S. Iwasaki and H. Kimura
Nagoya Math. J.
Vol. 37 (1970), 25–32

ON SOME DOUBLY TRANSITIVE PERMUTATION GROUPS OF DEGREE N AND ORDER $6n(n-1)$

SHIRO IWASAKI AND HIROSHI KIMURA[1)]

Dedicated to Professor K. Ono on his 60th birthday

The purpose of this paper is to prove the following result.

THEOREM. *Let Ω be the set of symbols $1, 2, \cdots, n$. Let $\mathfrak{G}$ be a doubly transitive group on Ω of order $6n(n-1)$ not containing a regular normal subgroup and let $\mathfrak{K}$ be the stabilizer of the set of symbols 1 and 2. Assume that $\mathfrak{K}$ is cyclic and independent, i.e., $\mathfrak{K} \cap G^{-1}\mathfrak{K}G = 1$ or $\mathfrak{K}$ for every element G of $\mathfrak{G}$. Then $\mathfrak{G}$ is isomorphic to either $PGL(2,7)$ or $PSL(2,13)$.*

We use the standard notation;

$C_{\mathfrak{X}}(\mathfrak{T})$: the centralizer of a subset $\mathfrak{T}$ in a group $\mathfrak{X}$

$N_{\mathfrak{X}}(\mathfrak{T})$: the normalizer of $\mathfrak{T}$ in $\mathfrak{X}$

$\langle \cdots \rangle$: the subgroup generated by $\cdots$

$|\mathfrak{T}|$: the number of elements in $\mathfrak{T}$

$[\mathfrak{X} : \mathfrak{Y}]$: the index of a subgroup $\mathfrak{Y}$ in $\mathfrak{X}$

$\mathfrak{T}^G$: $G^{-1}\mathfrak{T}G$ where $G \in \mathfrak{X}$.

Proof of Theorem

1. Let $\mathfrak{H}$ be the stabilizer of the symbol 1. $\mathfrak{K}$ is of order 6 and it is generated by a permutation K whose cyclic structure has the form $(1)(2)\cdots$. Since $\mathfrak{G}$ is doubly transitive on Ω, it contains an involution I with the cyclic structure $(1,2)\cdots$ which is conjugate to K^3. Then we have the following decomposition of $\mathfrak{G}$;

Received November 18, 1968

1) This work was supported by The Sakkokai Foundation.

$$\mathfrak{G} = \mathfrak{H} + \mathfrak{H}I\mathfrak{H}.$$

Since I is contained in $N_{\mathfrak{G}}(\mathfrak{K})$, it induces an automorphism of $\mathfrak{K}$ and (i) $K^I = K$ i.e. $\langle K, I\rangle$ is abelian or (ii) $K^I = K^{-1}$ i.e. $\langle K, I\rangle$ is dihedral. If an element $H'IH$ of a coset $\mathfrak{H}IH$ of $\mathfrak{H}$ is an involution, then $I(HH')I = (HH')^{-1}$ is contained in $\mathfrak{K}$. Hence, in case (i) the coset $\mathfrak{H}IH$ contains just two involutions, namely $H^{-1}IH$ and $H^{-1}K^3IH$, and, in case (ii) it contains just six involutions, namely $H^{-1}K'IH$ for $K' \in \mathfrak{K}$. Let $g(2)$ and $h(2)$ denote the numbers of involutions in $\mathfrak{G}$ and $\mathfrak{H}$, respectively. Since the number of cosets of $\mathfrak{H}$ in $\mathfrak{H}I\mathfrak{H}$ is $n-1$, we have

$$g(2) = h(2) + \alpha(n-1). \tag{1}$$

where $\alpha = 2$ and 6 for cases (i) and (ii), respectively.

