new periods,

$$\mathfrak{M} = \omega_1 \mathcal{O}_F + \omega_2 \mathscr{D}_{F/\mathbb{Q}}^{-1} \tag{3.15}$$

remains true but now

$$t = (\omega_1 \overline{\omega_2} - \overline{\omega_1}\omega_2)^{-1}. \tag{3.16}$$

Moreover, the formula in Lemma 3.2(i) simplifies to

$$E(\Phi(\mu\omega_1 + \lambda\omega_2), \Phi(\mu'\omega_1 + \lambda'\omega_2)) = \mathrm{Tr}_{F/\mathbb{Q}}(\mu'\lambda - \mu\lambda'). \tag{3.17}$$

Next let us consider $\tau = \omega_2/\omega_1$. We compute

$$\varphi_l(t)^{-1} = |\varphi_l(\omega_1)|^2(\overline{\varphi_l(\tau)} - \varphi_l(\tau)) = -2i|\varphi_l(\omega_1)|^2 \operatorname{Im}(\varphi_l(\tau))$$

for all $1 \leqslant l \leqslant g$. Our t satisfies $\mathrm{Re}(\varphi_l(t)) = 0$ and $\mathrm{Im}(\varphi_l(t)) > 0$. We conclude that $\mathrm{Im}(\varphi_l(\tau)) > 0$ for all $1 \leqslant l \leqslant g$. In particular, $\Phi(\tau) \in \mathbb{H}^g$.

Recall that the group $\widehat{\Gamma}(\mathscr{D}_{F/\mathbb{Q}}^{-1})$, defined in (3.2), acts on $\mathbb{H}^g$ and that we described a fundamental set for this action in Proposition 3.1. In the proposition below we use this group to transform ω_2/ω_1 to the said fundamental set.

Let $V \subseteq \mathcal{O}_F^{\times,+}$ be a set of representatives of $\mathcal{O}_F^{\times,+}/(\mathcal{O}_F^\times)^2$. Note that V is finite.

Proposition 3.3. *There exist $\omega_1, \omega_2 \in K^\times$ as in (3.15), $\Phi(\omega_2/\omega_1) \in \mathcal{F}(\mathscr{D}_{F/\mathbb{Q}}^{-1})$, and such that there is $v \in V$ with*

$$E(\Phi(\mu\omega_1 + \lambda\omega_2), \Phi(\mu'\omega_1 + \lambda'\omega_2)) = \mathrm{Tr}_{F/\mathbb{Q}}(v(\lambda\mu' - \lambda'\mu)) \tag{3.18}$$

for all $\mu, \mu', \lambda, \lambda' \in F$.

Proof. According to Proposition 3.1, there is

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}^+(\mathcal{O}_F \oplus \mathscr{D}_{F/\mathbb{Q}}^{-1})$$

with $\gamma\Phi(\tau) \in \mathcal{F}(\mathscr{D}_{F/\mathbb{Q}}^{-1})$. Multiplying γ by a scalar matrix with diagonal entry $u \in \mathcal{O}_F^\times$ does not affect $\gamma\Phi(\tau)$ and replaces $\det\gamma$ by $u^2 \det\gamma$. So, we may assume that $\det\gamma \in V$. We set $\omega_1' = d\omega_1 + c\omega_2, \omega_2' = b\omega_1 + a\omega_2$ and find, using the definition (3.1), that (3.15) again remains true. Using (3.17), we obtain

$$E(\Phi(\mu\omega_1' + \lambda\omega_2', \mu'\omega_1' + \lambda'\omega_2')) = \mathrm{Tr}_{F/\mathbb{Q}}((\det\gamma)(\mu'\lambda - \mu\lambda'))$$

for all $\mu, \mu', \lambda, \lambda' \in F$. The proposition follows on replacing ω_1 and ω_2 by ω_1' and ω_2', respectively. $\square$

Let $(\mu_1, \ldots, \mu_g)$ be any $\mathbb{Z}$-basis of $\mathcal{O}_F$. We may find a $\mathbb{Z}$-basis $(\lambda_1, \ldots, \lambda_g)$ of $\mathscr{D}_{F/\mathbb{Q}}^{-1}$ such that $(\Phi(\mu_1)\omega_1, \ldots, \Phi(\lambda_g)\omega_2)$ is a symplectic basis for E. We note that the λ_l may depend on the symplectic form E and thus on $\mathfrak{M}$, whereas the μ_m depend only on F. Let us see how to

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retrieve the $\lambda_1, \dots, \lambda_g$ from the other data. We define $\Lambda, U \in \mathrm{Mat}_g(\mathbb{R})$ as the square matrices with columns $\Phi(\lambda_1), \dots, \Phi(\lambda_g)$ and $\Phi(\mu_1), \dots, \Phi(\mu_g)$, respectively. Relation (3.18) yields

$$\mathrm{Tr}_{F/\mathbb{Q}}(v\lambda_l\mu_m) = \begin{cases} 1 & \text{for } l = m, \\ 0 & \text{for } l \neq m. \end{cases} \tag{3.19}$$

So, ${}^t\Lambda \operatorname{diag}(\varphi_1(v), \dots, \varphi_g(v))U$ is the $g \times g$ unit matrix. The period matrix with respect to the symplectic basis is

$$\begin{aligned} Z &= U^{-1} \operatorname{diag}(\varphi_1(\tau), \dots, \varphi_g(\tau))\Lambda = U^{-1} \operatorname{diag}(\varphi_1(v\tau), \dots, \varphi_g(v\tau))^t U^{-1} \\ &= {}^t\Lambda \operatorname{diag}(\varphi_1(v\tau), \dots, \varphi_g(v\tau))\Lambda. \end{aligned} \tag{3.20}$$

It is well known that Z lies in the Siegel upper half-space $\mathbb{H}_g$.

Remark 3.4. Let us assume that $g = 2$ and $\mathcal{O}_F^{\times,+} = (\mathcal{O}_F^{\times})^2$. So, F is a real quadratic field of discriminant $\Delta > 0$, say, and we may take V as above Proposition 3.3 to contain only 1. Thus, $\mathcal{O}_F = \mathbb{Z} + \theta\mathbb{Z}$ with $\theta = (\Delta + \sqrt{\Delta})/2$. The conjugate of θ over $\mathbb{Q}$ is $\theta' = (\Delta - \sqrt{\Delta})/2$ and we consider θ, θ' as real numbers. So,

$$\begin{pmatrix} \theta & 1 \\ \theta' & 1 \end{pmatrix}$$

becomes an admissible choice for U as above. Say ω_1, ω_2 are as in Proposition 3.3 with $\tau_1 = \varphi_1(\omega_2/\omega_1)$ and $\tau_2 = \varphi_2(\omega_2/\omega_1) \in \mathbb{C}$. A brief calculation using $\det U = \theta - \theta' = \sqrt{\Delta}$ yields the period matrix

$$Z = U^{-1} \begin{pmatrix} \tau_1 & \\ & \tau_2 \end{pmatrix} {}^tU^{-1} = \frac{1}{\Delta} \begin{pmatrix} \tau_1 + \tau_2 & -\tau_1\theta' - \tau_2\theta \\ -\tau_1\theta' - \tau_2\theta & \tau_1\theta'^2 + \tau_2\theta^2 \end{pmatrix}.$$

For the remainder of this section we suppose that $\mathcal{O} = \mathcal{O}_K$ and thus that $\mathfrak{M} = \mathfrak{A}$ is a fractional ideal of $\mathcal{O}_K$.

Next we will bound how close the point represented by Z lies to the boundary of the coarse moduli space of principally polarised abelian varieties of dimension g. We will do the same for the point in $\widehat{\Gamma}(\mathscr{D}_{F/\mathbb{Q}}^{-1})\backslash\mathbb{H}^g$ represented by τ from Proposition 3.3.

We define the norm of any ideal class $[\mathfrak{A}] \in \mathrm{Cl}_K$ as the least norm of an ideal representing the said class, i.e.,

$$\mathrm{N}([\mathfrak{A}]) = \min\{\mathrm{N}(\mathfrak{B}) : \mathfrak{B} \text{ is an ideal of } \mathcal{O}_K \text{ in } [\mathfrak{A}]\}.$$

Recall that $\mathcal{F}_g$ denotes Siegel's fundamental domain; see § 2.

Lemma 3.5. *Let ω_1 and ω_2 be as in Proposition 3.3. Then*

$$2^g \mathrm{N}_{K/\mathbb{Q}}(\omega_1) \prod_{l=1}^{g} \mathrm{Im}(\varphi_l(\omega_2/\omega_1)) = \mathrm{N}(\mathfrak{A})|\Delta_K|^{1/2} = |\mathrm{N}_{K/\mathbb{Q}}(t)|^{-1/2}.$$

Proof. We let $U, \Lambda \in \mathrm{Mat}_g(\mathbb{R})$ denote matrices as in (3.20). Let $\Omega_j = \operatorname{diag}(\varphi_1(\omega_j), \dots, \varphi_g(\omega_j))$ for $1 \leqslant j \leqslant 2$. Then the columns of

$$\begin{pmatrix} \Omega_1 & \Omega_2 \\ \overline{\Omega_1} & \overline{\Omega_2} \end{pmatrix} \begin{pmatrix} U & \\ & \Lambda \end{pmatrix}$$

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constitute a $\mathbb{Z}$-basis of $\Phi \times \overline{\Phi}(\mathfrak{A}) \subseteq \mathbb{C}^{2g}$. The determinant of this product has modulus

$$\mathrm{N}(\mathfrak{A})|\Delta_K|^{1/2} = |\det(\Omega_1\overline{\Omega_2} - \overline{\Omega_1}\Omega_2)||\det U||\det \Lambda|.$$

The first equality follows since $|\det U| = |\Delta_F|^{1/2}$ and $|\det \Lambda| = \mathrm{N}(\mathscr{D}_{F/\mathbb{Q}}^{-1})|\Delta_F|^{1/2} = |\Delta_F|^{-1/2}$.

To prove the second equality, let $(\alpha_1, \ldots, \alpha_{2g})$ be a $\mathbb{Z}$-basis of $\mathfrak{A}$. The entries of the matrix $(E(\Phi(\alpha_l), \Phi(\alpha_m)))_{1\leqslant l,m\leqslant 2g}$ can be expressed using the trace by (3.14). We find that

$$|\det(E(\Phi(\alpha_l), \Phi(\alpha_m)))_{1\leqslant l,m\leqslant 2g}| = |\det(\mathrm{Tr}_{K/\mathbb{Q}}(t\overline{\alpha_l}\alpha_m))_{1\leqslant l,m\leqslant 2g}| = |\mathrm{N}_{K/\mathbb{Q}}(t)|\mathrm{N}(\mathfrak{A})^2|\Delta_K|.$$

The absolute value on the left is 1 as the polarisation on A is principal. Our claim follows after taking the square root and rearranging terms. □

For the next lemma we fix representatives $\eta_m \in \mathbb{P}^1(F)$ of the $\#\,\mathrm{Cl}_F$ cusps of $\widehat{\Gamma}(\mathscr{D}_{F/\mathbb{Q}}^{-1})\backslash\mathbb{H}^g$ as in § 3.1.

Lemma 3.6. *Let Z be the period matrix (3.20), let ω_1, ω_2 be as in Proposition 3.3, and set $\tau = \omega_2/\omega_1$.*

(i) *There exists a constant $c = c(g) > 0$ which depends only on g with the following property. If $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ with $\gamma Z \in \mathcal{F}_g$, then*

$$\mathrm{Tr}(\mathrm{Im}(\gamma Z)) \leqslant c\left(\frac{|\Delta_K|^{1/2}}{\mathrm{N}([\mathfrak{A}^{-1}])}\right)^{1/g}.$$

(ii) *There exists a constant $c > 0$ which depends only on F and the η_m such that*

$$\mu(\eta_m, \Phi(\tau)) \leqslant c\frac{|\Delta_K|^{1/2}}{\mathrm{N}([\mathfrak{A}^{-1}])}$$

for all m.

Proof. Let $\omega \in \mathfrak{A} \smallsetminus \{0\}$ witness the injectivity diameter

$$\rho = \min\{H(\omega', \omega')^{1/2} : \omega' \in \Pi \smallsetminus \{0\}\} > 0$$

of A with its polarisation, i.e., $\rho^2 = H(\Phi(\omega), \Phi(\omega))$. Then

$$\rho^2 = E(i\Phi(\omega), \Phi(\omega)) = 2\sum_{l=1}^{g} |\varphi_l(t)||\varphi_l(\omega)|^2$$

by (3.13). The inequality between the arithmetic mean and the geometric mean implies that

$$\rho^2 \geqslant 2g\left(\prod_{l=1}^{n} |\varphi_l(t)||\varphi_l(\omega)|^2\right)^{1/g} = 2g(|\mathrm{N}_{K/\mathbb{Q}}(t)|^{1/2}|\mathrm{N}_{K/\mathbb{Q}}(\omega)|)^{1/g}.$$

By the second equality in Lemma 3.5, we deduce that

$$\rho^2 \geqslant 2g\left(\frac{|\mathrm{N}_{K/\mathbb{Q}}(\omega)|}{\mathrm{N}(\mathfrak{A})|\Delta_K|^{1/2}}\right)^{1/g}.$$

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Since $\omega \in \mathfrak{A}$ is non-zero, there is an ideal $\mathfrak{B}$ of $\mathcal{O}_K$ with $\mathfrak{A}\mathfrak{B} = \omega\mathcal{O}_K$. Thus,

$$\rho^2 \geqslant 2g(\mathrm{N}(\mathfrak{B})/|\Delta_K|^{1/2})^{1/g}$$

since $\mathrm{N}(\mathfrak{A})\mathrm{N}(\mathfrak{B}) = |\mathrm{N}_{K/\mathbb{Q}}(\omega)|$. So,

$$\rho^{-2} \leqslant (2g)^{-1}\left(\frac{|\Delta_K|^{1/2}}{\mathrm{N}([\mathfrak{A}^{-1}])}\right)^{1/g} \tag{3.21}$$

since $\mathfrak{B}$ is in the class $[\mathfrak{A}^{-1}]$.

Next we write $Z_{\mathrm{red}} = \gamma Z$ with γ as in (i). As Z_{red} lies in Siegel's fundamental domain, its imaginary part is Minkowski reduced. The matrix $\mathrm{Im}(Z_{\mathrm{red}})^{-1}$ represents the hermitian form H with respect to the standard basis on $\mathbb{C}^g$. If $y_1', \ldots, y_g'$ are the diagonal elements of $\mathrm{Im}(Z_{\mathrm{red}})^{-1}$, then we find that $\rho^2 \leqslant \min\{y_1', \ldots, y_g'\}$ on testing with standard basis vectors. The diagonal entries $y_1, \ldots, y_g$ of $\mathrm{Im}(Z_{\mathrm{red}})$ are positive and satisfy $\det \mathrm{Im}(Z_{\mathrm{red}}) \geqslant c y_1 \cdots y_g$ by [Kli90, Proposition 1(iii), § 2], where $c > 0$ depends only on g. Part (ii) of the same proposition provides a bound for the off-diagonal entries of $\mathrm{Im}(Z_{\mathrm{red}})$ in terms of the y_l. This allows us to bound the cofactors of $\mathrm{Im}(Z_{\mathrm{red}})$ using the triangle inequality. The lower bound for $\deg \mathrm{Im}(Z_{\mathrm{red}})$ then leads to $y_l' \leqslant c/y_l$ after possibly increasing c.

So,

$$\rho^{-2} \geqslant \max\{y_1, \ldots, y_g\}/c \geqslant \mathrm{Tr}(\mathrm{Im}(Y))/(cg).$$

We combine this inequality with (3.21) to deduce part (i).

