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ORTHOGONAL DIAGONAL LATIN SQUARES OF ORDER FOURTEEN

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Abstract

Orthogonal diagonal latin squares of order n, $ODLS(n)$, are orthogonal latin squares of order n with transversals on both the main diagonal and the back diagonal of each square. It has been proven that $ODLS(n)$ exist for all n except $n = 2, 3, 6, 10, 14, 15, 18$ and 26, in which the first three are impossible. In this note an example of $ODLS(14)$ is given.

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Orthogonal diagonal latin squares of order n, $ODLS(n)$, are orthogonal latin squares of order n with transversals on both the main diagonal and the back diagonal of each square. The following open question is asked by J. Denes and A. D. Keedwell [1]: Do there exist orthogonal diagonal latin squares of order n distinct from the impossible orders 2, 3 and 6? Recently several authors [2], [3], [4], [5] have proved that $ODLS(n)$ exist for all n except $n = 2, 3, 6, 10, 14, 15, 18$ and 26, in which the first three are impossible. In this note an example of $ODLS(14)$ is given.

First we construct the orthogonal latin squares of order 14 as follows. The first latin square $L_1 = (a_{ij})$ has

$$(a_{0,0}, a_{0,1}, \ldots, a_{0,9}) = (0, 3, A, 4, 7, 9, 5, C, B, D),$$
$$(a_{0,10}, a_{0,11}, a_{0,12}, a_{0,13}) = (1, 8, 2, 6),$$
$$(a_{10,0}, a_{11,0}, a_{12,0}, a_{13,0}) = (6, 8, 5, 7),$$

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and a latin square Q_1 of order 4 based on $\{A, B, C, D\}$ in its lower right corner. The other elements are determined by the following rules:

(i) when $0 \leqslant i, j \leqslant 9$, $a_{ij} = a_{i-1,j-1} + 1$, where all calculations (including those performed on indices) are performed modulo 10 on the residues $0, 1, 2, \ldots, 9$, and $x + 1 = x$ for the "infinite" element $x \in \{A, B, C, D\}$;

(ii) when $1 \leqslant r \leqslant 9$, $10 \leqslant j \leqslant 13$, $a_{r,j} = a_{0,j} + r$, the calculations being reduced by modulo 10;

(iii) when $1 \leqslant r \leqslant 9$, $10 \leqslant i \leqslant 13$, $a_{i,r} = a_{i,0} + r$, the calculations being reduced by modulo 10.

The second latin square $L_2 = (b_{ij})$ has

$$(b_{0,0}, b_{0,1}, \ldots, b_{0,9}) = (0, C, 9, D, A, 4, B, 1, 3, 2),$$
$$(b_{0,10}, b_{0,11}, b_{0,12}, b_{0,13}) = (5, 6, 8, 7),$$
$$(b_{10,0}, b_{11,0}, b_{12,0}, b_{13,0}) = (8, 1, 2, 6),$$

and a latin square Q_2 orthogonal to Q_1 in its lower right corner. The other elements are determined by the same rules. It is easy to see that L_1 and L_2 are orthogonal latin squares of order 14. If we take Q_1 and Q_2 to be orthogonal diagonal latin squares of order 4, we obtain the squares L_1 and L_2 shown in Table 1.

0	3	A	4	7	9	5	C	B	D	1	8	2	6
D	1	4	A	5	8	0	6	C	B	2	9	3	7
B	D	2	5	A	6	9	1	7	C	3	0	4	8
C	B	D	3	6	A	7	0	2	8	4	1	5	9
9	C	B	D	4	7	A	8	1	3	5	2	6	0
4	0	C	B	D	5	8	A	9	2	6	3	7	1
3	5	1	C	B	D	6	9	A	0	7	4	8	2
1	4	6	2	C	B	D	7	0	A	8	5	9	3
A	2	5	7	3	C	B	D	8	1	9	6	0	4
2	A	3	6	8	4	C	B	D	9	0	7	1	5
6	7	8	9	0	1	2	3	4	5	A	C	D	B
8	9	0	1	2	3	4	5	6	7	D	B	A	C
5	6	7	8	9	0	1	2	3	4	B	D	C	A
7	8	9	0	1	2	3	4	5	6	C	A	B	D