2. Let $\mathfrak{K}$ keep $i (i \geqq 2)$ symbols of Ω, say $1, 2, \cdots, i$, unchanged. By the assumption of the independence of $\mathfrak{K}$, K has neither 2-cycle nor 3-cycle in its cyclic decomposition, i.e., it has only 1-cycles and 6-cycles and $N_{\mathfrak{G}}\mathfrak{K} = C_{\mathfrak{G}}(K^3)$. Put $\mathfrak{J} = \{1, 2, \cdots, i\}$. Then by a theorem of Witt ([9, Th. 9. 4]), $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ can be considered as a doubly transitive permutation group on $\mathfrak{J}$. Since every permutation of $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ distinct from $\mathfrak{K}$ leaves by the definition of $\mathfrak{K}$ at most one symbol of $\mathfrak{J}$ fixed, $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is a complete Frobenius group on $\mathfrak{J}$. Therefore i equals a power of a prime number, say p^m, and the orders of $N_{\mathfrak{G}}\mathfrak{K}$ and $\mathfrak{H} \cap N_{\mathfrak{G}}\mathfrak{K}$ are equal to $6i(i-1)$ and $6(i-1)$, respectively. By the double transitivity of $\mathfrak{G}$, any involution in $\mathfrak{G}$ which leaves at least two symbols in Ω fixed is conjugate to K^3 and the number of such involutions is equal to $n(n-1)/i(i-1)$. Similarly, any involution in $\mathfrak{H}$ which leaves at least two symbols in Ω fixed is conjugate to K^3 in $\mathfrak{H}$ and its number is equal to $n-1/i-1$.

At first, let us assume that n is odd. Let $h^*(2)$ be the number of involutions in $\mathfrak{H}$ leaving only the symbol 1 fixed. Then from (1) and the above argument the following equality is obtained;

$$h^*(2)n + n(n-1)/i(i-1) = h^*(2) + (n-1)/(i-1) + \alpha(n-1). \tag{2}$$

Since i is less than n, it follows from (2) that $h^*(2) < \alpha$.

Now we shall prove that if $h^*(2) \neq 0$ and $K^I = K^{-1}$, then $h^*(2) = 3$. Let ζ be any involution in $\mathfrak{G}$ which leaves only one symbol of Ω fixed and

https://doi.org/10.1017/S0027763000013258 Published online by Cambridge University Press

assume that $C_{\mathfrak{G}}(\zeta)$ contains an element Q of order 3. At first we shall show that Q leaves only one symbol of Ω fixed. If Q leaves at least two symbols of Ω fixed, then, since $\mathfrak{G}$ is doubly transitive on Ω, there exists an element G in $\mathfrak{G}$ such that $Q^G = K^2$ and $\zeta^G = (1,2)\cdots$ is contained in $N_{\mathfrak{G}}\langle K^2\rangle$. Since $\langle I, K^2\rangle$ is dihedral, $\langle \zeta^G, K^2\rangle$ must be dihedral. In fact, since n, i and $h^*(2)$ are dependent on only $\mathfrak{G}$ and independent of the choice of $I = (1,2)\cdots$, from (2) so is α. But $\langle \zeta, Q\rangle$ is abelian, a contradiction. Thus if $|C_{\mathfrak{G}}(\zeta)|$ is divisible by 3, then 3 is a factor of $n-1$. Therefore $|C_{\mathfrak{G}}(\zeta)|$ and n are relatively prime and hence $[\mathfrak{G}: C_{\mathfrak{G}}(\zeta)]$ is divisible by $3n$. Even if $|C_{\mathfrak{G}}(\zeta)|$ is not divisible by 3, $|C_{\mathfrak{G}}(\zeta)|$ and n are relatively prime and hence the same conclusion is obtained. On the other hand, the number of involutions in $\mathfrak{G}$ which leaves only one symbol of Ω fixed is equal to $h^*(2)\cdot n$ and $h^*(2) < \alpha = 6$, hence we obtain $h^*(2) = 3$.

Furthermore, in the same way as in [6, 2. 2] $h^*(2) \neq 1$. (By the way, note that the core of $\mathfrak{G}$ is identy 1.) Thus there are three cases;

$$\text{(A)} \quad \alpha - h^*(2) = 2, \quad \text{(B)} \quad \alpha - h^*(2) = 3 \quad \text{and} \quad \text{(C)} \quad \alpha - h^*(2) = 6.$$

The following equalities are obtained from (2) for cases (A), (B) and (C), respectively.

