J. Aust. Math. Soc. **93** (2012), 77–83
doi:10.1017/S1446788712000250

A NOTE ON THE DISTRIBUTION FUNCTION OF $\varphi(p-1)/(p-1)$

JEAN-MARC DESHOUILLERS ✉ and MEHDI HASSANI

(Received 13 May 2011; accepted 1 February 2012; first published online 16 January 2013)

Communicated by F. Pappalardi

To the memory of Alf van der Poorten, an inspiring mathematician and a friend

Abstract

We study the differentiability of the limiting distribution function associated to the normalized Euler function defined on the shifted primes.

2010 *Mathematics subject classification*: primary 11N60; secondary 11K65.
Keywords and phrases: distribution function, Euler totient function, shifted primes.

1. Introduction

In 1968, Kátai [3] gave general conditions under which additive and multiplicative functions, taken over the set of shifted primes, have a continuous limiting distribution function. Moreover, this distribution function F is purely singular: this was proved by Erdős [1] for the distribution over all integers, and extensions to shifted primes can be found in Tenenbaum's book [4, Exercise 256, p. 423].

The nature of the support of the limiting distribution is not known, even in the case of the distribution over the integers. The question of the local behaviour of this limiting distribution, say G, seems to have been first addressed by Tjan [5] and the precise order of magnitude for the modulus of continuity of G (the quantity $Q(h) = \sup_x(G(x+h) - G(x))$ is also called the concentration by some authors) has been given by Erdős [2]. For more recent information on the local behaviour of G, we refer the reader to the recent paper by Toulmonde [6] and the references therein.

Supported by *Service Culturel de l'Ambassade de France en Iran*.

In this note, we consider the distribution of the numbers $(\varphi(p-1)/(p-1))$ as an example, where φ is Euler's function and p runs over prime numbers. We show that the limiting distribution function, defined by

$$F(x)=\lim_{N\to\infty}\frac{1}{\pi(N)}\mathrm{Card}\Big\{p\le N:\frac{\varphi(p-1)}{p-1}\le x\Big\},$$

is differentiable at no point $\varphi(m)/m$, where m is an even integer. Besides an appeal to the Bombieri–Vinogradov theorem, which could be replaced by the Siegel–Walfisz theorem, the proof is completely elementary, namely a mere moment computation. More precisely, we show the following result.

THEOREM 1.1. *Assume that m is any positive even integer and denote $x_m=\varphi(m)/m$. Then for all $A>0$ and $\delta>0$ there exists $y\in[x_m-\delta,x_m)$ such that*

$$F(x_m)-F(y)\ge A(x_m-y).$$

Hence, F is not differentiable from the left at the point x_m.

Throughout the paper we shall abbreviate $p\equiv 1$ modulo m as $p\equiv 1[m]$.

2. On Kátai's three-series theorem

The method used by Kátai [3] immediately leads to the following result.

PROPOSITION 2.1. *Assume that g is a positive-valued multiplicative number-theoretic function such that the three series*

$$\sum_{|\log g(p)|\le 1}\frac{\log g(p)}{p},\qquad \sum_{|\log g(p)|\le 1}\frac{\log^2 g(p)}{p}\qquad\text{and}\qquad \sum_{|\log g(p)|>1}\frac{1}{p}$$

converge. Then for every $m\ge 1$ there exists a distribution function G_m such that at all points y of continuity of G_m,

$$\lim_{N\to\infty}\frac{1}{\pi(N;m,1)}\sum_{\substack{p\le N\\ p\equiv 1[m]\\ g(p-1)\le y}}1=G_m(y).$$

Moreover, G_m is continuous if and only if the series

$$\sum_{\substack{p\equiv 1[m]\\ g(p)\ne 1}}\frac{1}{p}$$

diverges.

