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ON THE TWO-PARAMETER ERDŐS–FALCONER DISTANCE PROBLEM IN FINITE FIELDS

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Abstract

Given $E \subseteq \mathbb{F}_q^d \times \mathbb{F}_q^d$, with the finite field $\mathbb{F}_q$ of order q and the integer $d \geq 2$, we define the two-parameter distance set $\Delta_{d,d}(E) = \{(\|x-y\|, \|z-t\|) : (x,z),(y,t) \in E\}$. Birklbauer and Iosevich ['A two-parameter finite field Erdős–Falconer distance problem', *Bull. Hellenic Math. Soc.* **61** (2017), 21–30] proved that if $|E| \gg q^{(3d+1)/2}$, then $|\Delta_{d,d}(E)| = q^2$. For $d = 2$, they showed that if $|E| \gg q^{10/3}$, then $|\Delta_{2,2}(E)| \gg q^2$. In this paper, we give extensions and improvements of these results. Given the diagonal polynomial $P(x) = \sum_{i=1}^d a_i x_i^s \in \mathbb{F}_q[x_1, \ldots, x_d]$, the distance induced by P over $\mathbb{F}_q^d$ is $\|x-y\|_s := P(x-y)$, with the corresponding distance set $\Delta^s_{d,d}(E) = \{(\|x-y\|_s, \|z-t\|_s) : (x,z),(y,t) \in E\}$. We show that if $|E| \gg q^{(3d+1)/2}$, then $|\Delta^s_{d,d}(E)| \gg q^2$. For $d = 2$ and the Euclidean distance, we improve the former result over prime fields by showing that $|\Delta_{2,2}(E)| \gg p^2$ for $|E| \gg p^{13/4}$.

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1. Introduction

The general Erdős distance problem is to determine the number of distinct distances spanned by a finite set of points. In the Euclidean space, it is conjectured that for any finite set $E \subset \mathbb{R}^d$, $d \geq 2$, we have $|\Delta(E)| \gtrapprox |E|^{2/d}$, where $\Delta(E) = \{\|x-y\| : x, y \in E\}$. Here and throughout, $X \ll Y$ means that there exists $C > 0$ such that $X \leq CY$, and $X \lessapprox Y$ with the parameter N means that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that $X \leq C_\varepsilon N^\varepsilon Y$.

The finite field analogue of the distance problem was first studied by Bourgain *et al.* [2] over prime fields. In this setting, the Euclidean distance between any two points $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{F}_q^d$, the d-dimensional vector space over the finite field $\mathbb{F}_q$ of order q, is $\|x-y\| = \sum_{i=1}^d (x_i - y_i)^2 \in \mathbb{F}_q$. For prime fields $\mathbb{F}_p$ with $p \equiv 1 \pmod 4$, they showed that if $E \subset \mathbb{F}_p^2$ with $|E| = p^\delta$ for some $0 < \delta < 2$, then the distance set satisfies $|\Delta(E)| \gg |E|^{1/2+\varepsilon}$ for some $\varepsilon > 0$ depending only on δ.

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This bound does not hold in general for arbitrary finite fields $\mathbb{F}_q$, as shown by Iosevich and Rudnev [9]. In this general setting, they considered the Erdős–Falconer distance problem to determine how large $E \subset \mathbb{F}_q^d$ needs to be so that $\Delta(E)$ spans all possible distances or at least a positive proportion of them. More precisely, they proved that $\Delta(E) = \mathbb{F}_q$ if $|E| > 2q^{(d+1)/2}$ in the all distances case, and also conjectured that $|\Delta(E)| \gg q$ if $|E| \gg_\varepsilon q^{d/2+\varepsilon}$ in the positive proportion case. In [6], it was shown that the exponent in the all distances case is sharp for odd d, and the conjecture for the positive proportion case holds for all $E \subset \{x \in \mathbb{F}_q^d : \|x\| = 1\}$. It is conjectured that in even dimensions, the optimal exponent is $d/2$ for the all distances case. In particular for $d = 2$, it was shown in [3] that if $E \subseteq \mathbb{F}_q^2$ satisfies $|E| \gg q^{4/3}$, then $|\Delta(E)| \gg q$, improving the positive proportion case. The proofs in [3] use extension estimates for circles. Therefore, one would expect to get improvements for distance problems if one can obtain improved estimates for extension problems.