(A) $$n = i(2i-1) = p^m(2p^m - 1) \qquad (p\colon \text{odd}),$$

(B) $$n = i(3i-2) = p^m(3p^m - 2) \qquad (p\colon \text{odd}),$$

and

(C) $$n = i(6i-5) = p^m(6p^m - 5) \qquad (p\colon \text{odd}).$$

Next let us assume that n is even. Let $g^*(2)$ be the number of involutions in $\mathfrak{G}$ leaving no symbol of Ω fixed. Then corresponding to (2) the following equality is obtained from (1);

(3) $$g^*(2) + n(n-1)/i(i-1) = n - 1/i - 1 + \alpha(n-1).$$

Let J be an involution in $\mathfrak{G}$ leaving no symbol of Ω fixed. Assume that $|C_{\mathfrak{G}}(J)|$ is divisible by a prime factor q of $n-1$. Then $C_{\mathfrak{G}}(J)$ contains a permutation Q of order q and Q leaves at least two symbols of Ω fixed. Hence $q = 3$ and the common prime factor of $n-1$ and $|C_{\mathfrak{G}}(J)|$ is 3. Next assume that $|C_{\mathfrak{G}}(J)|$ is divisible by 3^2. Let $\mathfrak{P}$ be a Sylow 3-subgroup of $C_{\mathfrak{G}}(J)$. Since n is not divisible by 3, $\mathfrak{P}$ leaves just one symbol of Ω fixed.

Since J leaves no symbol of Ω fixed, this is a contradiction. Thus $[\mathfrak{G}: C_{\mathfrak{G}}(J)]$ is divisible by $n-1$ and hence $g^*(2)$ is so. On the other hand, it follows from (3) that $g^*(2) < \alpha(n-1)$. Thus we have $n = i(\beta i - \beta + 1)$, where $\beta = \alpha - g^*(2)/n-1$. Since n is even, i must be even and $i = 2^m$.

3. Case (A) for $p \neq 3$. Let $\mathfrak{P}$ be a Sylow p-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$. Since the group of automorphisms of $\mathfrak{K}$ is of order 2, we may assume that $\mathfrak{P}$ is a Sylow p-subgroup of $C_{\mathfrak{G}}\mathfrak{K}$. Then, since $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is a complete Frobenius group of degree p^m, $\mathfrak{P}$ is elementary abelian and normal in $N_{\mathfrak{G}}\mathfrak{K}$. In this case, $\mathfrak{P}$ is also a Sylow p-subgroup of $\mathfrak{G}$. Let the orders of $N_{\mathfrak{G}}\mathfrak{P}$ and $C_{\mathfrak{G}}\mathfrak{P}$ be $6p^m(p^m-1)x$ and $6p^m y$, respectively. If $x=1$, then from Sylow's theorem it should hold that $[\mathfrak{G}: N_{\mathfrak{G}}\mathfrak{P}] = (2p^m-1)(2p^m+1) \equiv 1 \pmod{p}$, which, since p is odd, is a contradiction. Thus x is greater than one. If $y=1$, then $C_{\mathfrak{G}}\mathfrak{P} = \mathfrak{K} \times \mathfrak{P}$ and $\mathfrak{K}$ would be normal in $N_{\mathfrak{G}}\mathfrak{P}$, and this would imply that $x=1$. Thus y is greater than one. Let $\mathfrak{S}$ be a Sylow 2-subgroup of $C_{\mathfrak{G}}\mathfrak{P}$. Since any permutation $(\neq 1)$ of $\mathfrak{P}$ leaves no symbol of Ω fixed, $\mathfrak{S}$ must leave at least two symbols of Ω fixed and hence $\mathfrak{S}$ is conjugate to $\langle K^3 \rangle$. Thus y is odd. If y is divisible by 3, then let $\mathfrak{R}$ be a Sylow 3-subgroup of $C_{\mathfrak{G}}\mathfrak{P}$. From the cyclic structure of K $n-i = 2i(i-1)$ is divisible by 6 and so n is not divisible by 3. Hence, as above, $\mathfrak{R}$ is conjugate to $\langle K^2 \rangle$. Thus y is relatively prime to 2, 3 and p. Therefore y is a factor of n and hence of $2p^m-1$. $\mathfrak{P}$ has a normal p-complement $\mathfrak{A}$ of order $6y$ in $C_{\mathfrak{G}}\mathfrak{P}$ and $\mathfrak{K}$ has a normal complement $\mathfrak{Y}$ of order y in $\mathfrak{A}$. Then $\mathfrak{Y}$ is normal even in $N_{\mathfrak{G}}\mathfrak{P}$. Any permutation $(\neq 1)$ of $\mathfrak{Y}$ does not leave any symbol of Ω fixed. Put $\mathfrak{V} = \mathfrak{H} \cap N_{\mathfrak{G}}\mathfrak{K}$. Then $\mathfrak{V}$ is contained in $N_{\mathfrak{G}}\mathfrak{Y}$. Assume that $\mathfrak{V}$ contains a permutation V of a prime order q which is commutative with a permutation $Y(\neq 1)$ of $\mathfrak{Y}$. Since V fixes at least two symbols of Ω, $q=2$ or 3. If $q=2$, then V is conjugate to K^3. Since $|C_{\mathfrak{G}}(K^3)|$ and y are relatively prime, this is a contradiction. Thus $q \neq 2$. Similarly, $q \neq 3$. Thus every permutation $(\neq 1)$ of $\mathfrak{V}$ is not commutative with any permutation $(\neq 1)$ of $\mathfrak{Y}$. This implies that y is not less than $|\mathfrak{V}|+1 = 6p^m-5$, which is a contradiction, for y is a factor of $2p^m-1$. Thus there exists no group satisfying the conditions of the theorem in Case (A) for $p \neq 3$.