For the proof of (ii), we abbreviate $\eta = \eta_m$ and fix $\alpha \in \mathcal{O}_F, \beta \in \mathscr{D}_{F/\mathbb{Q}}^{-1}$ with $\eta = [\alpha : \beta]$. Then

$$\mu(\eta, \Phi(\tau)) = \mathrm{N}(\alpha\mathcal{O}_F + \beta\mathscr{D}_{F/\mathbb{Q}})^2 |\mathrm{N}_{K/\mathbb{Q}}(\omega_1)| \prod_{l=1}^{g} \frac{\mathrm{Im}(\varphi_l(\tau))}{|\varphi_l(\omega_1\alpha - \omega_2\beta)|^2}$$

and so

$$\mu(\eta, \Phi(\tau)) = 2^{-g}\mathrm{N}(\alpha\mathcal{O}_F + \beta\mathscr{D}_{F/\mathbb{Q}})^2 \frac{\mathrm{N}(\mathfrak{A})|\Delta_K|^{1/2}}{|\mathrm{N}_{K/\mathbb{Q}}(\omega_1\alpha - \omega_2\beta)|}$$

by the first equality of Lemma 3.5. We observe that $\omega_1\alpha - \omega_2\beta \in \mathfrak{A}$ is non-zero. As above, $(\omega_1\alpha - \omega_2\beta) = \mathfrak{A}\mathfrak{B}$ for some ideal $\mathfrak{B} \in [\mathfrak{A}^{-1}]$. We conclude that

$$\mu(\eta, \Phi(\tau)) = 2^{-g}\mathrm{N}(\alpha\mathcal{O}_F + \beta\mathscr{D}_{F/\mathbb{Q}})^2 \frac{|\Delta_K|^{1/2}}{\mathrm{N}(\mathfrak{B})}.$$

With this, part (ii) follows since $\mathrm{N}(\mathfrak{B}) \geqslant \mathrm{N}([\mathfrak{A}^{-1}])$ and because α, β depend only on F and the η_m. □

The fact that the exponent $1/g$ in (i) is strictly less than one for the jacobian of a genus $g = 2$ curve will prove crucial later on.

The period matrix Z we constructed above may not lie in Siegel's fundamental domain $\mathcal{F}_g \subseteq \mathbb{H}_g$ defined in § 2. We rectify this in the next lemma by using Minkowski and Siegel's reduction theory.

Lemma 3.7. *Let ω_1, ω_2 be as in Proposition 3.3 and set $\tau = \omega_2/\omega_1$. For given $M > 0$, there is a finite set $\Sigma \subseteq \mathrm{Sp}_{2g}(\mathbb{Z})$ such that if $\max_m \mu(\eta_m, \Phi(\tau)) \leqslant M$, then there exists $\gamma \in \Sigma$ with $\gamma Z \in \mathcal{F}_g$.*

Proof. In this proof, all constants implicit in $\ll$ and $\gg$ depend on F, the set V, the matrix U, the choice of cusp representatives η_m, and M. So, $\mu(\eta_m, \Phi(\tau)) \ll 1$ for all cusp representatives η_m. Recall that $\Phi(\tau)$ lies in the fundamental set $\mathcal{F}(\mathscr{D}_{F/\mathbb{Q}}^{-1})$ coming from Proposition 3.1. If $\Phi(\tau)$ is equal to $(\tau_1, \ldots, \tau_g)$, then $|\mathrm{Re}(\tau_l)| \ll 1$ and $\mathrm{Im}(\tau_l) \gg 1$ for all $1 \leqslant l \leqslant g$.

There are at most finitely many possible Λ as in (3.20). Let us write z_{lm} for the entries of Z. The entries of Λ are $\varphi_l(\lambda_m)$. So,

$$z_{lm} = \sum_{j=1}^{g} \varphi_j(v\lambda_l\lambda_m)\tau_j \quad \text{and in particular } z_{ll} = \sum_{j=1}^{g} \varphi_j(v\lambda_l^2)\tau_j$$

for some $v \in V$. We observe that $\varphi_j(v) > 0$ as $V \subseteq \mathcal{O}_F^{\times,+}$ and $\varphi_j(\lambda_l) \in \mathbb{R} \smallsetminus \{0\}$. So,

$$|\mathrm{Im}(z_{lm})| \leqslant \sum_{j=1}^{g} |\varphi_j(v\lambda_l\lambda_m)| \,\mathrm{Im}(\tau_j) \ll \sum_{j=1}^{g} \varphi_j(v\lambda_l^2) \,\mathrm{Im}(\tau_j) = \mathrm{Im}(z_{ll})$$

for all $1 \leqslant l, m \leqslant g$. Taking the determinant of the imaginary part of (3.20) yields

$$1 \ll \prod_{j=1}^{g} \mathrm{Im}(\tau_j) = (\det \Lambda)^{-2} \det \mathrm{Im}(Z) \ll \det \mathrm{Im}(Z).$$

So, $\mathrm{Im}(Z)$ lies in the set $Q_g(t)$ from [Kli90, Definition 2, § 2] for all sufficiently large t.

On considering the real part, we obtain $|\mathrm{Re}(z_{lm})| \ll 1$ from (3.20) and $|\mathrm{Re}(\tau_l)| \ll 1$. Moreover, $\mathrm{Im}(z_{11}) \gg \mathrm{Im}(\tau_1) \gg 1$.

Hence, Z lies in $L_g(t)$ as in [Kli90, Definition 2, § 3] for all large t. The existence of the finite set Σ now follows from [Kli90, Theorem 1]. □

By our relation (3.20), the entries of Z are contained in the normal closure of $K/\mathbb{Q}$. In particular, the entries of Z are contained in a number field whose degree over $\mathbb{Q}$ is bounded by a constant depending only on g.

We use a recent result of Pila and Tsimerman to bound the height of a reduced period matrix.

Lemma 3.8. *Let us suppose that A is simple. If $\gamma \in \mathrm{Sp}_{2g}(\mathbb{Z})$ with $\gamma Z \in \mathcal{F}_g$, then $H(\gamma Z) \leqslant |\Delta_K|^c$ for a constant $c = c(g) > 0$ that depends only on g.*

Proof. This follows from Pila and Tsimerman [PT13, Theorem 3.1] as the endomorphism ring of A equals $\mathcal{O}_K$ under the simplicity assumption on A. □

3.3 The Galois orbit

We keep the notation of the previous two sections.

Any field automorphism $\sigma : \mathbb{C} \to \mathbb{C}$ determines a new abelian variety A^σ with complex multiplication. Let $\mathrm{Aut}(\mathbb{C}/K^*)$ denote the group of automorphisms that restrict to the identity on K^*, the reflex field of (K, Φ). Shimura [Shi97, Theorem 18.6] described how to recover a period lattice of A^σ if $\sigma \in \mathrm{Aut}(\mathbb{C}/K^*)$. We only state a special case of Shimura's theorem and avoid the language of idèles. Indeed, by the assumptions of this section $\mathfrak{A}$ is a fractional ideal in K and the ideal-theoretic formulation suffices.

To this extent let H^* denote the Hilbert class field of K^* and

$$\mathrm{art} : \mathrm{Cl}_{K^*} \to \mathrm{Gal}(H^*/K^*)$$

the group isomorphism coming from class field theory.

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Let Φ^* denote the CM type attached to the reflex of (K,Φ). The reflex norm $\mathrm{N}_\Phi : (K^*)^\times \to K^\times$ attached to (K,Φ) is

$$\mathrm{N}_\Phi(a) = \prod_{\varphi\in\Phi^*} \varphi(a);$$

cf. [Shi97, § 8.3] for standard properties including the fact that the target is indeed $K^\times$. If $\mathfrak{B}^*$ is a fractional ideal of K^*, then $\prod_{\varphi\in\Phi^*}\varphi(\mathfrak{B}^*)$ is a fractional ideal of K, which we denote with $\mathrm{N}_\Phi(\mathfrak{B}^*)$. Observe that N_Φ also induces a homomorphism of class groups $\mathrm{Cl}_{K^*} \to \mathrm{Cl}_K$, which we also denote by N_Φ.

Theorem 3.9 (Shimura). *Let $A, K, \Phi, K^*, \mathfrak{A}$, and t be as above and as in the last section.*

Suppose that $\sigma \in \mathrm{Aut}(\mathbb{C}/K^*)$; we consider A^σ as an abelian variety over $\mathbb{C}$. Let $\mathfrak{B}^*$ be a fractional ideal of K^* with $\mathrm{art}([\mathfrak{B}^*]) = \sigma|_{H^*}$. Then $A^\sigma(\mathbb{C}) \cong \mathbb{C}^g/\Phi(\mathfrak{A}^\sigma)$, where $\mathfrak{A}^\sigma = \mathrm{N}_\Phi(\mathfrak{B}^*)^{-1}\mathfrak{A}$ and t transforms to $\mathrm{N}(\mathfrak{B}^*)t$.

In particular, the set of period lattices in the $\mathrm{Aut}(\mathbb{C}/K^*)$-orbit are represented by

$$\{[\mathfrak{A}^\sigma] : \sigma \in \mathrm{Aut}(\mathbb{C}/K^*)\} = \mathrm{N}_\Phi(\mathrm{Cl}_{K^*})[\mathfrak{A}] = \{\mathrm{N}_\Phi([\mathfrak{B}^*])^{-1}[\mathfrak{A}] : [\mathfrak{B}^*] \in \mathrm{Cl}_{K^*}\}.$$

Proof. The first statement follows from [Shi97, Theorem 18.6 part (1)]. Observe that $\mathfrak{A}$ is a fractional ideal, so the action by the finite idèles factors through the maximal compact subgroup. The second statement is a consequence of the fact that the Artin homomorphism is bijective. □

If G is an abelian group, then $G[2]$ denotes its subgroup of elements that have order dividing 2.

We now specialise to the case we are interested in. The following lemma is well known.

Lemma 3.10. *Suppose that $K/\mathbb{Q}$ is cyclic of degree 4. Then (K,Φ) is primitive, A is a simple abelian variety, $K^* = K$, and*

$$\#\mathrm{N}_\Phi(\mathrm{Cl}_{K^*}) \geqslant \frac{\#\,\mathrm{Cl}_K}{\#\,\mathrm{Cl}_K[2]\cdot\#\,\mathrm{Cl}_F}. \tag{3.22}$$

Proof. In this lemma we identify the embeddings in the CM type $\Phi = \{\sigma_1, \sigma_2\}$ with automorphisms of K. By hypothesis, $\mathrm{Gal}(K/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$. As $\sigma_2\sigma_1^{-1}$ is neither the identity nor complex conjugation, it must generate the Galois group. So, (K,Φ) is primitive by [Shi97, Proposition 26, ch. II]. Therefore, A is simple.

Further down in [Shi97, Example 8.4], Shimura remarks that $K^* = K$ and $\Phi^* = \{\sigma_1^{-1}, \sigma_2^{-1}\}$ under the assumption that (K,Φ) is primitive and $K/\mathbb{Q}$ is abelian.

Observe that $\mathrm{N}_\Phi([\mathfrak{B}]) = \sigma_1^{-1}([\mathfrak{B}])\sigma_2^{-1}([\mathfrak{B}])$ if $[\mathfrak{B}] \in \mathrm{Cl}_{K^*}$ and recall that $\sigma_2\sigma_1^{-1}$ generates $\mathrm{Gal}(K/\mathbb{Q})$. To prove the final claim, it suffices to consider the case where σ_1 is the identity and σ_2 generates the Galois group. We abbreviate $\theta = \sigma_2^{-1}$. Let $\alpha \in \mathrm{Cl}_K$ be arbitrary. As K is a CM field with totally real subfield F, the class $\alpha\overline{\alpha}$ is represented by an ideal generated by an ideal of $\mathcal{O}_F$. Thus, there are at most $\#\,\mathrm{Cl}_F$ different possibilities for the class $\alpha\overline{\alpha}$. On the other hand, $\alpha\theta(\alpha)(\theta(\alpha)\theta^2(\alpha))^{-1} = \alpha\theta^2(\alpha)^{-1} = \alpha\overline{\alpha}^{-1}$ lies in $\mathrm{N}_\Phi(\mathrm{Cl}_{K^*})$. So, $\alpha^2 = (\alpha\overline{\alpha})(\alpha\overline{\alpha}^{-1})$ lies in at most $\#\,\mathrm{Cl}_F$ translates of $\mathrm{N}_\Phi(\mathrm{Cl}_{K^*})$. The bound (3.22) follows because Cl_K contains precisely $\#\,\mathrm{Cl}_K/\#\,\mathrm{Cl}_K[2]$ squares. □

Lemma 3.11. *Let $\epsilon > 0$. There exists a constant $c = c(\epsilon, F) > 0$ depending only on ϵ and the totally real field F with the following property. Suppose that $K/\mathbb{Q}$ is cyclic of degree 4; then*

$$\#\mathrm{N}_\Phi(\mathrm{Cl}_{K^*}) \geqslant c|\Delta_K|^{1/2-\epsilon}.$$

Proof. Zhang's proposition [Zha05, Proposition 6.3(2)] implies that $\#\,\mathrm{Cl}_K[2] \leqslant c|\Delta_K|^{\epsilon}$, where c depends only on $\epsilon > 0$.

Next we bound the class number of K from below using the Brauer–Siegel theorem. For any imaginary quadratic extension K of F that is Galois over $\mathbb{Q}$, we have $R_K \#\,\mathrm{Cl}_K \geqslant c|\Delta_K|^{1/2-\epsilon}$ with a possibly smaller constant $c > 0$; here $R_K > 0$ denotes the regulator of K. By [Was82, Proposition 4.16], R_K is at most twice the regulator of F. As we allow c to depend on F, our lemma follows from Lemma 3.10 on decreasing this constant c if necessary. □

4. Faltings height

We begin by recalling the definition of the Faltings height of an abelian variety. Then we apply a known case of Colmez's conjecture to compute the Faltings height of certain CM abelian varieties in terms of the L-functions. Finally, we will give an alternative formula for the Faltings height for an abelian variety that has good reduction everywhere and that is the jacobian of a hyperelliptic curve in genus 2.

4.1 General abelian varieties

Let A be an abelian variety of dimension $g \geqslant 1$ defined over a number field k. After extending k, we may suppose that A has semi-stable reduction at all finite places of k.

Put $S = \mathrm{Spec}\,\mathcal{O}_k$, where $\mathcal{O}_k$ is the ring of integers of k. Let $\mathcal{A} \longrightarrow S$ be the Néron model of A. We shall denote by $\varepsilon : S \to \mathcal{A}$ the zero section.

We write $\Omega^g_{\mathcal{A}/S}$ for the gth exterior power of the sheaf of relative differentials of the smooth morphism $\mathcal{A} \to S$. This is an invertible sheaf on $\mathcal{A}$ and its pull-back $\varepsilon^*\Omega^g_{\mathcal{A}/S}$ is an invertible sheaf on $\mathrm{Spec}\,\mathcal{O}_k$.

For any embedding σ of k in $\mathbb{C}$, the base change $\mathcal{A}_\sigma = \mathcal{A} \otimes_\sigma \mathbb{C}$ is an abelian variety over $\mathrm{Spec}\,\mathbb{C}$. There is a canonical isomorphism

$$\varepsilon^*\Omega^g_{\mathcal{A}/S} \otimes_\sigma \mathbb{C} \simeq H^0(\mathcal{A}_\sigma, \Omega^g_{\mathcal{A}_\sigma})$$

as vector spaces over $\mathbb{C}$. So, we can equip the first vector space with the L^2-metric $\|\cdot\|_\sigma$ defined by

$$\|\alpha\|^2_\sigma = \frac{i^{g^2}}{c_0^g} \int_{\mathcal{A}_\sigma(\mathbb{C})} \alpha \wedge \overline{\alpha}$$

for a normalising universal constant $c_0 > 0$.

The rank-one $\mathcal{O}_k$-module $\varepsilon^*\Omega^g_{\mathcal{A}/S}$, together with the hermitian norms $\|\cdot\|_\sigma$ at infinity, defines a hermitian line bundle over S.