There have been a recent series of other improvements and generalisations on the Erdős–Falconer distance problem. In [7], a generalisation for subsets of regular varieties was studied. Extension theorems and their connection to the Erdős–Falconer problem are the main focus of [10]. The exponents $(d + 1)/2$ and $d/2$ were improved in [13, 14] for subsets E with Cartesian product structure in the all distances case for $|\Delta(E)|$ and in the positive proportion case for the quotient distance set $|\Delta(E)/\Delta(E)|$.

A two-parameter variant of the Erdős–Falconer distance problem for the Euclidean distance was studied by Birklbauer and Iosevich in [1]. More precisely, given $E \subseteq \mathbb{F}_q^d \times \mathbb{F}_q^d$, where $d \geq 2$, define the two-parameter distance set as

$$\Delta_{d,d}(E) = \{(\|x - y\|, \|z - t\|) : (x, z), (y, t) \in E\} \subseteq \mathbb{F}_q \times \mathbb{F}_q.$$

They proved the following results.

THEOREM 1.1 [1]. *Let E be a subset in $\mathbb{F}_q^d \times \mathbb{F}_q^d$. If $|E| \gg q^{(3d+1)/2}$, then $|\Delta_{d,d}(E)| = q^2$.*

THEOREM 1.2 [1]. *Let E be a subset in $\mathbb{F}_q^2 \times \mathbb{F}_q^2$. If $|E| \gg q^{10/3}$, then $|\Delta_{2,2}(E)| \gg q^2$.*

In this short note, we provide an extension and an improvement of these results. Unlike [1], which relies heavily on Fourier analytic techniques, we use an elementary counting approach.

Let $P(x) = \sum_{i=1}^d a_i x_i^s \in \mathbb{F}_q[x_1, \ldots, x_d]$ be a fixed diagonal polynomial in d variables of degree $s \geq 2$. For $x = (x_1, \ldots, x_d), y = (y_1, \ldots, y_d) \in \mathbb{F}_q^d$, we introduce

$$\|x - y\|_s := P(x - y) = \sum_{i=1}^d a_i (x_i - y_i)^s \in \mathbb{F}_q.$$

For any set $E \subset \mathbb{F}_q^d \times \mathbb{F}_q^d$, define

$$\Delta_{d,d}^s(E) = \{(\|x - y\|_s, \|z - t\|_s) : (x, z), (y, t) \in E\} \in \mathbb{F}_q \times \mathbb{F}_q.$$

Our first result reads as follows.

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THEOREM 1.3. *Let E be a subset in $\mathbb{F}_q^d \times \mathbb{F}_q^d$. If $|E| \gg q^{(3d+1)/2}$, then $|\Delta^s_{d,d}(E)| \gg q^2$.*

Our method also works for the multi-parameter distance set for $E \subseteq \mathbb{F}_q^{d_1+\cdots+d_k}$, but we do not discuss such extensions here. For $d = 2$, we get an improved version of Theorem 1.2 for the Euclidean distance function over prime fields.

THEOREM 1.4. *Let $E \subseteq \mathbb{F}_p^2 \times \mathbb{F}_p^2$. If $|E| \gg p^{13/4}$, then $|\Delta_{2,2}(E)| \gg p^2$.*

The continuous versions of Theorems 1.3 and 1.4 have been studied in [4, 5, 8]. We do not know whether our method can be extended to that setting. It follows from our approach that the conjectured exponent $d/2$ of the (one-parameter) distance problem would imply the sharp exponent for the two-parameter analogue, namely $3d/2$, for even dimensions. We refer to [1] for constructions and more discussions.

2. Proof of Theorem 1.3

By using the following auxiliary result whose proof relies on Fourier analytic methods (see [15, Theorem 2.3] and [11, Corollaries 3.1 and 3.4]), we are able to give an elegant proof for Theorem 1.3. Compared with the method in [1], ours is more elementary.

LEMMA 2.1. *Let $X, Y \subseteq \mathbb{F}_q^d$. Define $\Delta^s(X, Y) = \{\|x - y\|_s : x \in X, y \in Y\}$. If $|X||Y| \gg q^{d+1}$, then $|\Delta^s(X, Y)| \gg q$.*