4. Case (A) for $p=3$. Let $\mathfrak{P}$ be a Sylow 3-subgroup of $C_{\mathfrak{G}}\mathfrak{K}$ containing K^2. It is also a Sylow 3-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$ and $\mathfrak{G}$. Let $\mathfrak{Q}$ be a

https://doi.org/10.1017/S0027763000013258 Published online by Cambridge University Press

subgroup of $N_{\mathfrak{G}}\mathfrak{K}$ containing $\mathfrak{K}$ such that $\mathfrak{Q}/\mathfrak{K}$ is a regular normal subgroup of $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$. Then $\mathfrak{P}$ is normal in $\mathfrak{Q} = \mathfrak{P}\mathfrak{K}$ and so in $N_{\mathfrak{G}}\mathfrak{K}$. Clearly $N_{\mathfrak{G}}\mathfrak{K} \supseteq C_{\mathfrak{G}}(K^2) \supseteq C_{\mathfrak{G}}\mathfrak{P}$. Let $3^{m'}(m' \geqq 1)$ be the order of the center of $\mathfrak{P}$, $Z(\mathfrak{P})$. Then we shall prove that $|C_{\mathfrak{G}}\mathfrak{P}| = 2\cdot 3^{m'+m''}(m'' > 0)$. Since $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is Frobenius group on $\mathfrak{J}$ with Frobenius kernel $\mathfrak{Q}/\mathfrak{K}$ and a complement $\mathfrak{H} \cap N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$, every permutation $(\neq \mathfrak{K})$ of $\mathfrak{Q}/\mathfrak{K}$ is not commutative with any permutation $(\neq \mathfrak{K})$ of $\mathfrak{H} \cap N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ and hence $C_{\mathfrak{G}}\mathfrak{P} \cap (\mathfrak{H} \cap N_{\mathfrak{G}}\mathfrak{K}) = \mathfrak{K}$. Since $C_{\mathfrak{G}}\mathfrak{P}$ is normal in $N_{\mathfrak{G}}\mathfrak{K}$, $C_{\mathfrak{G}}\mathfrak{P} \subseteq \mathfrak{Q}$ or $C_{\mathfrak{G}}\mathfrak{P} \supseteq \mathfrak{Q}$. If $C_{\mathfrak{G}}\mathfrak{P} \supseteq \mathfrak{Q}$, $(\mathfrak{H} \cap N_{\mathfrak{G}}\mathfrak{K})\, C_{\mathfrak{G}}\mathfrak{P} = N_{\mathfrak{G}}\mathfrak{K}$ and $|C_{\mathfrak{G}}\mathfrak{P}/\mathfrak{K}| = 3^m$. Thus we have $|C_{\mathfrak{G}}\mathfrak{P}| = 2\cdot 3^{m'+m''}(m'' \geqq 0)$. If $m'' = 0$ then $C_{\mathfrak{G}}\mathfrak{P}$ is the direct product of $\langle K^3\rangle$ and $Z(\mathfrak{P})$ and $\langle K^3\rangle$ is normal in $N_{\mathfrak{G}}\mathfrak{P}$. Hence $N_{\mathfrak{G}}\mathfrak{K} = N_{\mathfrak{G}}\mathfrak{P}$ and from Sylow's theorem it should hold that $[\mathfrak{G} : N_{\mathfrak{G}}\mathfrak{P}] = (2\cdot 3^m - 1)(2\cdot 3^m + 1) \equiv 1 \pmod{3}$, which is a contradiction. Thus it is obtained that $|C_{\mathfrak{G}}\mathfrak{P}| = 2\cdot 3^{m'+m''}\,(m'' > 0)$. Let $\mathfrak{P}'$ be a Sylow 3-subgroup of $C_{\mathfrak{G}}\mathfrak{P}$. Since $\mathfrak{P}'\mathfrak{P}/\mathfrak{P}$ is isomorphic to $\mathfrak{P}'/Z(\mathfrak{P})$, $\mathfrak{P}'\mathfrak{P}$ is a 3-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$. Further, since $\mathfrak{P}$ is a normal Sylow 3-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$, $\mathfrak{P}'\mathfrak{P} \subseteq \mathfrak{P}$ and so $\mathfrak{P}' \subseteq \mathfrak{P}$. Hence $\mathfrak{P}' \subseteq C_{\mathfrak{G}}\mathfrak{P} \cap \mathfrak{P} = Z(\mathfrak{P})$, which is contradictory to those orders. Thus there exists no group satisfying the conditions of the theorem in Case (A) for $p = 3$.