We apply Kátai's extended proposition to the function g defined by $g(n)=\varphi(n)/n$. It is easily seen that g satisfies all the conditions of Proposition 2.1 and thus, for any

https://doi.org/10.1017/S1446788712000250 Published online by Cambridge University Press

positive integer m, which we further assume to be even, and any $y \in \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{\pi(N;m,1)} \sum_{\substack{p\le N\\ p\equiv 1\ [m]\\ g(p-1)\le y}} 1 = G_m(y), \tag{2.1}$$

where G_m is a continuous distribution function.

Relation (2.1) indeed means that the sequence (in N) of the empirical measures

$$\nu_{N,m} = \frac{1}{\pi(N;m,1)} \sum_{\substack{p\le N\\ p\equiv 1\ [m]}} \delta_{(g(p-1))}, \tag{2.2}$$

where $\delta_{(a)}$ denotes the Dirac measure at positive a, weakly converges to the Lebesgue–Stieltjes measure with density dG/dx. For $p \equiv 1\ [m]$,

$$0 \le g(p-1) = \prod_{q|p-1}\Big(1-\frac{1}{q}\Big) = \frac{\varphi(m)}{m} \prod_{\substack{q|p-1\\ q\nmid m}}\Big(1-\frac{1}{q}\Big) \le \frac{\varphi(m)}{m}.$$

Thus, the support of the measure dG_m is indeed in $[0, x_m]$ with $x_m = \varphi(m)/m$.

We consider the continuous function $t \mapsto t^k$, with support in the compact set $[0, x_m]$. Then for all $k \ge 1$,

$$\lim_{N\to\infty} \int_0^{x_m} t^k \, d\nu_{N,m} = \int_0^{x_m} t^k \, dG_m(t). \tag{2.3}$$

3. Moments analysis

In this section, we compute the left-hand side of (2.3) by number-theoretic methods and obtain a lower bound for the right-hand side; more precisely, we show that the following is valid for all positive even integers m, and $k \ge 2$:

$$c_{m,k} := \int_0^{x_m} t^k \, dG_m(t) \gg_m \frac{x_m^k}{\log k}. \tag{3.1}$$

By the definition (2.2) of the measure $\nu_{N,m}$,

$$\int_0^{x_m} t^k \, d\nu_{N,m} = \frac{1}{\pi(N;m,1)} \sum_{\substack{p\le N\\ p\equiv 1\ [m]}} \Big(\frac{\varphi(p-1)}{p-1}\Big)^k.$$

In order to compute the sum, we introduce the multiplicative function $h_{m,k}$ defined for ℓ prime and $\alpha \ge 1$ by

$$h_{m,k}(\ell^\alpha) = \begin{cases} 1-(1-1/\ell)^k & \text{if } \ell \nmid m \text{ and } \alpha = 1,\\ 0 & \text{if } \ell \mid m \text{ or } \alpha \ge 2. \end{cases} \tag{3.2}$$

https://doi.org/10.1017/S1446788712000250 Published online by Cambridge University Press

By using the Möbius inversion formula, the function $f_{m,k}$ defined by

$$f_{m,k}(n) = \sum_{d|n} \mu(d) h_{m,k}(d)$$

is multiplicative. Obviously, for any prime ℓ and $\alpha \geq 1$,

$$f_{m,k}(\ell^\alpha) = 1 - h_{m,k}(\ell) = \begin{cases} (\varphi(\ell)/\ell)^k & \text{if } \ell \nmid m, \\ 1 & \text{if } \ell \mid m. \end{cases}$$

Thus

$$\begin{aligned} \sum_{\substack{p \leq N \\ p \equiv 1 [m]}} \left(\frac{\varphi(p-1)}{p-1} \right)^k &= \left(\frac{\varphi(m)}{m} \right)^k \sum_{\substack{p \leq N \\ p \equiv 1 [m]}} f_{m,k}(p-1) \\ &= x_m^k \sum_{\substack{p \leq N \\ p \equiv 1\ [m]}} \sum_{\substack{d | p-1 \\ (d,m)=1}} \mu(d) h_{m,k}(d) \\ &= x_m^k \sum_{\substack{d \leq N-1 \\ (d,m)=1}} \mu(d) h_{m,k}(d) \sum_{\substack{p \leq N \\ p \equiv 1\ [d] \\ p \equiv 1\ [m]}} 1 \\ &= x_m^k \sum_{\substack{d \leq N-1 \\ (d,m)=1}} \mu(d) h_{m,k}(d) \pi(N; md, 1). \end{aligned}$$

