LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS: IDENTIFICATION OF BOOLE'S INTEGRAL WITH THAT OF CAUCHY

by D. H. PARSONS

We consider an equation with constant coefficients

$$P(D).y \equiv (aD^n+a_1D^{n-1}+\dots+a_{n-1}D+a_n)y = f(x),$$

where $a \neq 0$ and $f(x)$ is continuous in a suitable interval. Suppose that the symbolic polynomial $P(D)$ has been fully decomposed into its (real or complex) linear factors, so that the equation may be written

$$P(D).y \equiv \left\{a\prod_{r=1}^{q}(D-b_r)^{m_r}\right\}.y = f(x), \quad \dots\dots(1)$$

where $b_1, \dots, b_q$ are distinct, and $m_1+\dots+m_q = n$. The Complementary Function being now known, we may write down a particular integral of (1) by Cauchy's method. This integral is

$$y = \int_c^x \phi(x-t).f(t)dt, \quad \dots\dots(2)$$

where $\phi(x)$ satisfies the conditions

$$\left.\begin{aligned} &P(D).\phi(x) = 0,\\ &\phi(0) = \phi'(0) = \dots = \phi^{n-2}(0) = 0,\\ &\phi^{n-1}(0) = 1/a. \end{aligned}\right\} \quad \dots\dots(3)$$

We shall refer to the integral (2) as Cauchy's integral.

But the most popular, and undoubtedly the simplest way of obtaining a particular integral of (1) is Boole's method, which proceeds formally as follows. First of all, we develop $1/P(X)$ in partial fractions, say

$$\frac{1}{a\prod_{r=1}^{q}(X-b_r)^{m_r}} \equiv \sum_{r=1}^{q}\sum_{s=1}^{m_r}\frac{a_{rs}}{(X-b_r)^s}. \quad \dots\dots(4)$$

We then write, from (1)

$$y = \frac{1}{P(D)}.f(x) = \left\{\sum_{r=1}^{q}\sum_{s=1}^{m_r}\frac{a_{rs}}{(D-b_r)^s}\right\}.f(x) = \sum_{r=1}^{q}\sum_{s=1}^{m_r}\left\{\frac{a_{rs}}{(D-b_r)^s}.f(x)\right\} \quad \dots(5)$$

$$= \sum_{r=1}^{q}\sum_{s=1}^{m_r}\left\{\frac{a_{rs}}{(D-b_r)^s}.e^{b_rx}e^{-b_rx}f(x)\right\} = \sum_{r=1}^{q}\sum_{s=1}^{m_r}\left\{a_{rs}e^{b_rx}\frac{1}{D^s}.e^{-b_rx}f(x)\right\},$$

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whence, by a well-known formula,

$$y = \sum_{r=1}^{q} \sum_{s=1}^{m_r} a_{rs} e^{b_r x} \int_c^x \frac{(x-t)^{s-1}}{(s-1)!} \cdot e^{-b_r t} f(t)dt$$

$$= \sum_{r=1}^{q} \sum_{s=1}^{m_r} \int_c^x \left\{ a_{rs} e^{b_r(x-t)} \frac{(x-t)^{s-1}}{(s-1)!} \right\} f(t)dt, \qquad \text{.........(6)}$$

(c being a conveniently chosen constant), and we have that

$$y = \int_c^x \left[\sum_{r=1}^{q} e^{b_r(x-t)} \left\{ \sum_{s=1}^{m_r} a_{rs} \frac{(x-t)^{s-1}}{(s-1)!} \right\} \right] f(t)dt. \qquad \text{.........(7)}$$

We shall refer to this as Boole's integral.