Recall that for any hermitian line bundle $\overline{\omega}$ over S, the Arakelov degree of $\overline{\omega}$ is defined as

$$\widehat{\deg}(\overline{\omega}) = \log \#(\omega/\mathcal{O}_k \eta) - \sum_{\sigma:\, k \to \mathbb{C}} \log \|\eta\|_\sigma,$$

where η is any non-zero section of $\overline{\omega}$. The Arakelov degree is independent of the choice of η by the product formula.

We now give the definition of the Faltings height, see [Fal83, p. 354], which is sometimes also called the differential height.

DEFINITION 4.1. The stable Faltings height of A is

$$h(A) := \frac{1}{[k:\mathbb{Q}]}\widehat{\deg}(\overline{\omega}) \quad \text{where } \omega = \varepsilon^*\Omega^g_{\mathcal{A}/S}$$

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becomes a hermitian line bundle $\overline{\omega}$ when equipped with the metrics mentioned above and $c_0 = 2\pi$.

To see that it satisfies a Northcott theorem, see for instance Faltings's Satz 1 [Fal83, pp. 356 and 357] or the theorem of Bost–David made available in [Paz12a].

For a discussion on some interesting values for c_0, see [Paz12b]. Faltings uses $c_0 = 2$. In this paper, the choice will be $c_0 = 2\pi$, following Deligne and Bost. This choice removes the π in the expression derived from the Chowla–Selberg formula in the CM case. The choice $c_0 = (2\pi)^2$ leads to a non-negative height $h_{F^+}(A)$ due to a result of Bost. In any case, one has the easy relations

$$\begin{aligned} h(A) = h_{\mathrm{Deligne}}(A) = h_{\mathrm{Bost}}(A) &= \frac{g}{2}\log 2\pi + h_{\mathrm{Colmez}}(A) \\ &= \frac{g}{2}\log \pi + h_{\mathrm{Faltings}}(A) = -\frac{g}{2}\log 2\pi + h_{F^+}(A). \end{aligned}$$

4.2 Colmez's conjecture

Colmez's conjecture [Col93] relates the Faltings height of an abelian variety with complex multiplication and the logarithmic derivative of certain Artin L-functions at $s = 0$. In the same paper Colmez proved his conjecture for CM fields that are abelian extensions of $\mathbb{Q}$ and satisfy a ramification condition above 2. Obus [Obu13] then generalised the result by dropping the ramification restriction. Yang [Yan10] verified the conjecture for certain abelian surfaces where the CM field is not Galois over $\mathbb{Q}$. Our work will rely only on the case when the CM field is a cyclic, quartic extension of the rationals.

Let us briefly recall Colmez's conjecture when the CM field K is an abelian extension of $\mathbb{Q}$ of degree $2g$.

Let $\Phi = \{\varphi_1, \ldots, \varphi_g\}$ be a CM type of K. If $\varphi : K \to \overline{\mathbb{Q}}$ is an embedding, Colmez sets

$$a_{K,\varphi,\Phi}(g_0) = \begin{cases} 1 & \text{if } g_0\varphi \in \Phi, \\ 0 & \text{otherwise} \end{cases}$$

for all $g_0 \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $A_{K,\Phi} = \sum_{\varphi\in\Phi} a_{K,\varphi,\Phi}$. Then $A_{K,\Phi}$ factors through $\mathrm{Gal}(K/\mathbb{Q})$ and by abuse of notation we sometimes consider $A_{K,\Phi}$ as a function on this Galois group. It is a $\mathbb{C}$-linear combination of the irreducible characters of $\mathrm{Gal}(K/\mathbb{Q})$. Moreover, the Artin L-series attached to any character that contributes to this sum is holomorphic and non-zero at $s = 0$. If χ is any character of $\mathrm{Gal}(K/\mathbb{Q})$, then f_χ denotes the conductor of χ.

In the following result we use the normalisation of the Faltings height used in § 4.1.

Theorem 4.2 (Colmez, Obus). *Let A be an abelian variety defined over a number field that is a subfield of $\mathbb{C}$. We suppose that A has complex multiplication by the ring of integers of a CM field K of degree $2\dim A$ over $\mathbb{Q}$. This data provides a CM type Φ of K. Suppose in addition that $K/\mathbb{Q}$ is an abelian extension. We decompose $A_{K,\Phi} = \sum_m c_m\chi_m$, where the χ_m denote the irreducible characters of* $\mathrm{Gal}(K/\mathbb{Q})$. *Then*

$$h(A) = \left(-\sum_m c_m \left(\frac{L'(\chi_m, s)}{L(\chi_m, s)} + \frac{1}{2}\log f_{\chi_m}\right) + \frac{g}{2}\log 2\pi\right)\Bigg|_{s=0},$$

where the right-hand side is evaluated at $s = 0$.

Proof. We refer to Colmez [Col93, Théorèmes 0.3(ii) and III.2.9], from which the result follows modulo a rational multiple of $\log 2$. Subsequent work of Obus [Obu13] removed this ambiguity. □

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Let us consider what happens for an abelian surface when $K/\mathbb{Q}$ is cyclic.

Proposition 4.3. *Suppose that K is a CM field with $K/\mathbb{Q}$ cyclic of degree 4 and let F be the real quadratic subfield of K.*

(i) *Let Φ be any CM type of K. If $g_0 \in \mathrm{Gal}(K/\mathbb{Q})$, then*

$$A_{K,\Phi}(g_0) = \begin{cases} 2 & \text{if } g_0 = 1, \\ 0 & \text{if } g_0 \text{ has order } 2, \\ 1 & \text{if } g_0 \text{ has order } 4. \end{cases}$$

(ii) *As a function on $\mathrm{Gal}(K/\mathbb{Q})$, we can decompose $A_{K,\Phi} = \chi_0 + \frac{1}{2}\chi$, where χ_0 is the trivial character and χ is induced by the non-trivial character of $\mathrm{Gal}(K/F)$. Moreover, the conductor f_χ of χ is Δ_K/Δ_F.*

(iii) *Let A be a simple abelian surface with endomorphism ring $\mathcal{O}_K$. Then*

$$h(A) = -\frac{1}{2}\frac{L'(0)}{L(0)} - \frac{1}{4}\log\frac{\Delta_K}{\Delta_F} = \frac{1}{2}\frac{L'(1)}{L(1)} + \frac{1}{4}\log\frac{\Delta_K}{\Delta_F} - \log(2\pi) - \gamma_{\mathbb{Q}},$$

where $L(s) = \zeta_K(s)/\zeta_F(s)$ is a quotient of the Dedekind ζ-functions of K and F, respectively, and $\gamma_{\mathbb{Q}} = 0.577215\ldots$ denotes Euler's constant.

(iv) *Let A be as in (iii). Then*

$$h(A) \geqslant -c + \frac{\sqrt{5}}{20}\log\Delta_K,$$

where c is a constant that depends only on F.

Proof. Let us write $\mathrm{Gal}(K/\mathbb{Q}) = \{1, h, h^2, h^3\}$. Then h has order 4 and h^2 is complex conjugation on K. By definition, we have $a_{K,\varphi,\Phi}(1) = 1$ for all $\varphi \in \Phi$. So, $A_{K,\Phi}(1) = 2$. On the other hand, no two elements of Φ are equal modulo complex conjugation. So, $A_{K,\Phi}(h^2) = 0$. Finally, $A_{K,\varphi,\Phi}(h) \in \{0, 1, 2\}$. If $\Phi = \{\varphi_1, \varphi_2\}$, then simultaneous equalities $h\varphi_1 = \varphi_2$ and $h\varphi_2 = \varphi_1$ are impossible. This rules out 2. We can also rule out 0 since $h\varphi_1 = h^2\varphi_2$ and $h\varphi_2 = h^2\varphi_1$ are impossible too. Thus, $A_{K,\varphi,\Phi}(h) = 1$ and by symmetry we also find that $A_{K,\varphi,\Phi}(h^3) = 1$. This completes the proof of part (i).

If χ is the character of $\mathrm{Gal}(K/\mathbb{Q})$ induced by the non-trivial irreducible representation of $\mathrm{Gal}(K/F)$, then

$$\chi(h^k) = \begin{cases} 2 & \text{if } k = 0, \\ 0 & \text{if } k = 1, \\ -2 & \text{if } k = 2, \\ 0 & \text{if } k = 3. \end{cases}$$

We observe that $A_{K,\Phi} = \chi_0 + \frac{1}{2}\chi$ and this yields the first part of (ii).

The conductor f_χ equals $\Delta_F \mathrm{N}(\mathfrak{d}_{K/F})$ by [Neu99, Proposition VII.11.7(iii)], where $\mathfrak{d}_{K/F}$ is the relative discriminant of K/F. The final statement of part (ii) follows as $\Delta_K = \Delta_F^2 \mathrm{N}(\mathfrak{d}_{K/F})$.

To prove (iii), we first remark that $L(s, \chi_0)$ is the Riemann ζ-function and that $\zeta_K(s)$ factors as $\zeta_F(s)L(s,\chi)$ with $L(\cdot,\chi)$ the Artin L-function attached to the character χ.

The first equality in (iii) now follows from Theorem 4.2 applied to (ii) and since $\zeta'(0)/\zeta(0) = \log 2\pi$. The second equality follows from the functional equation of the Dedekind ζ-function.

If M is any number field, then γ_M denotes the constant term in the Taylor expansion around $s = 1$ of the logarithmic derivative of the Dedekind ζ-function of M. Then γ_M is called the

Euler–Kronecker constant of M and generalises Euler's constant $\gamma_{\mathbb{Q}}$ to number fields. Badzyan's theorem [Bad10, Theorem 1] yields the lower bound

$$\gamma_M \geqslant -\tfrac{1}{2}(1 - \tfrac{\sqrt{5}}{5}) \log |\Delta_M|.$$

We have $L'(1)/L(1) = \gamma_K - \gamma_F$ and so part (iv) follows from Badzyan's bound together with part (iii). $\square$

Colmez [Col98] also obtained a lower bound for the Faltings height related to (iv) above.

Part (i) together with Theorem 4.2 implies that the Faltings height does not depend on the CM type of K. This was originally observed by Yang [Yan10] and is specific to degree 4.

4.3 Models of curves of genus 2

This section explains the choice of models used in the next section to give an explicit formula for the Faltings height of abelian surfaces. We will use Weierstrass models of degree 5 and of degree 6 for our curves of genus 2. To be able to choose models of degree 5 for a curve C, one needs to have at least one rational Weierstrass point on C, which can be obtained by passing to a finite field extension if needed. As plane models they are singular at infinity; one will recover the curve C through desingularisation.

We work with hyperelliptic equations for a curve C of genus 2 defined over a field k of characteristic 0.

Suppose that $C(k)$ contains a Weierstrass point of C. By Lockhart [Loc94, Proposition 1.2], there is a monic polynomial $P \in k[x]$ of degree 5 such that an open, affine subset of C is isomorphic to the affine curve determined by the equation $\mathcal{E} : y^2 = P$. We call $\mathcal{E}$ a restricted Weierstrass equation for C. Lockhart defined the discriminant of $\mathcal{E}$ as $2^8 \operatorname{disc}_5(P) \in k^\times$.

Say k is a subfield of $\mathbb{C}$. As on [Loc94, p. 740], we fix an ordering on the roots of P and attach a rank-4 discrete subgroup Λ of $\mathbb{C}^2$ and a period matrix $Z_{\mathcal{E}} \in \mathbb{H}_2$ to $\mathcal{E}$. We write $V_{\mathcal{E}} > 0$ for the covolume of Λ in $\mathbb{C}^2$.

Now we define a larger class of Weierstrass equations.

DEFINITION 4.4. A Weierstrass equation $\mathcal{E}$ for C/k is an equation

$$\mathcal{E} : y^2 + Qy = P$$

that describes an open, affine subset of C, where $P, Q \in k[x]$ with $\deg P \leqslant 6$ and $\deg Q \leqslant 3$. The discriminant of $\mathcal{E}$ is defined as $\Delta_{\mathcal{E}} = 2^{-12} \operatorname{disc}_6(4P + Q^2)$; it is a non-zero element of k.

Suppose that k is the field of fractions of a discrete valuation ring. We call $\mathcal{E}$ integral if P and Q have integral coefficients. The minimal discriminant $\Delta^0_{\min}(C)$ of C is the ideal of the ring of integers generated by a discriminant with minimal valuation among the discriminants of the integral equations of C. In Liu's terminology [Liu94], $\Delta^0_{\min}(C)$ is called the naive minimal discriminant. If k is a number field, then by abuse of notation we let $\Delta^0_{\min}(C)$ denote the ideal of $\mathcal{O}_k$ that is minimal at each finite place of k.

If $\mathcal{E} : y^2 = P(x)$ is a restricted Weierstrass equation as in Lockhart's work, then his notation of discriminant coincides with the one used above, i.e., $2^8 \operatorname{disc}_5(P) = \Delta_{\mathcal{E}}$ follows from basic properties of the discriminant.

Weierstrass equations are unique up to the following change of variables, see [Liu02, Corollary 4.33],

$$(**) \quad x = \frac{ax' + b}{cx' + d} \quad \text{and} \quad y = \frac{H(x') + ey'}{(cx' + d)^3},$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$, $e \in k^*$, $H \in k[x']$ with $\deg H \leqslant 3$.

4.4 Hyperelliptic jacobians in genus 2

In this section we state a formula for the Faltings height of the jacobian of a genus 2 curve if the said jacobian has potentially good reduction at all finite places. Ueno [Uen88] had a related expression for the Faltings height, but with a restriction on reduction type which is incommensurable with ours. The second-named author proved [Paz12b] another formula for the Faltings height of the jacobian of a hyperelliptic curve of arbitrary genus.

In order to formulate our result, we recall the definition of relevant theta functions and the 10 non-trivial theta constants in dimension 2. The latter correspond precisely to the even characteristics

$$\Theta_1 = \left\{ \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\0\\1/2 \end{bmatrix}, \begin{bmatrix} 0\\0\\1/2\\0 \end{bmatrix}, \begin{bmatrix} 0\\0\\1/2\\1/2 \end{bmatrix} \right\},$$
$$\Theta_2 = \left\{ \begin{bmatrix} 1/2\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1/2\\0\\0 \end{bmatrix}, \begin{bmatrix} 1/2\\1/2\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\1/2\\1/2\\0 \end{bmatrix}, \begin{bmatrix} 1/2\\0\\0\\1/2 \end{bmatrix}, \begin{bmatrix} 1/2\\1/2\\1/2\\1/2 \end{bmatrix} \right\}.$$

We abbreviate $\mathcal{Z}_2 = \Theta_1 \cup \Theta_2$, the union being disjoint. Say ${}^t(a,b) \in \mathcal{Z}_2$ with $a,b \in \frac{1}{2}\mathbb{Z}^2$. We denote

$$Q_{ab}(n) = {}^t(n+a)Z(n+a) + 2\,{}^t(n+a)b \tag{4.1}$$

for $n \in \mathbb{Z}^2$ and thus get a theta function

$$\theta_{ab}(0,Z) = \sum_{n\in\mathbb{Z}^2} e^{i\pi Q_{ab}(n)}.$$

We will use the classical Siegel cusp form

$$\chi_{10}(Z) = \prod_{m\in\mathcal{Z}_2} \theta_m(0,Z)^2, \quad \text{where } Z \in \mathbb{H}_2, \tag{4.2}$$

of weight 10; cf. the second Remark after Proposition 2, [Kli90, § 9]. So,

$$Z \mapsto |\chi_{10}(Z)| \det \operatorname{Im}(Z)^5$$

is an $\mathrm{Sp}_4(\mathbb{Z})$-invariant, real analytic map $\mathbb{H}_2 \to \mathbb{R}$.

For a finite place ν of a number field, we write

$$\iota(\nu) = \begin{cases} 4 & \text{if } \nu \mid 2,\\ 3 & \text{if } \nu \mid 3,\\ 1 & \text{otherwise.} \end{cases} \tag{4.3}$$

We recall that d_ν denotes a local degree and is defined in § 2.