5. Case (B) and (C). We shall examine a Sylow 2-subgroup of $\mathfrak{G}$. Since $K^I = K^{-1}$ in these cases, $[N_{\mathfrak{G}}\mathfrak{K} : C_{\mathfrak{G}}\mathfrak{K}] = 2$ and $|C_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}| = i(i-1)/2$. If $|C_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}|$ is even, then there exists an involution $\tau\mathfrak{K}$ in $C_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$. Since $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is a Frobenius group of order $i(i-1)$, a Sylow 2-subgroup of $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ contains only one involution. Hence $\tau\mathfrak{K}$ is conjugate to $I\mathfrak{K}$ in $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$. This contradicts that $K^I = K^{-1}$. Thus $|C_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}|$ is odd and $i - 1 = 2\cdot$(odd number).

In Case (B) $n - 1 = \{3(i-1) + 4\}(i-1) = 4\cdot$(odd number) and hence $|\mathfrak{G}| = 6n(n-1) = 8\cdot$ (odd number). Let $\mathfrak{S}$ be a Sylow 2-subgroup of $\mathfrak{G}$ containing $\langle K^3, I\rangle$. Then $\mathfrak{S}$ is neither abelian nor quaternion since $|N_{\mathfrak{G}}\mathfrak{K}| = |C_{\mathfrak{G}}(K^3)| = 4\cdot$(odd number). Thus $\mathfrak{S}$ is dihedral. Similarly, in Case (C) $|\mathfrak{G}| = 4\cdot$(odd number) and a Sylow 2-subgroup of $\mathfrak{G}$ is dihedral. Therefore, by [2] in both cases (B) and (C) $\mathfrak{G}$ is isomorphic to either

a subgroup of $P\Gamma L(2, q)$ containing $PSL(2, q)$, q odd,

or

the alternating group A_7.

But by [8, Satz 1, p. 422], in both cases (B) and (C) the former cannot happen and hence $\mathfrak{G}$ must be isomorphic to A_7. In Case (C) $|\mathfrak{G}|=4\cdot$(odd number) and this is imposible. Thus there exists no group satisfying the conditions of the theorem in Case (C). Since in Case (B) $h^*(2)=3$, $\mathfrak{G}$ has at least two conjugate classes of involutions. But all involutions of A_7 are conjugate in A_7. Thus there exists no group satisfying the conditions of the theorem in Case (B).