PROOF OF THEOREM 1.3. By assumption, $|E| \geq Cq^{d+(d+1)/2}$ for some constant $C > 0$. For $y \in \mathbb{F}_q^d$, let $E_y := \{x \in \mathbb{F}_q^d : (x, y) \in E\}$ and define

$$Y := \{y \in \mathbb{F}_q^d : |E_y| > \tfrac{1}{2}Cq^{(d+1)/2}\}.$$

We first show that $|Y| \geq \frac{1}{2}Cq^{(d+1)/2}$. Note that

$$|E| = \sum_{y\in Y} |E_y| + \sum_{y\in\mathbb{F}_q^d\setminus Y} |E_y| \leq q^d|Y| + \sum_{y\in\mathbb{F}_q^d\setminus Y} |E_y|,$$

where the last inequality holds since $|E_y| \leq q^d$ for $y \in \mathbb{F}_q^d$. Combining it with the assumption on $|E|$ gives the lower bound

$$\sum_{y\in\mathbb{F}_q^d\setminus Y} |E_y| \geq Cq^{d+(d+1)/2} - q^d|Y|.$$

However, by definition, $|E_y| \leq \frac{1}{2}Cq^{(d+1)/2}$ for $y \in \mathbb{F}_q^d \setminus Y$, yielding the upper bound

$$\sum_{y\in\mathbb{F}_q^d\setminus Y} |E_y| \leq \frac{1}{2}Cq^{d+(d+1)/2}.$$

These two bounds together give $Cq^{d+(d+1)/2} - q^d|Y| \leq \frac{1}{2}Cq^{d+(d+1)/2}$, proving the claimed bound $|Y| \geq \frac{1}{2}Cq^{(d+1)/2}$.

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In particular, Lemma 2.1 implies $|\Delta^s(Y, Y)| \gg q$, as $|Y||Y| \gg q^{d+1}$. However, for each $u \in \Delta^s(Y, Y)$, there are $z, t \in Y$ such that $||z - t||_s = u$. One has $|E_z|, |E_t| \gg q^{(d+1)/2}$, therefore, again by Lemma 2.1, $|\Delta^s(E_z, E_t)| \gg q$. Furthermore, for $v \in \Delta^s(E_z, E_t)$, there are $x \in E_z$ and $y \in E_t$ satisfying $||x - y||_s = v$. Note that $x \in E_z$ and $y \in E_t$ mean that $(x, z), (y, t) \in E$. Thus, $(v, u) = (||x - y||_s, ||z - t||_s) \in \Delta^s_{d,d}(E)$. From this, we conclude that $|\Delta^s_{d,d}(E)| \gg q|\Delta^s(Y, Y)| \gg q^2$, which completes the proof. □

3. Proof of Theorem 1.4

To improve the exponent over prime fields $\mathbb{F}_p$, we strengthen Lemma 2.1 as shown in Lemma 3.1 below. Following the proof of Theorem 1.3 and using Lemma 3.1 proves Theorem 1.4.

LEMMA 3.1. *Let $X, Y \subseteq \mathbb{F}_p^2$. If $|X|, |Y| \gg p^{5/4}$, then $|\Delta(X, Y)| \gg p$.*

PROOF. It is clear that if $X' \subseteq X$ and $Y' \subseteq Y$, then $\Delta(X', Y') \subseteq \Delta(X, Y)$. Thus, without loss of generality, we may assume that $|X| = |Y| = N$ with $N \gg p^{5/4}$. Let Q be the number of quadruples $(x, y, x', y') \in X \times Y \times X \times Y$ such that $||x - y|| = ||x' - y'||$. It follows easily from the Cauchy–Schwarz inequality that

$$|\Delta(X, Y)| \gg \frac{|X|^2|Y|^2}{Q}.$$

Let T be the number of triples $(x, y, y') \in X \times Y \times Y$ such that $||x - y|| = ||x - y'||$. By the Cauchy–Schwarz inequality again, one gets $Q \ll |X| \cdot T$. Next, we need to bound T. For this, denote $Z = X \cup Y$, so that $N \leq |Z| \leq 2N$. Let T' be the number of triples $(a, b, c) \in Z \times Z \times Z$ such that $||a - b|| = ||a - c||$. Obviously, $T \leq T'$. However, it was recently proved (see [12, Theorem 4]) that

$$T' \ll \frac{|Z|^3}{p} + p^{2/3}|Z|^{5/3} + p^{1/4}|Z|^2,$$

which gives

$$T \ll \frac{N^3}{p} + p^{2/3}N^{5/3} + p^{1/4}N^2,$$

and then $T \ll N^3/p$ (since $N \gg p^{5/4}$). Putting all the bounds together, we obtain

$$\frac{N^3}{|\Delta(X, Y)|} = \frac{|X||Y|^2}{|\Delta(X, Y)|} \ll \frac{Q}{|X|} \ll T \ll \frac{N^3}{p},$$

or equivalently, $|\Delta(X, Y)| \gg p$, as required. □

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