Now, the well-known justification of the form (5), or the entirely equivalent form (6), depends upon the existence of a certain number of derivatives of $f(x)$. For let $P_{rs}(X)$ denote the polynomial obtained by removing the factor $(X-b_r)^s$ from $P(X)$, so that from (4) we have that

$$1 \equiv \sum_{r=1}^{q} \sum_{s=1}^{m_r} a_{rs} P_{rs}(X). \qquad \text{.........(8)}$$

Then, carrying out the operation $P(D)$ on both sides of (5), we obtain

$$P(D) \, . \, y = \sum_{r=1}^{q} \sum_{s=1}^{m_r} \{ a_{rs} P_{rs}(D) \, . \, f(x) \} \qquad \text{.........(9)}$$

or

$$P(D) \, . \, y = \left\{ \sum_{r=1}^{q} \sum_{s=1}^{m_r} a_{rs} P_{rs}(D) \right\} . \, f(x)$$

$$= f(x), \text{ by (8)}.$$

In order that this proof may be valid, it is essential that $f(x)$ should admit enough derivatives for the expressions $P_{rs}(D) . f(x)$ on the right-hand side of (9) to exist.

2. We shall now show that the alternative form (7) of Boole's integral is, in fact, identical with Cauchy's integral, which is valid provided only that $f(x)$ is continuous in the neighbourhood of $x = c$.

Multiplying the identity (4) by X^n, and then writing $1/Y$ for X, we obtain the identity

$$\frac{1}{a \prod_{r=1}^{q} (1-b_r Y)^{m_r}} \equiv \sum_{r=1}^{q} \sum_{s=1}^{m_r} \frac{1}{Y^{n-s}} \cdot \frac{a_{rs}}{(1-b_r Y)^s}. \qquad \text{.........(10)}$$

For sufficiently small $|Y|$, the left-hand side of (10) may be expanded as a convergent power series in Y. But the right-hand side, for sufficiently small $|Y|$, is equal to

$$\sum_{r=1}^{q} \sum_{s=1}^{m_r} \frac{a_{rs}}{Y^{n-s}} \sum_{t=0}^{\infty} \frac{(s+t-1)!}{(s-1)!\,t!} b_r^t Y^t.$$

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Hence, equating to zero the coefficients of the negative powers of Y, we have

$$\sum_{r=1}^{q} \sum_{s=1}^{S_{rp}} \frac{p!}{(s-1)!(p-s+1)!} a_{rs} b_r^{p-s+1} = 0, \quad (p = 0, 1, \ldots, n-2) \quad \text{.........(11)}$$

where $S_{rp} = m_r$ if $p+1 \geqq m_r$, and $S_{rp} = p+1$ if $p+1 < m_r$. Again, equating constant terms on the right and left sides of (10), we have that

$$\sum_{r=1}^{q} \sum_{s=1}^{m_r} \frac{(n-1)!}{(s-1)!(n-s)!} a_{rs} b_r^{n-s} = \frac{1}{a}. \quad \text{........................(12)}$$

Now let

$$F(x) = \sum_{r=1}^{q} e^{b_r x} \left\{ \sum_{s=1}^{m_r} a_{rs} x^{s-1}/(s-1)! \right\},$$

then, clearly, $P(D).F(x) = 0$, while

$$D^p . F(x) = \sum_{r=1}^{q} e^{b_r x} \left[\sum_{s=1}^{m_r} (D + b_r)^p . a_{rs} x^{s-1}/(s-1)! \right] \quad (p = 0, 1, 2, \ldots).$$

Putting $x = 0$, and using (11) and (12), we see that

$$\left. \begin{array}{l} F(0) = F'(0) = \ldots = F^{n-2}(0) = 0, \\ F^{n-1}(0) = 1/a. \end{array} \right\}$$

It follows, therefore, that $F(x)$ is identical with the function $\phi(x)$ in (3) and, writing $(x-t)$ for x, we see that (7) is identical with (2).

This result justifies the use of Boole's simple and elegant procedure for writing down the particular integral, without making any assumption about the existence of derivatives of $f(x)$, provided only that the result be expressed in the form of a single quadrature of the form (7).

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