When d is large, say $d \geq D = \lfloor N^{1/3} \rfloor$, we use the trivial upper bound $N/(md)$ for $\pi(N; md, 1)$, as well as the upper bound k/ℓ for $h_{m,k}(\ell)$. We get

$$\sum_{\substack{d \geq D \\ (d,m)=1}} |\mu(d) h_{m,k}(d)| \frac{N}{md} \leq N \sum_{d \geq D} \frac{k^{\omega(d)}}{d^2} \ll_{m,k} N^{5/6}.$$

We now consider small d, that is to say, $d < D = \lfloor N^{1/3} \rfloor$. We write

$$\pi(N; md, 1) = \frac{\pi(N)}{\varphi(m)\varphi(d)} + E(N; md, 1).$$

The Bombieri–Vinogradov theorem implies that

$$\sum_{\substack{d \leq D \\ (d,m)=1}} |E(N; md, 1)| = O_m\left(\frac{\pi(N)}{\log N} \right).$$

This relation, combined with the trivial upper bound $|h_{m,k}(d)| \leq 1$, leads to

$$\sum_{\substack{d \leq D \\ (d,m)=1}} |\mu(d) h_{m,k}(d) E(N; md, 1)| = O_m\left(\frac{\pi(N)}{\log N} \right).$$

https://doi.org/10.1017/S1446788712000250 Published online by Cambridge University Press

We are left with the main contribution

$$\frac{\pi(N)}{\varphi(m)}\sum_{\substack{d\le D\\(d,m)=1}}\frac{\mu(d)h_{m,k}(d)}{\varphi(d)}=\frac{\pi(N)}{\varphi(m)}\sum_{\substack{d=1\\(d,m)=1}}^{\infty}\frac{\mu(d)h_{m,k}(d)}{\varphi(d)}+o_m(\pi(N)),$$

since, as above, the upper bound k/ℓ for $|h_{m,k}(\ell)|$ implies the absolute convergence of the series. By the definition of $h_{m,k}$,

$$\sum_{\substack{d=1\\(d,m)=1}}^{\infty}\frac{\mu(d)h_{m,k}(d)}{\varphi(d)}=\prod_{\ell\nmid m}\Big(1-\frac{1-(1-1/\ell)^k}{\ell-1}\Big)\ge\prod_{\ell\ge 3}\Big(1-\frac{1-(1-1/\ell)^k}{\ell-1}\Big).$$

For $3\le\ell\le k^2$, we use the lower bound

$$1-\frac{1-(1-1/\ell)^k}{\ell-1}\ge 1-\frac{1}{\ell-1}=\Big(1-\frac{1}{\ell}\Big)\Big(1-\frac{1}{(\ell-1)^2}\Big),$$

and for $\ell>k^2$, we use

$$1-\frac{1-(1-1/\ell)^k}{\ell-1}\ge 1-\frac{k}{\ell(\ell-1)}\ge 1-\frac{1}{\ell^{1/2}(\ell-1)}.$$

Thus

$$\prod_{\ell\ge 3}\Big(1-\frac{1-(1-1/\ell)^k}{\ell-1}\Big)\ge\prod_{3\le\ell<k^2}\Big(1-\frac{1}{\ell}\Big)\prod_{\ell>3}\Big(1-\frac{1}{(\ell-1)^2}\Big)\prod_{\ell>3}\Big(1-\frac{1}{\ell^{1/2}(\ell-1)}\Big)$$
$$\gg\frac{1}{\log k},$$

where the last inequality comes from Mertens' theorem and the absolute convergence of the two infinite products. This proves (3.1).