6. Case n is even and $\langle K, I\rangle$ is dihedral. Let $\mathfrak{Q}/\mathfrak{K}$ be a Frobenius kernel of Frobenius group $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ on $\mathfrak{I}$. Then $C_{\mathfrak{G}}\mathfrak{K}$ contains $\mathfrak{Q}$ or is contained in $\mathfrak{Q}$. Since I is contained in $\mathfrak{Q}$ and not contained in $C_{\mathfrak{G}}\mathfrak{K}$, $\mathfrak{Q}$ contains $C_{\mathfrak{G}}\mathfrak{K}$. Also, since $[N_{\mathfrak{G}}\mathfrak{K}: C_{\mathfrak{G}}\mathfrak{K}]=2$ and $[N_{\mathfrak{G}}\mathfrak{K}: \mathfrak{Q}]=2^m-1$, we have $m=1$ and $i=2$. Therefore, in cases $\beta=3$ and 6 $\mathfrak{G}$ is a Zassenhaus group and it can be seen that $\mathfrak{G}$ is isomorphic to $PGL(2,7)$ in the case $\beta=3$ and that $\mathfrak{G}$ is isomorphic to $PSL(2,13)$ in the case $\beta=6$ ([1], [3] and [10]). In the other cases, since $n-i$ must be divisible by 6, there exists no group satisfying the conditions of the theorem.

7. Now only the case $\langle K, I\rangle$ is abelian remain. In this case we may assume that $N_{\mathfrak{G}}\mathfrak{K}=C_{\mathfrak{G}}\mathfrak{K}$. In fact, if $[N_{\mathfrak{G}}\mathfrak{K}: C_{\mathfrak{G}}\mathfrak{K}]=2$, then there exists an element in a Sylow 2-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$ and so in $\mathfrak{Q}$($\mathfrak{Q}$ is the same meaning as in 6.) but in no $C_{\mathfrak{G}}\mathfrak{K}$. For the same reason as in 6, $\mathfrak{Q}$ contains $C_{\mathfrak{G}}\mathfrak{K}$, which was dealt with in 6.

Let $\mathfrak{S}$ be a Sylow 2-subgroup of $N_{\mathfrak{G}}\mathfrak{K}$ containing K^3. Then, since $\mathfrak{Q}=\mathfrak{K}\mathfrak{S}$ and $N_{\mathfrak{G}}\mathfrak{K}=C_{\mathfrak{G}}\mathfrak{K}$, $\mathfrak{S}$ is a normal Hall subgroup of $\mathfrak{Q}$ and hence normal in $N_{\mathfrak{G}}\mathfrak{K}$. Since $|\mathfrak{H}|=6(n-1)=2\cdot$(odd number), $\mathfrak{H}$ contains a subgroup $\mathfrak{U}$ of order $3(n-1)$. Hence $\mathfrak{H}\cap N_{\mathfrak{G}}\mathfrak{K}$ contains a subgroup $\mathfrak{V}=\mathfrak{U}\cap N_{\mathfrak{G}}\mathfrak{K}$ of order $3(2^m-1)$. Let $\mathfrak{P}$ be a Sylow 3-subgroup of $\mathfrak{V}$ containing K^2. Since $N_{\mathfrak{G}}\mathfrak{K}/\mathfrak{K}$ is a Frobenius group on $\mathfrak{I}$, all the Sylow subgroups of $\mathfrak{V}\mathfrak{K}/\mathfrak{K}$ are cyclic. Therefore $\mathfrak{P}/\langle K^2\rangle$ is cyclic and $\mathfrak{P}$ is abelian.

Since every permutation $(\neq\mathfrak{K})$ of $\mathfrak{S}\mathfrak{K}/\mathfrak{K}$ is not commutative with any permutation $(\neq\mathfrak{K})$ of $\mathfrak{V}\mathfrak{K}/\mathfrak{K}$ and $\mathfrak{S}$ contains I, any element $(\neq\mathfrak{K})$ of $\mathfrak{S}\mathfrak{K}/\mathfrak{K}$ is conjugate to $I\mathfrak{K}$ under $\mathfrak{V}\mathfrak{K}/\mathfrak{K}$. Hence, noting that $\mathfrak{S}\cap\mathfrak{K}=\langle K^3\rangle$, every permutation $(\neq 1)$ of $\mathfrak{S}$ can be represented in the form K^3, I^V or I^VK^3, where V is any permutation of $\mathfrak{B}$. Therefore every element $(\neq 1)$ of $\mathfrak{S}$ is an involution and $\mathfrak{S}$ is elementary abelian.

From now on, we use the notations in this paragraph.