Theorem 4.5. *Let C be a curve of genus 2 defined over a number field k such that $C(k)$ contains a Weierstrass point of C and such that $\mathrm{Jac}(C)$ has good reduction at all finite places of k. Let $J_2, J_6, J_8, J_{10} \in k$ be Igusa's invariants attached to C. The following properties hold true.*

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(i) *For any embedding $\sigma : k \to \mathbb{C}$, let Z_σ be a period matrix coming from a restricted Weierstrass model of $C \otimes_\sigma \mathbb{C}$ as in § 4.3; then $\chi_{10}(Z_\sigma) \neq 0$. Moreover, we have*

$$h(\mathrm{Jac}(C)) = \frac{1}{[k:\mathbb{Q}]}\bigg(\frac{1}{60}\sum_{\nu \in M_k^0} \frac{d_\nu}{\iota(\nu)} \log \max\{1, |J_{10}^{-\iota(\nu)} J_{2\iota(\nu)}^5|_\nu\} \\ - \frac{1}{10}\sum_{\sigma : k \to \mathbb{C}} \log(2^8 \pi^{10} |\chi_{10}(Z_\sigma)| \det(\mathrm{Im}\, Z_\sigma)^5)\bigg).$$

(ii) *Let ν be a finite place of k; then*

$$\mathrm{ord}_\nu \Delta_{\min}^0(C) = \frac{1}{\iota(\nu)} \max\{0, -\mathrm{ord}_\nu(J_{10}^{-\iota} J_{2\iota}^5)\}.$$

We will prove this theorem after some preliminary work. But first we state an immediate corollary.

Corollary 4.6. *Let C, k, and the Z_σ be as in Theorem 4.5; then*

$$h(\mathrm{Jac}(C)) = \frac{1}{[k:\mathbb{Q}]}\bigg(\frac{1}{60}\log \mathrm{N}(\Delta_{\min}^0(C)) - \frac{1}{10}\sum_{\sigma : k \to \mathbb{C}} \log(2^8 \pi^{10} |\chi_{10}(Z_\sigma)| \det \mathrm{Im}(Z_\sigma)^5)\bigg).$$

Suppose that C is a curve of genus 2 defined over a number field k and presented by a Weierstrass equation as in Definition 4.4. There exists a classical basis for $H^0(C, \Omega^1_{C/k})$ given by

$$\omega_1 = \frac{dx}{2y + Q(x)} \quad \text{and} \quad \omega_2 = \frac{x\, dx}{2y + Q(x)}.$$

Consider the section $\omega_1 \wedge \omega_2 \in \det H^0(C, \Omega^1_{C/k})$. A change of variables in the Weierstrass models of C leaves

$$\eta = \Delta_{\mathcal{E}}^2 (\omega_1 \wedge \omega_2)^{\otimes 20} \tag{4.4}$$

invariant; cf. [Liu94, § 1.3].

We now show how to use η, a differential form on the curve C, to compute the Faltings height of the jacobian $\mathrm{Jac}(C)$.

Suppose that $p : \mathcal{C} \to S$ is a regular semi-stable model of C over $S = \mathrm{Spec}\, \mathcal{O}_k$. We now prove that

$$h(\mathrm{Jac}(C)) = \frac{1}{[k:\mathbb{Q}]} \widehat{\deg}(\det p_* \omega_{\mathcal{C}/S}), \tag{4.5}$$

where $\omega_{\mathcal{C}/S}$ denotes the relative canonical bundle and where the hermitian metrics on $\det p_* \omega_{\mathcal{C}/S}$ are determined by

$$\|\omega_1 \wedge \omega_2\|_\sigma^2 = \det\bigg(\frac{i}{2\pi} \int_{(C \otimes_\sigma \mathbb{C})(\mathbb{C})} \omega_l \wedge \overline{\omega_m}\bigg)_{1 \leqslant l, m \leqslant 2} \tag{4.6}$$

with $\sigma : k \to \mathbb{C}$ an embedding. Indeed, suppose that ε is a section of $\mathcal{C} \to S$ and let $\mathrm{Pic}^0_{\mathcal{C}/S}$ be the relative Picard scheme of degree 0. Then $\mathrm{Pic}^0_{\mathcal{C}/S}$ is the identity component of the Néron model $\mathcal{A}$ of the jacobian of C by [BLR90, Theorem 4, ch. 9.5]. This is an open subscheme of $\mathcal{A}$ which contains the image of ε. Therefore, $\varepsilon^* \Omega_{\mathrm{Pic}^0_{\mathcal{C}/S}/S} = \varepsilon^* \Omega_{\mathcal{A}/S}$, which allows us to replace $\mathcal{A}$ by $\mathrm{Pic}^0_{\mathcal{C}/S}$ in the computations below. Then

$$\mathrm{Lie}(\mathcal{A}) \simeq R^1 p_* \mathcal{O}_{\mathcal{C}}.$$

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Moreover, by Grothendieck duality (see [Liu02, 6.4.3, p. 243]), we have

$$(R^1 p_* \mathcal{O}_{\mathcal{C}})^\vee \simeq p_* \omega_{\mathcal{C}/S}.$$

Then

$$\varepsilon^* \Omega^1_{\mathcal{A}/S} \simeq \mathrm{Lie}(\mathcal{A})^\vee \simeq p_* \omega_{\mathcal{C}/S}$$

and hence

$$\varepsilon^* \Omega^2_{\mathcal{A}/S} \simeq \det p_* \omega_{\mathcal{C}/S},$$

which turns out to be an isometry by 4.15 of the second lecture of [Szp85]. We conclude (4.5) by taking the Arakelov degree.

4.4.1 *Archimedean places.* We use Lockhart [Loc94] and Mumford [Mum84] as references for these places. If T is a subset of $\{1,2,3,4,5\}$, one then defines $m_T = \sum_{i \in T} m_i \in \frac{1}{2}\mathbb{Z}^4$ with

$$m_1 = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \\ 0 \end{bmatrix}, \quad m_3 = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}, \quad m_4 = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad m_5 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \end{bmatrix}.$$

In the notation of Lockhart's definition [Loc94, Definition 3.1], $\varphi(Z)$ is the fourth power of

$$\chi_{10}(Z) = \prod_{\substack{T \subseteq \{1,2,3,4,5\} \\ \#T = 3}} \theta_{m_{T \circ \{1,3,5\}}}(0, Z)^2, \tag{4.7}$$

where $\circ$ denotes the symmetric difference of sets and θ_m are theta functions from § 4.4.

The following result is from Lockhart [Loc94, Proposition 3.3]. In his notation, we have $r = \binom{5}{3} = 10$ and $n = \binom{4}{3} = 4$.

Proposition 4.7. *Let C be a curve of genus 2 defined over $\mathbb{C}$ and suppose that $\mathcal{E}$ is a restricted Weierstrass equation for C. One has the uniformisation $\mathrm{Jac}(C)(\mathbb{C}) \simeq \mathbb{C}^2/\Lambda_{\mathcal{E}}$ with the lattice $\Lambda_{\mathcal{E}} \subseteq \mathbb{C}^2$, its period matrix $Z_{\mathcal{E}} \in \mathbb{H}_2$ and covolume $V(\Lambda_{\mathcal{E}})$, both as near the beginning of § 4.3. Then $|\Delta_{\mathcal{E}}| V(\Lambda_{\mathcal{E}})^5$ is independent of the equation $\mathcal{E}$ and*

$$|\Delta_{\mathcal{E}}| V(\Lambda_{\mathcal{E}})^5 = 2^8 \pi^{20} |\chi_{10}(Z_{\mathcal{E}})| \det \mathrm{Im}(Z_{\mathcal{E}})^5.$$

Next comes the archimedean contribution of the section η from (4.4).

Proposition 4.8. *Let C be a curve of genus 2 defined over a number field k. Let $\sigma : k \to \mathbb{C}$ be an embedding and suppose that $\mathcal{E}$ is a restricted Weierstrass equation for $C \otimes_\sigma \mathbb{C}$. We write $\omega_1 = dx/(2y), \omega_2 = x\, dx/(2y)$, and $\eta = \Delta_{\mathcal{E}}^2 (\omega_1 \wedge \omega_2)^{\otimes 20}$. Then $\chi_{10}(Z_{\mathcal{E}}) \neq 0$ and*

$$\log \|\eta\|_\sigma = 2 \log(2^8 \pi^{10} |\chi_{10}(Z_{\mathcal{E}})| \det \mathrm{Im}(Z_{\mathcal{E}})^5).$$

Proof. We use Proposition 4.7 to compute

$$\begin{aligned} \|\eta\|_\sigma^2 &= |\Delta_{\mathcal{E}}|_\sigma^4 (\|\omega_1 \wedge \omega_2\|_\sigma^2)^{20} \\ &= |\Delta_{\mathcal{E}}|_\sigma^4 \det\left(\frac{i}{2\pi} \int_{(C \otimes_\sigma \mathbb{C})(\mathbb{C})} \omega_l \wedge \overline{\omega_m} \right)_{1 \leqslant l,m \leqslant 2}^{20} \\ &= \frac{1}{\pi^{40}} |\Delta_{\mathcal{E}}|_\sigma^4 V(\Lambda_{\mathcal{E}})^{20} \\ &= \frac{1}{\pi^{40}} 2^{32} \pi^{80} |\chi_{10}(Z_{\mathcal{E}})|^4 \det(\mathrm{Im}\, Z_{\mathcal{E}})^{20}; \end{aligned}$$

here the second equality requires the definition (4.6), the next one is classical, cf. [GH78, ch. 2.2], and the fourth one is Proposition 4.7.

Hence, $\|\eta\|_\sigma = 2^{16} \pi^{20} |\chi_{10}(Z_{\mathcal{E}})|^2 \det(\mathrm{Im}\, Z)^{10}$ and it follows in particular that $\chi_{10}(Z_{\mathcal{E}}) \neq 0$. $\square$

4.4.2 *Non-archimedean places.*

PROPOSITION 4.9. *Let C be a curve of genus 2 defined over a number field k. Let ν be a finite place of k at which $\mathrm{Jac}(C)$ has good reduction. If η is as in (4.4), then*

$$\mathrm{ord}_\nu(\eta) = \frac{1}{3\iota}\max\{0, \mathrm{ord}_\nu(J_{10}^{\iota}J_{2\iota}^{-5})\} = \frac{1}{3}\mathrm{ord}_\nu \Delta_{\min}^0(C),$$

where $\iota = \iota(\nu)$ is as in (4.3) and where the J_2, J_6, J_8, J_{10} are as in Theorem 4.5.

Proof. Let k_ν^{unr} be the maximal unramified extension of k_ν inside a fixed algebraic closure of k_ν. This is a strictly henselian field equipped with a discrete valuation whose ring of integers $\mathcal{O}$ has an algebraically closed residue field. Thus, $\mathcal{O}$ satisfies the hypothesis needed for the references below.

Recall that $\mathrm{Jac}(C \otimes_k k_\nu^{\mathrm{unr}})$ has good reduction by hypothesis; it has in particular semi-stable reduction. So, the curve $C \otimes_k k_\nu^{\mathrm{unr}}$ has semi-stable reduction by Deligne and Mumford's theorem cited in the introduction. The minimal regular model $f : \mathcal{C}_{\min} \to S$ of $C \otimes_k k_\nu^{\mathrm{unr}}$ over $S = \mathrm{Spec}\,\mathcal{O}$ is semi-stable by [Liu02, Theorem 10.3.34(a)]. The canonical model $\mathcal{C}_{\mathrm{st}}$, obtained via a contraction $\mathcal{C}_{\min} \to \mathcal{C}_{\mathrm{st}}$, is stable by part (b) of the same theorem [Liu02]. It is well known that exactly seven geometric configurations can arise for the geometric special fibre of the stable model. They are pictured in [Liu02, Example 10.3.6].

We infer from a theorem of Raynaud that the special fibre of $\mathcal{C}_{\mathrm{st}} \to S$ is either smooth or a union of two elliptic curves meeting at a point; see the paragraph before [Liu93, Proposition 2].

Later on, we will consider these two cases separately. But first let us fix a Weierstrass equation $\mathcal{E} : y^2 + Qy = P$ for $C \otimes_k k_\nu^{\mathrm{unr}}$ such that

$$\omega_1 = \frac{dx}{2y+Q} \quad \text{and} \quad \omega_2 = \frac{x\,dx}{2y+Q}$$

constitute an $\mathcal{O}$-basis of $H^0(\mathcal{C}_{\min}, \omega_{\mathcal{C}_{\min}/S})$; its existence is guaranteed by [Liu94, Proposition 2(a)].

Then

$$\eta = \Delta_{\mathcal{E}}^2(\omega_1 \wedge \omega_2)^{\otimes 20} \in \det H^0(\mathcal{C}_{\min}, \omega_{\mathcal{C}_{\min}/S})^{\otimes 20}$$

by the invariance under coordinate change mentioned after Theorem 4.5. The equation $\mathcal{E}$ is minimal in Ueno's sense [Liu94, Definition 1], and we use $\mathrm{ord}_\nu(\Delta_{\min})$ to denote the order of Ueno's minimal discriminant; cf. the same definition. Observe that this order is non-negative, but may be less than the order of the minimal discriminant $\Delta_{\min}^0(C)$. By Proposition 3 and its corollary, both in [Liu94], we find that

$$\mathrm{ord}(\eta) = 2\,\mathrm{ord}_\nu(\Delta_{\min}). \tag{4.8}$$

First, let us suppose that the special fibre of the stable model is not smooth. Then we are in [Liu93, case (V) of Théorème 1] and by [Liu93, Proposition 2] $\mathcal{C}_{\min}$ is of type $[I_0\text{-}I_0\text{-}m]$ in Namikawa and Ueno's classification [NU73]. The value

$$m = \frac{1}{12\iota}\mathrm{ord}_\nu(J_{10}^{\iota}J_{2\iota}^{-5}) \geqslant 1, \tag{4.9}$$

computed in part (v) of the proposition, is the thickness of the singular point in the special fibre of $\mathcal{C}_{\mathrm{st}}$, see [Liu02, Definition 3.23]; here we used that $I_2 = J_2/12$, $I_6 = J_6$, and $I_8 = J_8$ as in Liu's paper [Liu93].

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The fibre of $\mathcal{C}_{\min} \to \mathcal{C}_{\mathrm{st}}$ above the unique singular point is a chain of $m-1$ copies of the projective line. We shall use the Artin conductor of $\mathcal{C}_{\min}/S$; see the introduction [Liu94] for a definition. By [Liu94, Proposition 1],

$$-\mathrm{Art}(\mathcal{C}_{\min}/S) = m; \tag{4.10}$$

indeed, the conductor mentioned in the reference has exponent 0 because $\mathrm{Jac}(C \otimes_k k_\nu^{\mathrm{unr}})$ has good reduction at ν.

Saito proved in [Sai88, Theorem 1] that $-\mathrm{Art}(\mathcal{C}_{\min}/S)$ equals the order of yet a further discriminant attached to $\mathcal{C}_{\min}/S$; its definition is given in [Sai88] and relies on unpublished work of Deligne. Saito attributed this equality to Deligne in the semi-stable case, which covers our application.

Proposition [Sai89] yields

$$\mathrm{ord}_\nu(\Delta_{\min}) = -\mathrm{Art}(\mathcal{C}_{\min}/S) + m = 2m; \tag{4.11}$$

cf. also [Liu94, § 2.1]. Using this we can relate the section η to the Igusa invariants as follows:

$$\mathrm{ord}(\eta) = 2\,\mathrm{ord}_\nu(\Delta_{\min}) = 4m = \frac{1}{3\iota}\,\mathrm{ord}_\nu(J_{10}^{\iota} I_{2\iota}^{-5}), \tag{4.12}$$

where the first equality used (4.8) and the last equality used (4.9). We obtain

$$\mathrm{ord}(\eta) = \frac{1}{3\iota}\,\mathrm{ord}_\nu(J_{10}^{\iota} J_{2\iota}^{-5}) = \frac{1}{3\iota}\max\{0, \mathrm{ord}_\nu(J_{10}^{\iota} J_{2\iota}^{-5})\} \tag{4.13}$$

and hence the first equality of this proposition in the current case.