4. Completing the proof of Theorem 1.1

In this section, we assume that Theorem 1.1 does not hold and we deduce an upper bound for $\int_0^{x_m} t^k\,dG_m(t)$ that contradicts (3.1), thus proving Theorem 1.1. The negation of Theorem 1.1 is

$$\exists A>0,\ \exists\delta>0,\ \forall y\in[x_m-\delta,x_m),\qquad F(x_m)-F(y)<A(x_m-y).$$

Thus, by the definition of F, for all $y\in[x_m-\delta,x_m)$,

$$\lim_{N\to\infty}\frac{1}{\pi(N)}\mathrm{Card}\Big\{p\le N:\frac{\varphi(p-1)}{p-1}\in[y,x_m)\Big\}<A(x_m-y).$$

https://doi.org/10.1017/S1446788712000250 Published online by Cambridge University Press

This implies that, in the same range for y,

$$\limsup_{N\to\infty} \frac{1}{\pi(N)} \mathrm{Card}\Big\{p \le N : p \equiv 1[m], \frac{\varphi(p-1)}{p-1} \in [y, x_m)\Big\} < A(x_m - y).$$

By the definition of G_m, the left-hand side of this inequality is $(G_m(x_m) - G_m(y))\varphi(m)$. Thus, for all $y \in [x_m - \delta, x_m]$,

$$G_m(x_m) - G_m(y) \le A_m(x_m - y),$$

where $A_m = A/\varphi(m)$. Integrating by parts the integral expression of $c_{m,k}$,

$$\begin{aligned}
c_{m,k} &= \int_0^{x_m} t^k \, dG_m(t) \\
&= [t^k(G_m(t) - G_m(x_m))]_0^{x_m} - \int_0^{x_m} kt^{k-1}(G_m(t) - G_m(x_m))\, dt \\
&\le \int_0^{x_m-\delta} kt^{k-1}\, dt + A_m \int_{x_m-\delta}^{x_m} kt^{k-1}(x_m - t)\, dt \\
&= (x_m - \delta)^k + A_m[t^k(x_m - t)]_{x_m-\delta}^{x_m} + A_m \int_{x_m-\delta}^{x_m} t^k\, dt \\
&= A_m \frac{x_m^k}{k+1} + o_m\Big(\frac{x_m^k}{k}\Big) = O_m\Big(\frac{x_m^k}{k}\Big),
\end{aligned}$$

which contradicts the inequality (3.1). Thus, Theorem 1.1 is proved, as well as the nondifferentiability of F from the left at any point $x_m = \varphi(m)/m$, where m is an even integer.

Acknowledgement

We express our thanks to Gérald Tenenbaum who considered the question addressed here and provided a shorter proof using deeper analytical tools.

References

[1] P. Erdős, 'Some remarks about additive and multiplicative functions', *Bull. Amer. Math. Soc.* **52** (1946), 527–537.

[2] P. Erdős, 'On the distribution of numbers of the form $\sigma(n)/n$ and on some related questions', *Pacific J. Math.* **52** (1974), 59–65.

[3] I. Kátai, 'On distribution of arithmetical functions on the set prime plus one', *Compositio Math.* **19** (1968), 278–289.

[4] G. Tenenbaum, *Introduction à la théorie Analytique et Probabiliste des Nombres* (Belin, Paris, 2008).

[5] M. M. Tjan, 'On the question of the distribution of values of the Euler function $\varphi(n)$', *Litovsk. Mat. Sb.* **6** (1966), 105–119.

[6] V. Toulmonde, 'Comportement au voisinage de 1 de la fonction de répartition de $\phi(n)/n$', *Int. J. Number Theory* **5** (2009), 1347–1384.

JEAN-MARC DESHOUILLERS, Institut Mathématique de Bordeaux, UMR 5251, Université de Bordeaux et CNRS, 33405 TALENCE Cedex, France
e-mail: jean-marc.deshouillers@math.u-bordeaux.fr

MEHDI HASSANI, Department of Mathematics, University of Zanjan, University Blvd., 45371-38791 Zanjan, Iran
e-mail: mehdi.hassani@znu.ac.ir