8. Case $\beta=1$ and $\langle K, I\rangle$ is abelian. Since $n-i=2^m(2^m-1)$ is divisible by 6, 3 is a factor of 2^m-1. Hence $|\mathfrak{P}|$ is not less than 3^2 and $\mathfrak{P}$ leaves only the symbol 1 fixed and $N_{\mathfrak{G}}\mathfrak{P}$ is contained in $\mathfrak{H}$. Since $\langle K^3\rangle$ is a Sylow 2-subgroup of $C_{\mathfrak{G}}\mathfrak{P}$, we obtain that $N_{\mathfrak{G}}\mathfrak{P} = C_{\mathfrak{G}}\mathfrak{P}(N_{\mathfrak{G}}\mathfrak{P} \cap C_{\mathfrak{G}}(K^3))$. Hence $N_{\mathfrak{G}}\mathfrak{P} = C_{\mathfrak{G}}\mathfrak{P}(N_{\mathfrak{G}}\mathfrak{P} \cap \mathfrak{H} \cap N_{\mathfrak{G}}\mathfrak{K}) = C_{\mathfrak{G}}\mathfrak{P}(N_{\mathfrak{G}}\mathfrak{P} \cap \mathfrak{K}\mathfrak{B}) = C_{\mathfrak{G}}\mathfrak{P}(N_{\mathfrak{G}}\mathfrak{P} \cap \mathfrak{B})$. On the other hand, since 3 is the least prime factor of $|\mathfrak{B}/\langle K^2\rangle| = 2^m-1$ and a Sylow 3-subgroup $\mathfrak{P}/\langle K^2\rangle$ of $\mathfrak{B}/\langle K^2\rangle$ is cyclic, $N_{\mathfrak{B}/<K^2>}(\mathfrak{P}/\langle K^2\rangle) = C_{\mathfrak{B}/<K^2>}(\mathfrak{P}/\langle K^2\rangle)$. It is easily seen that $N_{\mathfrak{B}/<K^2>}(\mathfrak{P}/\langle K^2\rangle) = N_{\mathfrak{G}}\mathfrak{P} \cap \mathfrak{B}/\langle K^2\rangle$. Let X be any element of $N_{\mathfrak{G}}\mathfrak{P} \cap \mathfrak{B}$. Then, X induces trivial automorphisms of $\langle K^2\rangle$ and $\mathfrak{P}/\langle K^2\rangle$. Therefore $\langle X\rangle$ must be a 3-group and $\langle X\rangle \subseteq \mathfrak{P} \subseteq C_{\mathfrak{G}}\mathfrak{P}$. Hence $N_{\mathfrak{G}}\mathfrak{P} \cap \mathfrak{B} \subseteq C_{\mathfrak{G}}\mathfrak{P}$ and $N_{\mathfrak{G}}\mathfrak{P}=C_{\mathfrak{G}}\mathfrak{P}$. By the splitting theorem of Burnside $\mathfrak{P}$ has a normal complement in $\mathfrak{G}$. Since all the Sylow subgroups different from Sylow 3-subgroup of $\mathfrak{B}$ are cyclic, in the same way as in [4, Case C], it can be shown that $\mathfrak{G}$ has the normal subgroup $\mathfrak{N}$, which is a complement of $\mathfrak{B}$. In particular, $\mathfrak{N} \cap \mathfrak{U} = \mathfrak{D}$ is a normal subgroup of $\mathfrak{H}$. Since $|C_{\mathfrak{G}}(K^3)| = 6\cdot 2^m(2^m-1)$ and $|\mathfrak{D}| = 2^m+1$ are relatively prime, K^3 induces a fixed-point-free automorphism of $\mathfrak{D}$ of order 2 and so $\mathfrak{D}$ is abelian. $\mathfrak{N}$ is the product of $\mathfrak{D}$ and a Sylow 2-subgroup of $\mathfrak{G}$. Hence $\mathfrak{N}$, and therefore $\mathfrak{G}$ is solvable ([5]). Then $\mathfrak{G}$ must contain a regular normal subgroup. Thus there exists no group satisfying the conditions of the theorem in this case.

9. Case $\beta=2$ and $\langle K, I\rangle$ is abelian. In this case $\mathfrak{S}$ is a Sylow 2-subgroup of $\mathfrak{G}$ and an elementary abelian group of order 2^{m+1}. Since $g^*(2)=0$, every involution of $\mathfrak{G}$ is conjugate to K^3.