Next we apply a result of Liu to relate $\mathrm{ord}(\eta)$ to the order of the minimal discriminant $\Delta_{\min}^0(C)$. In Liu's notation [Liu94], we have $c(\mathcal{C}_{\min}) = m$. Théorème 2 of [Liu94] implies that

$$\mathrm{ord}_\nu(\Delta_{\min}^0(C)) = \mathrm{ord}_\nu(\Delta_{\min}) + 10m.$$

So, $\mathrm{ord}_\nu(\Delta_{\min}^0(C)) = 12m$ by (4.11). The second equality in the assertion follows from (4.12).

Second, suppose that the special fibre of $\mathcal{C}_{\mathrm{st}}$ is smooth. Using [Liu94], we find that the Artin conductor of $\mathcal{C}/S$ vanishes. Just as near (4.11), we find that $\mathrm{ord}_\nu(\Delta_{\min}) = 0$ and thus $\mathrm{ord}_\nu(\Delta_{\min}^0(C)) = 0$ as $0 \leqslant \mathrm{ord}_\nu(\Delta_{\min}^0(C)) \leqslant \mathrm{ord}_\nu(\Delta_{\min})$ holds in general by [Liu94, Proposition 2(d)]. Using (4.8), we conclude that $\mathrm{ord}(\eta) = 0$. Théorème 1 of [Liu93], attributed to Igusa, states that $\mathrm{ord}_\nu(J_{10}^{-\iota} J_{2\iota}^{5}) \geqslant 0$ for all $\iota \in \{1,2,3,4,5\}$. In particular, $\mathrm{ord}_\nu(J_{10}^{\iota} J_{2\iota}^{-5}) \leqslant 0$ and so the proposition holds true in this case too. □

4.4.3 *Proof of Theorem 4.5.* Since there is a k-rational Weierstrass point by hypothesis, there is a restricted Weierstrass equation as in Proposition 4.8, with coefficients in k. Part (i) of the theorem follows by studying the local contributions to the Faltings height. The infinite places are handled by Proposition 4.8 and the finite places are dealt with by Proposition 4.9. Observe that the Arakelov degree of η is 20 times the desired Faltings height of $\mathrm{Jac}(C)$.

Part (ii) is the second equality in Proposition 4.9.

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5. Archimedean estimates

5.1 Lower bounds for the Siegel modular form of weight 10 in degree 2

The contribution of the infinite places to the Faltings height in Theorem 4.5 involves the Siegel modular form χ_{10} of weight 10 and degree 2 defined in (4.2). A lower bound for the modulus of χ_{10} can be used to bound the height from below.

In this section the period matrix Z lies in Siegel's fundamental domain $\mathcal{F}_2$ described in § 2.

The modular form χ_{10} vanishes at those elements

$$Z = \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix} \tag{5.1}$$

of $\mathcal{F}_2$ for which z_{12} vanishes and only at those; cf. [Kli90, the proof of Proposition 2, § 9]. They correspond to abelian surfaces that are products of elliptic curves; thus, they are not jacobians of genus 2 curves.

In the following lemmas we implicitly use techniques from the second-named author's work [Paz13] and obtain some minor numerical improvements. We will use a and b to denote components of the even characteristic ${}^t(a,b) \in \mathcal{Z}_2$ from § 4.4 and abbreviate

$$T_{ab} = \Big\{ n \in \mathbb{Z}^2 : \operatorname{Im} Q_{ab}(n) = \min_{m \in \mathbb{Z}^2} \operatorname{Im} Q_{ab}(m) \Big\},$$

where Q_{ab} was defined in (4.1).

Lemma 5.1. *For all $n, n' \in T_{ab}$, we have*

$$e^{i\pi Q_{ab}(n)} = e^{i\pi Q_{ab}(n')}.$$

Moreover, T_{ab} is finite and

$$|\theta_{ab}(0,Z)| \geqslant 2\#T_{ab} \cdot e^{-\pi \min_{m\in\mathbb{Z}^2} \operatorname{Im} Q_{ab}(m)} - \sum_{n \in \mathbb{Z}^2} e^{-\pi \operatorname{Im} Q_{ab}(n)}.$$

Proof. This is [Paz13, Lemma 4.18]. □

Lemma 5.2. *If ${}^t(a,b) \in \Theta_1$, i.e., $a = 0$, one has*

$$|\theta_{ab}(0,Z)| \geqslant 0.44$$

for all $Z \in \mathcal{F}_2$.

Proof. This follows from [Paz13, Proposition 4.19]. □

Lemma 5.3. *If ${}^t(a,b) \in \Theta_2$ with $a \neq [0,0]$ and $a \neq [1/2,1/2]$, one has*

$$|\theta_{ab}(0,Z)| \geqslant 0.75 e^{-\pi {}^t a \operatorname{Im} Z a}.$$

Proof. This follows from [Paz13, Proposition 4.20]. □

The crucial case is $a = b = [1/2, 1/2]$ as the corresponding theta constant vanishes on diagonal matrices in Siegel's fundamental domain.

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Lemma 5.4. *If* ${}^t(a,b)=[1/2,1/2,\nu/2,\nu/2]$ *with* $\nu\in\{0,1\}$, *one has*

$$|\theta_{ab}(0,Z)|\geqslant 1.12|1+(-1)^\nu e^{\pi i z_{12}}|e^{-\pi({}^t a\,\mathrm{Im}\,Za-\mathrm{Im}\,z_{12})}.$$

Proof. This follows from [Paz13, Proposition 4.22] and from

$$2\bigg(2-\bigg(\sum_{m\geqslant 0}e^{-(\pi\sqrt{3}/4)m(m+1)}(2m+1)\bigg)^2\bigg)\geqslant 1.12. \qquad \square$$

Lemma 5.5. *Let* z *be a complex number with* $|\mathrm{Re}(z)|\leqslant\pi$. *Then*

$$|e^{iz/2}+1|\geqslant 1 \quad \textit{and} \quad |e^{iz}-1|\geqslant (1-e^{-1})\min\{1,|z|\}.$$

Proof. The first inequality follows from $\mathrm{Re}(e^{iz/2})\geqslant 0$. For the second inequality, we note that $z\mapsto (e^{iz}-1)/z$ is entire and does not vanish if $|\mathrm{Re}(z)|\leqslant\pi$ and $z\mapsto e^{iz}-1$ does not vanish if $|\mathrm{Re}(z)|\leqslant\pi$ and $|z|\geqslant 1$. By the maximum modulus principle applied to the reciprocals, we deduce that the minimum of $|e^{iz}-1|/\min\{1,|z|\}$ subject to $|\mathrm{Re}(z)|\leqslant\pi$ is attained on $|z|=1$ or $|\mathrm{Re}(z)|=\pi$. In the latter case the quotient is $|e^{-\mathrm{Im}(z)}+1|\geqslant 1$, which is better than the claim. Let us now suppose that $|z|=1$. We assume that $|t|<1-e^{-1}$ with $t=e^{iz}-1$; this will lead to a contradiction and will thus complete this proof. The logarithm $\log(1+t)=\sum_{n\geqslant 1}(-1)^{n+1}t^n/n$ converges and satisfies $e^{\log(1+t)}=1+t=e^{iz}$. So, $\log(1+t)=iz+2\pi i k$ for an integer k. We bound the modulus of $\log(1+t)$ from above using the triangle inequality and obtain

$$|z+2\pi k|=|iz+2\pi i k|\leqslant\sum_{n\geqslant 1}\frac{|t|^n}{n}=-\log(1-|t|)<-\log(1-(1-e^{-1}))=1.$$

This is impossible since $|z|=1$. $\square$

The next proposition combines the previous lemmas.

Proposition 5.6. *For any* $Z\in\mathcal{F}_2$ *as in (5.1), one has*

$$|\chi_{10}(Z)|\geqslant c_0\min\{1,\pi|z_{12}|\}^2e^{-2\pi(\mathrm{Tr}(\mathrm{Im}\,Z)-\mathrm{Im}\,z_{12})}\geqslant c_0\min\{1,\pi|z_{12}|\}^2e^{-2\pi\,\mathrm{Tr}\,\mathrm{Im}\,Z}$$

with $c_0=8\times 10^{-5}$.

Proof. We use Lemmas 5.2–5.4 in connection with the definition (4.2) to obtain

$$|\chi_{10}(Z)|\geqslant 0.44^8\cdot 0.75^8\cdot 1.12^4|e^{i\pi z_{12}}+1|^2|e^{i\pi z_{12}}-1|^2e^{4\pi\,\mathrm{Im}\,z_{12}}\prod_{{}^t(a,b)\in\Theta_2}e^{-2\pi{}^t a\,\mathrm{Im}\,Za}.$$

Observe that $|\mathrm{Re}(z_{12})|\leqslant 1/2$. The first inequality in the assertion follows from this, Lemma 5.5 applied to $2\pi z_{12}$ and πz_{12}, and since the product over Θ_2 equals $e^{-2\pi(\mathrm{Tr}\,\mathrm{Im}\,Z+\mathrm{Im}\,z_{12})}$. The second inequality follows as $Z\in\mathcal{F}_2$ entails $\mathrm{Im}\,z_{12}\geqslant 0$. $\square$

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5.2 Subconvexity

Let K be a number field. Say $\chi : \mathrm{Cl}_K \to \mathbb{C}^\times$ is a character of the class group. We may also think of χ as a Hecke character of conductor $\mathcal{O}_K$. The L-series attached to the character χ is

$$L(s,\chi) = \sum_{\mathfrak{A}} \frac{\chi([\mathfrak{A}])}{\mathrm{N}(\mathfrak{A})^s},$$

where here and below we sum over non-zero ideals $\mathfrak{A}$ of $\mathcal{O}_K$.

It is well known that this Dirichlet series determines a meromorphic function on $\mathbb{C}$ with at most a simple pole at $s = 1$ if χ is the trivial character.

The following subconvexity estimate follows from Michel and Venkatesh's deep Theorem 1.1 [MV10].

Theorem 5.7. *Let F be a totally real number field. There exist constants $c_1 > 0, N > 0$, and $\delta \in (0, 1/4)$ depending on F with the following property. If K/F is an imaginary quadratic extension and $\chi : \mathrm{Cl}_K \to \mathbb{C}^\times$ is a character, then*

$$|L(\tfrac{1}{2} + it, \chi)| \leqslant c_1(1 + |t|)^N |\Delta_K|^{1/4-\delta}.$$

The following lemma involves a well-known trick in analytic number theory; cf. work of Duke *et al.* [ID02, p. 574]. We shift a contour integral into the critical strip and apply the subconvexity result cited above.

Lemma 5.8. *Let F be a totally real number field and let δ be from Theorem 5.7. There is a constant $c_2 > 0$ depending only on F with the following property. Say K is a totally imaginary quadratic extension of F and let H be a coset of a subgroup of Cl_K. If $\epsilon \in (0, 1]$ and $x = \epsilon|\Delta_K|^{1/2}$, then*

$$\frac{1}{\#H} \sum_{\substack{\mathfrak{A} \\ \mathrm{N}(\mathfrak{A}) \leqslant x, [\mathfrak{A}] \in H}} \left(\frac{x}{\mathrm{N}(\mathfrak{A})}\right)^{1/2} \leqslant c_2 \epsilon^{1/2} \max\left\{1, \frac{|\Delta_K|^{1/2-\delta/2}}{\#H}\right\}. \tag{5.2}$$

Proof. We fix a smooth test function $f : (0, \infty) \to [0, \infty)$ that satisfies

$$f(y) = \begin{cases} y^{-1/2} & \text{if } y \in (0, 1], \\ 0 & \text{if } y \geqslant 2. \end{cases} \tag{5.3}$$

Its Mellin transform

$$\tilde{f}(s) = \int_0^\infty f(y) y^{s-1}\, dy$$

exists if $\mathrm{Re}(s) > 1/2$ and the Mellin inversion formula holds; cf. [Coh07, Proposition 9.7.7]. Using in addition [Coh07, Theorem 9.7.5(4)], we see that $\tilde{f}$ decays rapidly; here this means that if $\sigma > 1/2$ is fixed and $N \geqslant 1$, then $|\tilde{f}(\sigma + it)|(1 + |t|)^N$ is a bounded function in $t \in \mathbb{R}$.

For a real number $x > 0$ and a character $\chi : \mathrm{Cl}_K \to \mathbb{C}^\times$, we define

$$S(x, \chi) = \sum_{\mathfrak{A}} \chi([\mathfrak{A}]) f\left(\frac{\mathrm{N}(\mathfrak{A})}{x}\right). \tag{5.4}$$

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The sum is finite since f vanishes at large arguments. If $\sigma \in \mathbb{R}$, then $\int_{(\sigma)}$ signifies the integral along the vertical line $\mathrm{Re}(s) = \sigma$. The Mellin inversion formula leads to

$$S(x,\chi) = \frac{1}{2\pi i} \sum_{\mathfrak{A}} \chi([\mathfrak{A}]) \int_{(2)} \tilde{f}(s) \left(\frac{x}{\mathrm{N}(\mathfrak{A})} \right)^s ds = \frac{1}{2\pi i} \int_{(2)} \tilde{f}(s) \left(\sum_{\mathfrak{A}} \frac{\chi([\mathfrak{A}])}{\mathrm{N}(\mathfrak{A})^s} \right) x^s \, ds;$$

the sum and the integral commute by the dominant convergence theorem. The inner sum is the L-function $L(s,\chi)$ and hence

$$S(x,\chi) = \frac{1}{2\pi i} \int_{(2)} \tilde{f}(s) L(s,\chi) x^s \, ds.$$

Let H_0 denote the translate of H containing the unit element; it is a subgroup of Cl_K. Suppose that χ is any character with $\chi|_{H_0} = 1$. The function $|L(\sigma+it,\chi)|$ has at most polynomial growth in the imaginary part t if $\sigma \in (1/2,1)$ is fixed. By a contour shift and by the decay property of $\tilde{f}$, we arrive at

$$S(x,\chi) = \frac{1}{2\pi i} \int_{(\sigma)} \tilde{f}(s) L(s,\chi) x^s \, ds + \xi(\chi) \tilde{f}(1) (\mathrm{Res}_{s=1} \zeta_K(s)) x,$$

where $\xi(\chi_0) = 1$ for χ_0 the trivial character and $\xi(\chi) = 0$ if $\chi \neq \chi_0$.

Here and below c_3, c_4, c_5, c_6, and c_7 denote positive constants that depend only on F, f, δ, and σ but not on K, χ, ϵ, or H.

Let h_K denote the class number of K, R_K the regulator of K, and ω_K the number of roots of unity in K. The residue of ζ_K at $s = 1$ is positive and at most $c_3 h_K R_K / |\Delta_K|^{1/2}$ by the analytic class number formula. The unit groups of K and F have equal rank. As in the proof of Lemma 3.11, we have $R_K \leqslant c_4$, where c_4 may depend on F. Hence,

$$|S(x,\chi)| \leqslant \frac{1}{2\pi} \int_{(\sigma)} |\tilde{f}(s) L(s,\chi)| x^\sigma \, ds + c_5 \xi(\chi) \frac{h_K}{|\Delta_K|^{1/2}} x \tag{5.5}$$

with $c_5 = c_3 c_4 |\tilde{f}(1)|$.

Soon we will apply the Phragmén–Lindelöf principle, cf. [IK04, Theorem 5.53], to bound $|L(s,\chi)|$ from above in terms of $|L(1/2+it,\chi)|$ and $|L(2+it,\chi)|$; here $s = \sigma + it$. Indeed, the bound $|L(2+it,\chi)| \leqslant \zeta(2)^{[K:\mathbb{Q}]}$ is elementary but to bound $|L(1/2+it,\chi)|$ we need Theorem 5.7. We abbreviate

$$l(\sigma) = \tfrac{2}{3}(2-\sigma) \tag{5.6}$$

whose graph linearly interpolates $l(1/2) = 1$ and $l(2) = 0$.