L_1

0	C	9	D	A	4	B	1	3	2	5	6	8	7
3	1	C	0	D	A	5	B	2	4	6	7	9	8
5	4	2	C	1	D	A	6	B	3	7	8	0	9
4	6	5	3	C	2	D	A	7	B	8	9	1	0
B	5	7	6	4	C	3	D	A	8	9	0	2	1
9	B	6	8	7	5	C	4	D	A	0	1	3	2
A	0	B	7	9	8	6	C	5	D	1	2	4	3
D	A	1	B	8	0	9	7	C	6	2	3	5	4
7	D	A	2	B	9	1	0	8	C	3	4	6	5
C	8	D	A	3	B	0	2	1	9	4	5	7	6
8	9	0	1	2	3	4	5	6	7	A	D	B	C
1	2	3	4	5	6	7	8	9	0	C	B	D	A
2	3	4	5	6	7	8	9	0	1	D	A	C	B
6	7	8	9	0	1	2	3	4	5	B	C	A	D

L_2

TABLE 1

It is seen that these squares have a common transversal down the diagonal. Another common transversal is cells (0, 5), (1, 6), (2, 7), (3, 8) and (4, 9) and their transposes, together with the back diagonal of the subsquares Q_1 and Q_2. If rows and columns are permuted simultaneously, the diagonal is preserved, and it is possible to move the other transversal onto the back diagonal. Thus we get an example of orthogonal diagonal latin squares of order 14, illustrated in Table 2.

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0	3	*A*	4	7	1	8	2	6	*D*	*B*	*C*	5	9
D	1	4	*A*	5	2	9	3	7	*B*	*C*	6	0	8
B	*D*	2	5	*A*	3	0	4	8	*C*	7	1	9	6
C	*B*	*D*	3	6	4	1	5	9	8	2	0	7	*A*
9	*C*	*B*	*D*	4	5	2	6	0	3	1	8	*A*	7
6	7	8	9	0	*A*	*C*	*D*	*B*	5	4	3	2	1
8	9	0	1	2	*D*	*B*	*A*	*C*	7	6	5	4	3
5	6	7	8	9	*B*	*D*	*C*	*A*	4	3	2	1	0
7	8	9	0	1	*C*	*A*	*B*	*D*	6	5	4	3	2
2	*A*	3	6	8	0	7	1	5	9	*D*	*B*	*C*	4
A	2	5	7	3	9	6	0	4	1	8	*D*	*B*	*C*
1	4	6	2	*C*	8	5	9	3	*A*	0	7	*D*	*B*
3	5	1	*C*	*B*	7	4	8	2	0	*A*	9	6	*D*
4	0	*C*	*B*	*D*	6	3	7	1	2	9	*A*	8	5

0	*C*	9	*D*	*A*	5	6	8	7	2	3	1	*B*	4
3	1	*C*	0	*D*	6	7	9	8	4	2	*B*	5	*A*
5	4	2	*C*	1	7	8	0	9	3	*B*	6	*A*	*D*
4	6	5	3	*C*	8	9	1	0	*B*	7	*A*	*D*	2
B	5	7	6	4	9	0	2	1	8	*A*	*D*	3	*C*
8	9	0	1	2	*A*	*D*	*B*	*C*	7	6	5	4	3
1	2	3	4	5	*C*	*B*	*D*	*A*	0	9	8	7	6
2	3	4	5	6	*D*	*A*	*C*	*B*	1	0	9	8	7
6	7	8	9	0	*B*	*C*	*A*	*D*	5	4	3	2	1
C	8	*D*	*A*	3	4	5	7	6	9	1	2	0	*B*
7	*D*	*A*	2	*B*	3	4	6	5	*C*	8	0	1	9
D	*A*	1	*B*	8	2	3	5	4	6	*C*	7	9	0
A	0	*B*	7	9	1	2	4	3	*D*	5	*C*	6	8
9	*B*	6	8	7	0	1	3	2	*A*	*D*	4	*C*	5

TABLE 2

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[1] J. Denes and A. D. Keedwell, *Latin squares and their applications* (Academic Press, New York, 1974).

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[5] W. D. Wallis and L. Zhu, 'Four pairwise orthogonal diagonal latin squares of order 12', *Utilitas Math.* **21** C (1982), 205–207.

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