If $\mathfrak{S}^G$ contains K^3 for some $G \in \mathfrak{G}$, then $\mathfrak{S}^G = \mathfrak{S}$. In fact, since $\mathfrak{S}$ is abelian and normal in $N_{\mathfrak{G}}\mathfrak{K} = C_{\mathfrak{G}}(K^3)$, $\mathfrak{S}^G$ is contained in $N_{\mathfrak{G}}\mathfrak{K}$ and $\mathfrak{S}^G = \mathfrak{S}$. Thus we have

$$[\mathfrak{G}:\ C_{\mathfrak{G}}(K^3)] = (2^{m+1}-1)[\mathfrak{G}:\ N_{\mathfrak{G}}\mathfrak{S}],$$

namely

$$[N_{\mathfrak{G}}\mathfrak{S}:\ N_{\mathfrak{G}}\mathfrak{K}] = 2^{m+1}-1.$$

https://doi.org/10.1017/S0027763000013258 Published online by Cambridge University Press

Hence $|N_{\mathfrak{G}}\mathfrak{S}| = 2^{m+1}\cdot 3(2^m-1)(2^{m+1}-1)$ and $N_{\mathfrak{G}}\mathfrak{S}$ contains a subgroup $\mathfrak{A}$ of order $3(2^m-1)(2^{m+1}-1)$. Put $\mathfrak{B}_1 = \mathfrak{A}\cap\mathfrak{S}\mathfrak{B} = \mathfrak{A}\cap N_{\mathfrak{G}}\mathfrak{K}$. By a theorem of Schur-Zassenhaus $\mathfrak{B}$ and $\mathfrak{B}_1$ are conjugate in $\mathfrak{S}\mathfrak{B}$. A Sylow 3-subgroup of $\mathfrak{B}_1$ is abelian and all the other Sylow subgroups are cyclic. Therefore likewise in 8, it can be shown that $\mathfrak{A}$ has the normal subgroup $\mathfrak{B}$ of order $2^{m+1}-1$. Since $2^{m+1}-1$ and $|\mathfrak{H}| = 6(n-1)$ are relatively prime, every permutation $(\neq 1)$ of $\mathfrak{B}$ leaves no symbol of Ω fixed. If a permutation V of $\mathfrak{B}_1$ leaves at least two symbol of Ω fixed, then V is conjugate to K^2 and $|C_{\mathfrak{G}}(V)|$ is equal to $|N_{\mathfrak{G}}\mathfrak{K}|$. This implies that $C_{\mathfrak{G}}(V)\cap\mathfrak{B} = 1$, for $|\mathfrak{B}| = 2^{m+1}-1$ and $|N_{\mathfrak{G}}\mathfrak{K}| = 2^{m+1}\cdot 3(2^m-1)$ are relatively prime. Thus every permutation $(\neq 1)$ of $\mathfrak{B}$ is not commutative with any permutation $(\neq 1)$ of $\mathfrak{B}_1$. Hence $|\mathfrak{B}|-1 = 2^{m+1}-2 \geqq |\mathfrak{B}_1| = 3(2^m-1)$, a contradiction. Thus there exists no group satisfying the conditions of the theorem in this case.

Thus Theorem is proved.

References

[1] W. Feit, On a class of doubly transitive permutation groups, Ill. J. Math., **4** (1960), 170–186.

[2] D. Gorenstein and J.H. Walter, The characterization of finite groups with dihedral Sylow 2-subgroups, I, II, III, J. Alg., **2** (1965), 85–151, 218–270, 334–393.

[3] N. Ito, On a class of doubly transitive permutation groups, Ill. J. Math., **6** (1962), 341–352.

[4] ———, On doubly transitive groups of degree n and order $2(n-1)n$, Nagoya Math. J., **27** (1966), 409–417.

[5] O. Kegel, Produkte nilpotenter Gruppen, Arch. Math., **12** (1961), 90–93.

[6] H. Kimura, On doubly transitive permutation groups of degree n and order $4(n-1)n$, J. Math. Soc. Japan (to appear).

[7] ————, On some doubly transitive permutation groups of degree n and order $2^l(n-1)n$, (to appear).

[8] H. Lüneburg, Charakterisierungen der endlichen desarguesschen Ebenen, Math. Z. **85** (1964), 419–450.

[9] H. Wielandt, Finite permutation groups, Academic Press, New York, 1964.

[10] H. Zassenhaus, Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen, Hamb. Abh., **11** (1936), 17–40.

Mathematical Institute,
Hokkaido University