We suppose first that $\chi \neq \chi_0$. Then $L(\cdot,\chi)$ is an entire function and we may apply the Phragmén–Lindelöf principle directly. So,

$$|L(\sigma+it,\chi)| \leqslant c_1^{l(\sigma)} \zeta(2)^{[K:\mathbb{Q}](1-l(\sigma))} (1+|t|)^{N l(\sigma)} |\Delta_K|^{(1/4-\delta) l(\sigma)}$$

for all $t \in \mathbb{R}$, where we may assume that $c_1 \geqslant 1$. To treat the trivial character, we work with the entire function $L(s,\chi)(s-1)$. As $|\sigma+it-1| \geqslant 1-\sigma > 0$, we obtain

$$|L(\sigma+it,\chi_0)| \leqslant \frac{1}{1-\sigma} c_1^{l(\sigma)} \zeta(2)^{[K:\mathbb{Q}](1-l(\sigma))} (1+|t|)^{N l(\sigma)+1} |\Delta_K|^{(1/4-\delta) l(\sigma)},$$

where the additional $1+|t|$ appears since $|s-1| \leqslant 1 + |\mathrm{Im}(s)|$ if $\mathrm{Re}(s) \in \{1/2, 2\}$.

In any case, we have $|L(\sigma+it,\chi)| \leqslant c_6(1-\sigma)^{-1}(1+|t|)^{Nl(\sigma)+1}|\Delta_K|^{(1/4-\delta)l(\sigma)}$ with $c_6 = c_1\zeta(2)^{[K:\mathbb{Q}]}$. Together with (5.5) and the decay property of $\tilde{f}$, we obtain

$$|S(x,\chi)| \leqslant c_7\bigg(|\Delta_K|^{(1/4-\delta)l(\sigma)}x^\sigma + \xi(\chi)\frac{h_K}{|\Delta_K|^{1/2}}x\bigg).$$

We substitute $x = \epsilon|\Delta_K|^{1/2}$ to find

$$\begin{aligned}|S(x,\chi)| &\leqslant c_7(|\Delta_K|^{(1/4-\delta)l(\sigma)+\sigma/2}\epsilon^\sigma + \xi(\chi)h_K\epsilon)\\ &\leqslant c_7\epsilon^{1/2}(|\Delta_K|^{(1/4-\delta)l(\sigma)+\sigma/2} + \xi(\chi)h_K),\end{aligned} \tag{5.7}$$

where we used $\epsilon \leqslant \epsilon^\sigma \leqslant \epsilon^{1/2}$ as $\sigma \in (1/2,1)$ and $\epsilon \in (0,1]$.

We consider the mean

$$S(x) = \frac{1}{[\mathrm{Cl}_K : H_0]}\sum_{\chi|_{H_0}=1}\overline{\chi(H)}S(x,\chi) \tag{5.8}$$

over all characters χ of Cl_K that are constant on H. Since χ takes values on the unit circle, we may bound

$$|S(x)| \leqslant \frac{[\mathrm{Cl}_K : H_0]-1}{[\mathrm{Cl}_K : H_0]}\max\{|S(x,\chi)| : \chi|_{H_0} = 1 \text{ and } \chi \neq \chi_0\} + \frac{|S(x,\chi_0)|}{[\mathrm{Cl}_K : H_0]}.$$

We observe that $[\mathrm{Cl}_K : H_0] = h_K/\#H_0$. The bound (5.7) yields

$$|S(x)| \leqslant c_7\epsilon^{1/2}(|\Delta_K|^{(1/4-\delta)l(\sigma)+\sigma/2} + \#H_0). \tag{5.9}$$

We insert the finite sum (5.4) into (5.8) and rearrange the order of summation to obtain

$$S(x) = \frac{1}{[\mathrm{Cl}_K : H_0]}\sum_{\mathfrak{A}}\bigg(\sum_{\chi|_{H_0}=1}\overline{\chi(H)}\chi([\mathfrak{A}])\bigg)f\bigg(\frac{\mathrm{N}(\mathfrak{A})}{x}\bigg).$$

For χ from the inner sum, we have $\overline{\chi(H)}\chi([\mathfrak{A}]) = \chi([\mathfrak{B}])$ for a fractional ideal $\mathfrak{B}$ with $[\mathfrak{A}\mathfrak{B}^{-1}] \in H$. But $\sum_{\chi|_{H_0}=1}\chi([\mathfrak{B}])$ equals $[\mathrm{Cl}_K : H_0]$ if $[\mathfrak{B}] \in H_0$ and 0 otherwise. Hence,

$$S(x) = \sum_{\substack{\mathfrak{A}\\ [\mathfrak{A}]\in H}} f\bigg(\frac{\mathrm{N}(\mathfrak{A})}{x}\bigg) \geqslant \sum_{\substack{\mathfrak{A}\\ \mathrm{N}(\mathfrak{A})\leqslant x, [\mathfrak{A}]\in H}} f\bigg(\frac{\mathrm{N}(\mathfrak{A})}{x}\bigg) = \sum_{\substack{\mathfrak{A}\\ \mathrm{N}(\mathfrak{A})\leqslant x, [\mathfrak{A}]\in H}}\bigg(\frac{x}{\mathrm{N}(\mathfrak{A})}\bigg)^{1/2}$$

as f is non-negative and by (5.3). We divide by $\#H = \#H_0$ and use (5.9) to obtain

$$\frac{1}{\#H}\sum_{\substack{\mathfrak{A}\\ \mathrm{N}(\mathfrak{A})\leqslant x, [\mathfrak{A}]\in H}}\bigg(\frac{x}{\mathrm{N}(\mathfrak{A})}\bigg)^{1/2} \leqslant c_7\epsilon^{1/2}\bigg(\frac{|\Delta_K|^{(1/4-\delta)l(\sigma)+\sigma/2}}{\#H} + 1\bigg).$$

The lemma follows as we may fix $\sigma \in (1/2,1)$ with $(1/4-\delta)l(\sigma) + \sigma/2 \leqslant 1/2 - \delta/2$. □

Next we state two simple consequences of the previous proposition that we need for our main result. Recall that the norm $\mathrm{N}([\mathfrak{A}])$ of an ideal class in $[\mathfrak{A}] \in \mathrm{Cl}_K$ is the smallest norm of a representative.

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Proposition 5.9. *Let F and δ be as in Lemma 5.8. There is a constant $c_8 > 0$ depending only on F with the following property. Say K is a totally imaginary quadratic extension of F and let H be a coset of a subgroup of Cl_K; then the following two properties hold.*

(i) *We have*

$$\frac{1}{\#H}\sum_{[\mathfrak{A}]\in H}\left(\frac{|\Delta_K|^{1/2}}{\mathrm{N}([\mathfrak{A}^{-1}])}\right)^{1/2} \leqslant c_8 \max\left\{1, \frac{|\Delta_K|^{1/2-\delta/2}}{\#H}\right\}.$$

(ii) *Let $\epsilon \in (0,1]$; then*

$$\frac{1}{\#H}\#\{[\mathfrak{A}] \in H : \mathrm{N}([\mathfrak{A}^{-1}]) \leqslant \epsilon|\Delta_K|^{1/2}\} \leqslant c_2\epsilon^{1/2}\max\left\{1, \frac{|\Delta_K|^{1/2-\delta/2}}{\#H}\right\}.$$

Proof. Let $d = [K : \mathbb{Q}]$. By a theorem of Minkowski, any ideal class of K is represented by an ideal whose norm is at most $\epsilon|\Delta_K|^{1/2}$, where $\epsilon = (d!/d^d)(4/\pi)^{r_2}$ and $2r_2$ is the number of non-real embeddings $K \to \mathbb{C}$. It is well known that $\epsilon \leqslant 1$. Part (i) follows from Lemma 5.8 applied to $x = \epsilon|\Delta_K|^{1/2}$ and to the coset H^{-1} since ϵ depends only on F.

Part (ii) follows from Lemma 5.8 applied to H^{-1} since the terms in the sum on the left of (5.2) are at least 1. □

In our application, the coset H will generally have more than $|\Delta_K|^{1/2-\delta/2}$ elements. In this case, the upper bound in part (i) simplifies to c_8, which depends only on F. Also, if ϵ is sufficiently small in part (ii), then we find that only a small proportion of elements of H^{-1} will have norm less than $\epsilon|\Delta_K|^{1/2}$. Geometrically speaking, these ideal classes correspond to Galois conjugates of a CM abelian variety that lie close to the cusp in the moduli space. So, only a small proportion of said conjugates are near the cusp.

6. Proof of the theorems

We begin this section by proving Theorems 1.3 and 1.4.

Let F be a real quadratic field. We fix representatives $\eta_m \in \mathbb{P}^1(F)$ of cusps of $\widehat{\Gamma}(\mathscr{D}_{F/\mathbb{Q}}^{-1})\backslash\mathbb{H}^2$ as in § 3.1. In particular, $\eta_1 = \infty$. We will work with a parameter $\epsilon \in (0,1]$ that depends only on F and a second parameter $\kappa \in (0,1]$ that depends only on F and ϵ. We regard κ as small with respect to ϵ. We will see how to fix these parameters in due course.

Let C be as in Theorem 1.4 and suppose that $k \subseteq \mathbb{C}$ is a number field over which C is defined which we will increase at will. Let K be the CM field of $\mathrm{Jac}(C)$. We may suppose that $k \supseteq K$.

As discussed in greater detail in the introduction, the basic strategy is to let the lower bound coming from Proposition 4.3 compete with an upper bound of the Faltings height. To estimate the Faltings height from above we need its expression in Corollary 4.6. Observe that this corollary is applicable as, after possibly increasing k, the classical theorem of Serre and Tate [ST68] states that the CM abelian variety $\mathrm{Jac}(C)$ has good reduction everywhere. We will show that the archimedean contribution to the Faltings height is negligible when compared to the non-archimedean contribution. We use notation introduced in Theorem 4.5 and Corollary 4.6. Observe that k satisfies the hypothesis of the theorem after passing to a finite field extension. By part (ii) of the theorem, the normalised norm in (1.4) does not change after passing to a further finite extension of k. This settles the last statement of Theorem 1.4.

We thus decompose $h(\mathrm{Jac}(C)) = h^0 + h_1^\infty + h_2^\infty + h_3^\infty + h_4^\infty - \frac{4}{5}\log 2 - \log \pi$, where

$$h^0 = \frac{1}{[k:\mathbb{Q}]} \sum_{\nu \in M_k^0} \frac{1}{60} \log \mathrm{N}(\Delta^0_{\min}(C))$$

is the finite part and

$$h_1^\infty + h_2^\infty + h_3^\infty + h_4^\infty = -\frac{1}{[k:\mathbb{Q}]} \sum_{\sigma : k \to \mathbb{C}} \frac{1}{10} \log(|\chi_{10}(Z_\sigma)| \det(\operatorname{Im} Z_\nu)^5); \tag{6.1}$$

here the single h_m^∞ for $m \in \{1,2,3,4\}$ are determined as follows.

Shimura's theorem 3.9 describes the period matrices coming from a Galois orbit that fixes the reflex field K^*. Observe that $K^* = K$ by Lemma 3.10 because $K/\mathbb{Q}$ is cyclic. So, (6.1) holds, where each

$$h_m^\infty = -\frac{1}{10\#\mathrm{N}_{\Phi_m}(\mathrm{Cl}_K)} \sum_{[\mathfrak{A}] \in \mathrm{N}_{\Phi_m}(\mathrm{Cl}_K)[\mathfrak{B}_m]} \log(|\chi_{10}(Z_\mathfrak{A})| \det(\operatorname{Im} Z_\mathfrak{A})^5) \tag{6.2}$$

corresponds to one of the four cosets of $\mathrm{Aut}(\mathbb{C}/K)$ in $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$; here Φ_m is a CM type of K and $\mathfrak{B}_m \subseteq \mathcal{O}_K$ is a fractional ideal. Observe that the terms on the right of (6.2) do not depend on the choice of a representative $\mathfrak{A} \in [\mathfrak{A}]$. Indeed, we already observed that $Z \mapsto |\chi_{10}(Z)| \det(\operatorname{Im} Z)^5$ is $\mathrm{Sp}_4(\mathbb{Z})$-invariant in § 4.4.

For any $\mathfrak{A}$ as in the sum (6.2), Proposition 3.3 provides $\tau_\mathfrak{A}$ with $\Phi_m(\tau_\mathfrak{A})$ in the fundamental set $\mathcal{F}(\mathscr{D}_{F/\mathbb{Q}}^{-1})$ from § 3.1. The period matrices $Z_\mathfrak{A}$ are as described in (3.20).

Later on we will show that there exists $c(\epsilon, F) > 0$ depending only on ϵ and F such that

$$h_m^\infty \leqslant \epsilon^{1/2} \log \Delta_K + c(\epsilon, F) \quad \text{for each } 1 \leqslant m \leqslant 4. \tag{6.3}$$

Our Theorem 1.4 follows from this inequality and from Proposition 4.3.

But also Theorem 1.3 follows from (6.3) after adjusting ϵ. Indeed, the archimedean contribution to the Faltings height of $\mathrm{Jac}(C)$ is invariant under replacing the field of definition by a finite extension; so the extensions of k made before are harmless.

Of course, all m can be treated in a similar manner. So, we simplify notation by abbreviating $h^\infty = h_m^\infty$, writing H for $\mathrm{N}_{\Phi_m}(\mathrm{Cl}_K)[\mathfrak{B}_m]$ and Φ for the CM type Φ_m. Observe that H is a coset in the class group Cl_K.

In this new notation, we have

$$h^\infty = -\frac{1}{10\#H} \sum_{[\mathfrak{A}] \in H} \log(|\chi_{10}(Z_{\mathfrak{A},\mathrm{red}})| \det(\operatorname{Im} Z_{\mathfrak{A},\mathrm{red}})^5),$$

where $Z_{\mathfrak{A},\mathrm{red}} \in \mathcal{F}_2$ is in the $\mathrm{Sp}_4(\mathbb{Z})$-orbit of $Z_\mathfrak{A}$.

Below, $c_1, c_2, \ldots, c_8$ denote positive constants that depend only on the real quadratic field F.

Taking the sign in h^∞ into account, we would like to bound each logarithm in h^∞ from *below* using Proposition 5.6. If $z_{\mathfrak{A},12}$ is the off-diagonal entry of the Siegel reduced matrix $Z_{\mathfrak{A},\mathrm{red}}$, then $z_{\mathfrak{A},12} \neq 0$. Indeed, otherwise $Z_{\mathfrak{A},\mathrm{red}}$ is diagonal. But this is impossible because $\mathrm{Jac}(C)$ is not a product of elliptic curves due to the fact that $K/\mathbb{Q}$ is cyclic; see [BL04, Corollary 11.8.2] and Lemma 3.10. Another way to see that $z_{\mathfrak{A},12} \neq 0$ is by noting that χ_{10} restricted to $\mathcal{F}_2$ vanishes only on diagonal matrices and by using the proof of Proposition 4.8. By Proposition 5.6, we obtain

$$h^\infty \leqslant c_1 + \frac{1}{10\#H} \sum_{[\mathfrak{A}] \in H} (\log \max\{1, |z_{\mathfrak{A},12}|^{-2}\} + 2\pi \operatorname{Tr}(\operatorname{Im} Z_{\mathfrak{A},\mathrm{red}}) - 5 \log \det(\operatorname{Im} Z_{\mathfrak{A},\mathrm{red}})).$$

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Since $Z_{\mathfrak{A},\mathrm{red}}$ is Siegel reduced, we have $\det(\mathrm{Im}\, Z_{\mathfrak{A},\mathrm{red}}) \geqslant c_2$. So, the average value of the quantity $-\log \det(\mathrm{Im}\, Z_{\mathfrak{A},\mathrm{red}})$ is bounded from above uniformly. After possibly increasing c_1, we find

$$h^\infty \leqslant c_1 + \frac{1}{10\#H} \sum_{[\mathfrak{A}]\in H} (\log \max\{1, |z_{\mathfrak{A},12}|^{-2}\} + 2\pi \,\mathrm{Tr}(\mathrm{Im}\, Z_{\mathfrak{A},\mathrm{red}})).$$

Next we use Lemma 3.6(i) to bound each $\mathrm{Tr}(\mathrm{Im}\, Z_{\mathfrak{A},\mathrm{red}})$ from above to get

$$h^\infty \leqslant c_1 + c_3 \frac{1}{\#H} \sum_{[\mathfrak{A}]\in H} \biggl(\log \max\{1, |z_{\mathfrak{A},12}|^{-1}\} + \biggl(\frac{\Delta_K^{1/2}}{\mathrm{N}([\mathfrak{A}^{-1}])}\biggr)^{1/2}\biggr).$$

We continue by tackling the terms $\Delta_K^{1/2}/\mathrm{N}([\mathfrak{A}^{-1}])$. The trivial bound that follows from $\mathrm{N}([\mathfrak{A}^{-1}]) \geqslant 1$ is of little use here as it leads to an upper bound for h^∞ of the magnitude $\Delta_K^{1/4}$. When compared with the logarithmic lower bound coming from Proposition 4.3, this is not good enough to conclude (6.3). We need the subconvexity bound. Proposition 5.9(i) combined with the lower bound for $\#H$ from Lemma 3.11 implies that the average contribution of $(\Delta_K^{1/2}/\mathrm{N}(\mathfrak{A}))^{1/2}$ is bounded from above. Thus,

$$h^\infty \leqslant c_4 + c_3 \frac{1}{\#H} \sum_{[\mathfrak{A}]\in H} \log \max\{1, |z_{\mathfrak{A},12}|^{-1}\}. \tag{6.4}$$

Recall that $z_{\mathfrak{A},12}$ is a non-zero algebraic number of absolute logarithmic Weil height at most $H(Z_{\mathfrak{A},\mathrm{red}})$. As $K/\mathbb{Q}$ is Galois, we conclude that $[\mathbb{Q}(z_{\mathfrak{A},12}) : \mathbb{Q}] \leqslant 4$ using the expression (3.20). The fundamental inequality of Liouville found in [BG06, 1.5.19] thus implies that $|z_{\mathfrak{A},12}| \geqslant H(Z_{\mathfrak{A},\mathrm{red}})^{-4}$. The height of this reduced period matrix is bounded from above polynomially in Δ_K by Lemma 3.8. Therefore, taking the logarithm yields

$$\log |z_{\mathfrak{A},12}| \geqslant -c_5 \log \Delta_K. \tag{6.5}$$

We use this inequality to bound from above the terms in (6.4) for which $\tau_{\mathfrak{A}}$ is close to one of the cusps, i.e., $\max_m \mu(\eta_m, \Phi(\tau_{\mathfrak{A}})) > c_6\epsilon^{-1}$ with $c_6 = c$ the constant from Lemma 3.6(ii). We have

$$h^\infty \leqslant c_4 + c_5 \biggl(\frac{1}{\#H} \sum_{\max_m \mu(\eta_m, \Phi(\tau_{\mathfrak{A}}))> c_6\epsilon^{-1}} 1\biggr) \log \Delta_K + c_5 \frac{1}{\#H} \sum_{(*)} \log \max\{1, |z_{\mathfrak{A},12}|^{-1}\},$$

where $(*)$ abbreviates the condition $\max_m \mu(\eta_m, \Phi(\tau)) \leqslant c_6\epsilon^{-1}$ here and in the sums below. Observe that being close to a cusp entails $\mathrm{N}([\mathfrak{A}^{-1}]) < \epsilon\Delta_K^{1/2}$ by Lemma 3.6(ii). Part (ii) of Proposition 5.9 tells us that not too many $\tau_{\mathfrak{A}}$ are close to a cusp. We obtain

$$h^\infty \leqslant c_4 + c_7\epsilon^{1/2} \max\biggl\{1, \frac{\Delta_K^{1/2-\delta/2}}{\#H}\biggr\} \log \Delta_K + c_5 \frac{1}{\#H} \sum_{(*)} \log \max\{1, |z_{\mathfrak{A},12}|^{-1}\}.$$

We apply Lemma 3.11 again to bound $\Delta_K^{1/2-\delta/2}/\#H$ from above. Thus,

$$h^\infty \leqslant c_4 + c_7\epsilon^{1/2} \log \Delta_K + c_5 \frac{1}{\#H} \sum_{(*)} \log \max\{1, |z_{\mathfrak{A},12}|^{-1}\}.$$

It remains to bound the sum on the right. If some $|z_{\mathfrak{A},12}|$ is small, then the corresponding conjugate of $\mathrm{Jac}(C)$ is close to a product of elliptic curves in the appropriate coarse moduli

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space. To measure this proximity we require the second parameter $\kappa \in (0,1]$. We split the upper bound for h^∞ up into a subsum where $|z_{\mathfrak{A},12}| > \kappa$ holds and one where it does not. The first subsum is at most $|\log \kappa|$ and so

$$h^\infty \leqslant c_4 + c_7\epsilon^{1/2}\log\Delta_K + c_5|\log\kappa| + \frac{c_5}{\#H}\sum_{\substack{(*)\\ |z_{\mathfrak{A},12}|\leqslant\kappa}}(-\log|z_{\mathfrak{A},12}|).$$

We use (6.5) again to obtain

$$h^\infty \leqslant c_8(1+|\log\kappa|) + c_8\left(\epsilon^{1/2} + \frac{1}{\#H}\sum_{\substack{(*)\\ |z_{\mathfrak{A},12}|\leqslant\kappa}} 1\right)\log\Delta_K. \tag{6.6}$$

To conclude, we must bound the remaining sum in (6.6). So, say $[\mathfrak{A}]$ corresponds to one of its terms. The property $(*)$ implies that $\Phi(\tau_{\mathfrak{A}})$ is bounded away from all cusps. So, $\Phi(\tau_{\mathfrak{A}})$ lies in a compact subset $\mathcal{K}$ of $\mathbb{H}^2$, cf. Proposition 3.1, which depends only on ϵ. Being bounded away from the cusps entails that reducing $Z_{\mathfrak{A}}$ to $Z_{\mathfrak{A},\mathrm{red}}$ requires only a finite subset of $\mathrm{Sp}_4(\mathbb{Z})$. Indeed, we apply Lemma 3.7 to $M = c_6\epsilon^{-1}$ to obtain a finite set $\Sigma \subseteq \mathrm{Sp}_4(\mathbb{Z})$, which depends only on c_6 and ϵ, such that $Z_{\mathfrak{A},\mathrm{red}} = \gamma Z_{\mathfrak{A}}$ for some $\gamma \in \Sigma$. Therefore,

$$Z_{\mathfrak{A}} \in \bigcup_{\gamma\in\Sigma}\gamma^{-1}\mathcal{A}(\kappa),$$

where

$$\mathcal{A}(\kappa) = \left\{\begin{pmatrix} z_1 & z_{12}\\ z_{12} & z_2\end{pmatrix} \in \mathcal{F}_2 : |z_{12}| \leqslant \kappa\right\}.$$

Each $\mathcal{A}(\kappa)$ is closed in $\mathrm{Mat}_2(\mathbb{C})$ and $\bigcap_{\kappa>0}\mathcal{A}(\kappa)$ contains only diagonal elements.

We can reconstruct $\Phi(\tau_{\mathfrak{A}})$ from $Z_{\mathfrak{A}}$ as follows. The expression (3.20) determines $\#\mathcal{O}_{F,+}^\times/(\mathcal{O}_F^\times)^2$ holomorphic mappings $\mathbb{H}^2 \to \mathbb{H}_2$. So, $\Phi(\tau_{\mathfrak{A}})$ lies in the pre-image of $\bigcup_{\gamma\in\Sigma}\gamma^{-1}\mathcal{A}(\kappa)$ under one of them. Recall that $\Phi(\tau_{\mathfrak{A}})$ lies in the compact set $\mathcal{K}$. As $\kappa \to 0$, the hyperbolic measure of the intersection of the said pre-image and $\mathcal{K}$ tends to 0.

Galois orbits are equidistributed by Zhang's Corollary 3.3 [Zha05] and [MV10, Theorem 1.2] by Michel and Venkatesh. In particular,

$$\limsup_{\Delta_K\to+\infty}\frac{1}{\#H}\#\Big\{\tau_{\mathfrak{A}} : [\mathfrak{A}] \in H \text{ and } \max_m \mu(\eta_m, \Phi(\tau_{\mathfrak{A}})) \leqslant c_6\epsilon^{-1} \text{ and } |z_{\mathfrak{A},12}| \leqslant \kappa\Big\}$$

is bounded above by an expression that tends to 0 as $\kappa \to 0$. We fix κ sufficiently small in terms of ϵ such that this limit superior is at most $\epsilon^{1/2}$.

We can now continue bounding (6.6) from above. If Δ_K is sufficiently large with respect to ϵ, then the number of terms in the sum is at most $2\epsilon^{1/2}\#H$ by the last paragraph. Therefore,

$$h^\infty \leqslant c_8(1 + |\log\kappa| + 3\epsilon^{1/2}\log\Delta_K).$$

If Δ_K is not large enough, we have a similar bound with a possibly larger c_8. We have thus verified the inequality (6.3) and therefore Theorem 1.4. □

Proof of Theorem 1.1. We have seen essentially the same argument in the introduction; let us repeat it here again. Let F and C be as in the theorem. Then we take C as defined over a sufficiently large number field k with $\Delta^0_{\min}(C) = \mathcal{O}_k$. If K is the CM field of $\mathrm{Jac}(C)$, then its discriminant Δ_K is bounded from above by a constant depending only on F by Theorem 1.4. By the theorem of Hermite–Minkowski, there are only finitely many possibilities for K. As there are only finitely many abelian surfaces over $\overline{\mathbb{Q}}$ with CM by the maximal order of K, this leaves at most finitely many possibilities for $\mathrm{Jac}(C)$ as an abelian variety. But each abelian variety, such as $\mathrm{Jac}(C)$, carries only finitely many principal polarisations up to equivalence; this follows from the general Narasimhan–Nori theorem, or from more elementary considerations as $\mathrm{Jac}(C)$ is simple, or in a direct way using the arguments in § 3.2. Thus, up to $\overline{\mathbb{Q}}$-isomorphism, there are only finitely many possibilities for $\mathrm{Jac}(C)$ as a principally polarised abelian variety. By Torelli's theorem this leaves only finitely many $\overline{\mathbb{Q}}$-isomorphism classes for the curve C. □

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Appendix A. Numerical examples

In this section we provide some numerical examples for our expression of the Faltings height in Theorem 4.5. We will approximate $|\chi_{10}(Z_\nu)| \det \mathrm{Im}(Z_v)^5$ numerically and compare the resulting sum with the conclusion of Colmez's conjecture, Proposition 4.3(iii).

Let K be a CM field that is a quartic, cyclic extension of $\mathbb{Q}$ and has maximal totally real subfield F. Let A be an abelian surface defined over a number field whose endomorphism ring is $\mathcal{O}_K$.

First we describe how to compute $L'(0)/L(0)$, where L is as in Proposition 4.3. For this, let $f_K \geqslant 1$ be the finite part of the conductor of $K/\mathbb{Q}$. In other words, f_K is the least positive integer such that K is a subfield of the cyclotomic field generated by a root of unity of order f_K. Recall that $\Delta_K > 0$, as $K/\mathbb{Q}$ is a CM field of degree 4, and $\Delta_F > 0$, since $F/\mathbb{Q}$ is real quadratic. By [Neu99, Propositions 11.9 and 11.10, ch. VII], we have

$$\Delta_K = f_K^2 \Delta_F. \tag{A 1}$$

The L-function $L(s) = \zeta_K(s)/\zeta_F(s)$ is a product $L(s,\chi)L(s,\overline{\chi})$ of Dirichlet L-functions for some character $\chi : (\mathbb{Z}/f_K\mathbb{Z})^\times \to \mathbb{C}$ of order 4. If $(\mathbb{Z}/f_K\mathbb{Z})^\times$ is cyclic, e.g., if f_K is a prime, then χ is uniquely determined up to complex conjugation.

We use (A 1) and Proposition 4.3(iii) to compute

$$h(A) = -\frac{1}{2}\log f_K - \mathrm{Re}\,\frac{L'(0,\chi)}{L(0,\chi)}.$$

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Observe that χ is an odd character. Corollary 10.3.2 and Proposition 10.3.5(1) of Cohen [Coh07] allow us to compute $L(0,\chi)$ and $L'(0,\chi)$, respectively. We find

$$h(A) = \frac{1}{2}\log f_K + f_K \operatorname{Re}\biggl(\frac{\sum_{m=1}^{f_K-1}\chi(m)\log\Gamma(m/f_K)}{\sum_{m=1}^{f_K-1}\chi(m)m}\biggr), \tag{A 2}$$

where $\Gamma(\cdot)$ is the gamma function.

To compute the Igusa invariants $J_2, J_4, J_6, J_8, J_{10}$ of a hyperelliptic equation, we use the `pari/gp` package of Rodriguez–Villegas, based on work of Mestre and Liu. We used the same software to determine the places of potentially good reduction for the curves listed below.

We consider three curves of genus 2 defined over the rationals. The first quite obviously has a jacobian variety with CM. Van Wamelen [vWa99a, vWa99b] verified this in the remaining two cases. The source of the CM fields K in the second and third examples is van Wamelen's table [vWa99a]. For examples 2 and 3, van Wamelen does not prove that the endomorphism ring is the full ring of integers of K. But equality is compatible with our computations below. We use the symbol $\doteq$ to denote conditional equality, subject to the hypothesis that the endomorphism ring of the jacobian under consideration is indeed the full ring of integers of the CM field. In all three cases, K has trivial class group.

Example A.1. We consider the curve C defined by

$$y^2 = x^5 - 1.$$

Let $\zeta = e^{2\pi i/5}$ be a primitive 5th root of unity. Then $(x,y)\mapsto(\zeta x, y)$ is a automorphism of C of order 5 defined over the cyclotomic field $K = \mathbb{Q}(\zeta)$. So, the endomorphism ring of $\operatorname{Jac}(C)$ over the algebraic closure contains $\mathbb{Z}[\zeta] = \mathcal{O}_K$. The two must be equal. Observe that $F = \mathbb{Q}(\sqrt{5})$ is the maximal totally real subfield of K and $f_K = 5$. As a character χ near (A 2), we take for example $\chi(1) = 1, \chi(2) = i, \chi(3) = -i, \chi(4) = -1$. So,

$$h(\operatorname{Jac}(C)) = \tfrac{1}{2}\log 5 + \tfrac{1}{2}\log(\Gamma(\tfrac{1}{5})^{-3}\Gamma(\tfrac{2}{5})^{-1}\Gamma(\tfrac{3}{5})\Gamma(\tfrac{4}{5})^{3}) = -1.4525092396456\ldots. \tag{A 3}$$

Bost *et al.* [BMM90] computed this Faltings height using a different approach to be

$$h(\operatorname{Jac}(C)) = 2\log 2\pi - \tfrac{1}{2}\log(\Gamma(\tfrac{1}{5})^{5}\Gamma(\tfrac{2}{5})^{3}\Gamma(\tfrac{3}{5})\Gamma(\tfrac{4}{5})^{-1}).$$

This expression equals (A 3) by classical properties of the gamma function.

The Igusa invariants of C are

$$(J_2, J_4, J_6, J_8, J_{10}) = (0, 0, 0, 0, 2^{-12}\cdot 5^4).$$

So, there is no contribution to the finite places in Theorem 4.5. In fact, C has potentially good reduction everywhere. This was already observed by Bost *et al.*

The different ideal $\mathscr{D}_{F/\mathbb{Q}}$ equals $\sqrt{5}\mathcal{O}_F$. If $\omega_1 = 1$ and $\omega_2 = \sqrt{5}\zeta$, then

$$\omega_1\mathcal{O}_F + \omega_2\mathscr{D}_{F/\mathbb{Q}}^{-1} = \mathcal{O}_F + \zeta\mathcal{O}_F = \mathcal{O}_K.$$

The period matrix of $\mathcal{O}_K$ can be computed using Remark 3.4 with $\theta = (5+\sqrt{5})/2$,

$$\tau_1 = \sqrt{5}\zeta \quad\text{and}\quad \tau_2 = -\sqrt{5}\zeta^3$$

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as

$$Z = \begin{pmatrix} \sqrt{5}^{-1}(\zeta - \zeta^3) & -1 - \zeta\frac{1+\sqrt{5}}{2} \\ -1 - \zeta\frac{1+\sqrt{5}}{2} & 2\sqrt{5}\zeta + \frac{5+\sqrt{5}}{2} \end{pmatrix}.$$

We observe that $\det \mathrm{Im}(Z) = \sqrt{5}/4$ and use a computer to approximate

$$-\tfrac{1}{10}\log(|\chi_{10}(Z)|\det \mathrm{Im}(Z)^5) = 0.246738390651711\ldots.$$

We add $-\log(2^{4/5}\pi)$ in accordance with Theorem 4.5 and find that the sum approximates (A 3) up to the displayed digits.

Example A.2. The second example concerns the new curve C

$$y^2 = -103615x^6 - 41271x^5 + 17574x^4 + 197944x^3 + 67608x^2 - 103680x - 40824.$$

The endomorphism ring of the jacobian $\mathrm{Jac}(C)$ has complex multiplication by the ring of algebraic integers in $K = \mathbb{Q}(\sqrt{-61+6\sqrt{61}})$. The real quadratic subfield of K is $F = \mathbb{Q}(\sqrt{61})$. We have $\Delta_K = 61^3$ and $\Delta_F = 61$, so the conductor of K is $f_K = 61$. Now $\mathscr{D}_{F/\mathbb{Q}} = \sqrt{61}\mathcal{O}_F$. Let $\chi : (\mathbb{Z}/61\mathbb{Z})^\times \to \mathbb{C}^\times$ be the character of order 4 with $\chi(2) = i$; observe that 2 generates $(\mathbb{Z}/61\mathbb{Z})^\times$. Then

$$\sum_{m=1}^{60} \chi(m)m = -61(1-i)$$

and so

$$h(\mathrm{Jac}(C)) \doteq \frac{1}{2}\log 61 - \frac{1}{2}\sum_{m=1}^{60} \mathrm{Re}(\chi(m)(1+i))\log\Gamma\left(\frac{m}{61}\right) = 0.2688651723313\ldots \tag{A 4}$$

by (A 2).

The Igusa invariants satisfy

$$\frac{J_8^5}{J_{10}^4} = -2^{40}\cdot 3^{-91}\cdot 5^{-48}\cdot 41^{-48}\cdot 643^5\cdot 1871^5\cdot 197802923306762502646309935^5,$$

$$\frac{J_6^5}{J_{10}^3} = 2^{25}\cdot 3^{-72}\cdot 5^{-36}\cdot 7^5\cdot 41^{-36}\cdot 487^5\cdot 3449^5\cdot 3467^5\cdot 42488533591199^5, \tag{A 5}$$

and

$$\frac{J_2^5}{J_{10}} = -2^{25}\cdot 3^{-19}\cdot 5^{-12}\cdot 7^{15}\cdot 41^{-12}\cdot 39079^5. \tag{A 6}$$

The quotient (A 5) yields the contribution of 3 to the Faltings height and (A 6) the contributions of 5 and 41. Explicitly, the finite contribution to $h(\mathrm{Jac}(C))$ as in Theorem 4.5 is

$$\tfrac{2}{5}\log 3 + \tfrac{1}{5}\log 5 + \tfrac{1}{5}\log 41. \tag{A 7}$$

Our curve has potentially good reduction away from $3, 5$, and 41.

We fix roots $\tau_1, \tau_2 \in \mathbb{H}$ of $x^4 - 61x^3 + 6039x^2 - 137677x + 889319$. They are suitable diagonal elements as in Remark 3.4 and can be used to construct a period matrix Z with $\theta = (61+\sqrt{61})/2$. We approximate

$$-\tfrac{1}{10}\log(|\chi_{10}(Z)|\det \mathrm{Im}(Z)^5) = 0.464065891333779\ldots.$$

We add (A 7) and $-\log(2^{4/5}\pi)$ from Theorem 4.5 to this value and see that the resulting value approximates (A 4) well.

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Example A.3. Our final example has bad reduction above 2. Let C be given by

$$y^2 = -x^5 + 3x^4 + 2x^3 - 6x^2 - 3x + 1.$$

The endomorphism ring of $\mathrm{Jac}(C)$ is the ring of integers in $K = \mathbb{Q}(\sqrt{-2+\sqrt{2}})$ which contains $F = \mathbb{Q}(\sqrt{2})$. We have $\Delta_K = 2^{11}, \Delta_F = 2^3$, and $f_K = 2^4$. We must take slightly more care when finding χ as $(\mathbb{Z}/16\mathbb{Z})^\times \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is not cyclic and admits four characters of order 4. The kernel of χ we are interested in corresponds to the fixed field of K in the number field generated by a root of unity of order 16. The non-trivial element in $\ker \chi \subseteq (\mathbb{Z}/16\mathbb{Z})^\times$ is represented by $7, 9$, or -1. However, $a^2 \equiv 1$ or $9 \mod 16$ if a is odd. This rules our 9 as a representative because $K/\mathbb{Q}$ is cyclic of order 4. Moreover, -1 is also impossible because it represents complex conjugation in the Galois group. This leaves 7, i.e., $\chi(7) = 1$. We must have $\chi(15) = -1$ and $\chi(9) = \chi(7 \cdot 15) = -1$. Again up to complex conjugation there are at most two choices for χ. As $\chi(3) = \chi(3 \cdot 7) = \chi(5)$, one choice is

m	1	3	5	7	9	11	13	15
$\chi(m)$	1	i	i	1	-1	$-i$	$-i$	-1

Thus,

$$\sum_{m=1}^{15} \chi(m)m = -16(1+i)$$

and so

$$h(\mathrm{Jac}(C)) \doteq \log 4 + \frac{1}{2}\log\left(\frac{\Gamma(\frac{9}{16})\Gamma(\frac{11}{16})\Gamma(\frac{13}{16})\Gamma(\frac{15}{16})}{\Gamma(\frac{1}{16})\Gamma(\frac{3}{16})\Gamma(\frac{5}{16})\Gamma(\frac{7}{16})}\right)$$

by (A 2). Numerically, we find

$$h(\mathrm{Jac}(C)) \doteq -1.2016102497487\ldots. \tag{A 8}$$

The Igusa invariants satisfy

$$\frac{J_8^5}{J_{10}^4} = -2^{-24}\cdot 3^{10}\cdot 2029^5, \quad \frac{J_6^5}{J_{10}^3} = 2^{-8}\cdot 3^5 \cdot 47^5, \quad \text{and} \quad \frac{J_2^5}{J_{10}} = 2^4 \cdot 3^{15}.$$

So, only 2 contributes to the finite part of the height in Theorem 4.5. In fact, C has potentially good reduction outside of 2. The contribution to the finite part is

$$\tfrac{1}{10}\log 2.$$

We can take

$$\tau_1 = 2\sqrt{-2+\sqrt{2}}\sqrt{2} \quad \text{and} \quad \tau_2 = 2\sqrt{-2-\sqrt{2}}\sqrt{2}$$

to construct Z, now with $\theta = (2+\sqrt{2})/2$, and find

$$-\tfrac{1}{10}\log(|\chi_{10}(Z)|\det \mathrm{Im}(Z)^5) = 0.428322662492607\ldots.$$

We must add $(\log 2)/10$ to this value to compensate for bad reduction and $-\log(2^{4/5}\pi)$ due to the normalisation of the archimedean places. We end up with a good match with (A 8).

References

Aut06 P. Autissier, *Hauteur de Faltings et hauteur de Néron–Tate du diviseur thêta*, Compos. Math. **142** (2006), 1451–1458.

Bad10 A. I. Badzyan, *The Euler–Kronecker constant*, Mat. Zametki **87** (2010), 35–47.

BL04 C. Birkenhake and H. Lange, *Complex abelian varieties*, Grundlehren Math. Wiss., vol. 302 (Springer, Berlin, 2004).

BG06 E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs, vol. 4 (Cambridge University Press, Cambridge, 2006).

BLR90 S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 21 (Springer, Berlin, 1990).

BMM90 J.-B. Bost, J.-F. Mestre and L. Moret-Bailly, *Sur le calcul explicite des 'classes de Chern' des surfaces arithmétiques de genre* 2, Astérisque **183** (1990), 69–105; Séminaire sur les pinceaux de courbes elliptiques (Paris, 1988).

CU05 L. Clozel and E. Ullmo, *Équidistribution de mesures algébriques*, Compos. Math. **141** (2005), 1255–1309.

Coh05 P. B. Cohen, *Hyperbolic equidistribution problems on Siegel 3-folds and Hilbert modular varieties*, Duke Math. J. **129** (2005), 87–127.

Coh07 H. Cohen, *Number theory volume II: analytic and modern tools*, Graduate Texts in Mathematics, vol. 240 (Springer, New York, 2007).

Col93 P. Colmez, *Périodes des variétés abéliennes à multiplication complexe*, Ann. of Math. (2) **138** (1993), 625–683.

Col98 P. Colmez, *Sur la hauteur de Faltings des variétés abéliennes à multiplication complexe*, Compos. Math. **111** (1998), 359–368.

DM69 P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 75–109.

Fal83 G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. **73** (1983), 349–366.

Fon85 J.-M. Fontaine, *Il n'y a pas de variété abélienne sur* $\mathbb{Z}$, Invent. Math. **81** (1985), 515–538.

Gor97 E. Z. Goren, *On certain reduction problems concerning abelian surfaces*, Manuscripta Math. **94** (1997), 33–43.

GL06 E. Z. Goren and K. E. Lauter, *Evil primes and superspecial moduli*, Int. Math. Res. Not. IMRN **2006** (2006), Art. ID 53864.

GL07 E. Z. Goren and K. E. Lauter, *Class invariants for quartic CM fields*, Ann. Inst. Fourier (Grenoble) **57** (2007), 457–480.

GH78 P. Griffiths and J. Harris, *Principles of algebraic geometry* (Wiley-Interscience, New York, 1978).

Hab15 P. Habegger, *Singular moduli that are algebraic units*, Algebra Number Theory **9** (2015), 1515–1524.

IKO86 T. Ibukiyama, T. Katsura and F. Oort, *Supersingular curves of genus two and class numbers*, Compos. Math. **57** (1986), 127–152.

Igu60 J. Igusa, *Arithmetic variety of moduli for genus two*, Ann. of Math. (2) **72** (1960), 612–649.

ID02 H. Iwaniec, W. Duke and J. B. Friedlander, *The subconvexity problem for Artin L-functions*, Invent. Math. **149** (2002), 489–577.

IK04 H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, vol. 53 (American Mathematical Society, Providence, RI, 2004).

dJN91 J. de Jong and R. Noot, *Jacobians with complex multiplication*, in *Arithmetic algebraic geometry (Texel, 1989)*, Progress in Mathematics, vol. 89 (Birkhäuser, Boston, MA, 1991), 177–192.

Kli90 H. Klingen, *Introductory lectures on Siegel modular forms*, Cambridge Studies in Advanced Mathematics, vol. 20 (Cambridge University Press, Cambridge, 1990).

Liu93 Q. Liu, *Courbes stables de genre* 2 *et leur schéma de modules*, Math. Ann. **295** (1993), 201–222.

Liu94 Q. Liu, *Conducteur et discriminant minimal de courbes de genre* 2, Compos. Math. **94** (1994), 51–79.

Liu02 Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6 (Oxford University Press, Oxford, 2002); translated from the French by Reinie Erné, Oxford Science Publications.

Loc94 P. Lockhart, *On the discriminant of a hyperelliptic curve*, Trans. Amer. Math. Soc. **342** (1994), 729–752.

MV10 P. Michel and A. Venkatesh, *The subconvexity problem for* GL_2, Publ. Math. Inst. Hautes Études Sci. **111** (2010), 171–271.

MB01 L. Moret-Bailly, *Problèmes de Skolem sur les champs algébriques*, Compos. Math. **125** (2001), 1–30.

Mum84 D. Mumford, *Tata lectures on theta. II*, Progress in Mathematics, vol. 43 (Birkhäuser, Boston, MA, 1984).

NT91 Y. Nakkajima and Y. Taguchi, *A generalization of the Chowla–Selberg formula*, J. Reine Angew. Math. **419** (1991), 119–124.

NU73 Y. Namikawa and K. Ueno, *The complete classification of fibres in pencils of curves of genus two*, Manuscripta Math. **9** (1973), 143–186.

Neu99 J. Neukirch, *Algebraic number theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 322 (Springer, Berlin, 1999).

Obu13 A. Obus, *On Colmez's product formula for periods of CM-abelian varieties*, Math. Ann. **356** (2013), 401–418.

Paz12a F. Pazuki, *Theta height and Faltings height*, Bull. Soc. Math. France **140** (2012), 19–49.

Paz12b F. Pazuki, *Décompositions en hauteurs locales*, Preprint, 2012, arXiv:1205.4525.

Paz13 F. Pazuki, *Minoration de la hauteur de Néron–Tate sur les surfaces abéliennes*, Manuscripta Math. **142** (2013), 61–99.

PT13 J. Pila and J. Tsimerman, *The André–Oort conjecture for the moduli space of abelian surfaces*, Compos. Math. **149** (2013), 204–216.

PT14 J. Pila and J. Tsimerman, *Ax–Lindemann for* $\mathcal{A}_g$, Ann. of Math. (2) **179** (2014), 659–681.

Sai88 T. Saito, *Conductor, discriminant, and the Noether formula of arithmetic surfaces*, Duke Math. J. **57** (1988), 151–173.

Sai89 T. Saito, *The discriminants of curves of genus* 2, Compos. Math. **69** (1989), 229–240.

Sch03 R. Schoof, *Abelian varieties over cyclotomic fields with good reduction everywhere*, Math. Ann. **325** (2003), 413–448.

ST68 J.-P. Serre and J. T. Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.

Shi97 G. Shimura, *Abelian varieties with complex multiplication and modular functions*, Princeton Mathematical Series, vol. 46 (Princeton University Press, Princeton, NJ, 1997).

Szp85 L. Szpiro (ed.), *Séminaire sur les pinceaux arithmétiques: la conjecture de Mordell*, Astérisque, vol. 127 (Société Mathématique de France, Paris, 1985).

Uen88 K. Ueno, *Discriminants of curves of genus* 2 *and arithmetic surfaces*, in *Algebraic geometry and commutative algebra,* Vol. II (Kinokuniya, Tokyo, 1988), 749–770.

vdG88 G. van der Geer, *Hilbert modular surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 16 (Springer, Berlin, 1988).

vWa99a P. van Wamelen, *Examples of genus two CM curves defined over the rationals*, Math. Comp. **68** (1999), 307–320.

vWa99b P. van Wamelen, *Proving that a genus* 2 *curve has complex multiplication*, Math. Comp. **68** (1999), 1663–1677.

Voj99 P. Vojta, *Integral points on subvarieties of semiabelian varieties. II*, Amer. J. Math. **121** (1999), 283–313.

Was82 L. C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Mathematics, vol. 83 (Springer, New York, 1982).

Yan10 T. Yang, *The Chowla–Selberg formula and the Colmez conjecture*, Canad. J. Math. **62** (2010), 456–472.

Zha05 S. Zhang, *Equidistribution of CM-points on quaternion Shimura varieties*, Int. Math. Res. Not. IMRN **2005** (2005), 3657–3689.

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