Proceedings of the Edinburgh Mathematical Society (2011) **54**, 345–361
DOI:10.1017/S0013091509001643

HIGHER-ORDER SINGULAR MULTI-POINT BOUNDARY-VALUE PROBLEMS ON TIME SCALES

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(Received 12 November 2009)

Abstract We study classes of higher-order singular boundary-value problems on a time scale $\mathbb{T}$ with a positive parameter λ in the differential equations. A homeomorphism and homomorphism ϕ are involved both in the differential equation and in the boundary conditions. Criteria are obtained for the existence and uniqueness of positive solutions. The dependence of positive solutions on the parameter λ is studied. Applications of our results to special problems are also discussed. Our analysis mainly relies on the mixed monotone operator theory. The results here are new, even in the cases of second-order differential and difference equations.

Keywords: positive solutions; singular boundary-value problems; existence; uniqueness; dependence; mixed monotone operator

2010 *Mathematics subject classification:* Primary 34B15
Secondary 34B08; 34B10

1. Introduction

We will assume that the reader has a basic knowledge of time scales and the time-scale notation that was first introduced by Hilger [**7**] and later refined in the monographs of Bohner and Peterson [**2**,**3**]. Let $T > 0$ be fixed and let $\mathbb{T}$ be a time scale with $0, T \in \mathbb{T}$. We are concerned here with the positive solutions of the higher-order singular boundary-value problems (BVPs) on $\mathbb{T}$ consisting of the equation

$$(\phi(u^{\Delta^{n-1}}))^{\nabla} + \lambda a(t) f(u) = 0, \quad t \in (0, T)_{\mathbb{T}}, \tag{1.1}$$

and one of the two multi-point boundary conditions (BCs)

$$\left.\begin{aligned} u^{\Delta^i}(0) &= \sum_{j=1}^{m} \alpha_j u^{\Delta^i}(\xi_j), \quad i = 0, \dots, n-2, \\ \phi(u^{\Delta^{n-1}}(T)) &= \sum_{j=1}^{m} \beta_j \phi(u^{\Delta^{n-1}}(\xi_j)) \end{aligned}\right\} \tag{1.2}$$

and

$$\left.\begin{aligned} u^{\Delta^i}(0) &= \sum_{j=1}^{m} \alpha_j u^{\Delta^i}(\xi_j), \quad i = 0, \dots, n-3, \\ \phi(u^{\Delta^{n-1}}(0)) &= \sum_{j=1}^{m} \alpha_j \phi(u^{\Delta^{n-1}}(\xi_j)), \\ u^{\Delta^{n-2}}(T) &= \sum_{j=1}^{m} \beta_j u^{\Delta^{n-2}}(\xi_j), \end{aligned}\right\} \tag{1.3}$$

where $\lambda > 0$ is a parameter, $m \geqslant 1$ and $n \geqslant 2$ are integers, $a\colon (0,T)_{\mathbb{T}} \to [0,\infty)$ and $f\colon (0,\infty) \to [0,\infty)$ are continuous, $\xi_j \in [0,T]_{\mathbb{T}}$ with $0 < \xi_1 < \cdots < \xi_m < \rho^{n-1}(T)$, $\alpha_j, \beta_j \in [0,\infty)$ for $j = 1,\dots,m$, and $\phi\colon \mathbb{R} \to \mathbb{R}$ satisfies the following:

(i) $\phi(x)$ is increasing in x;

(ii) $\phi(x)$ is a continuous bijection and its inverse mapping is also continuous;

(iii) $\phi(xy) = \phi(x)\phi(y)$ for $x, y \in \mathbb{R}$.

Typical examples of ϕ satisfying the above conditions are $\phi(x) = |x|^{p-2}x$ and $\phi(x) = x^{q/r}$, where $p > 1$ is a real number and q and r are odd positive integers. By a *positive solution* of BVP (1.1), (1.2) we mean a function $u\colon [0,T]_{\mathbb{T}} \to (0,\infty)$ satisfying (1.1), (1.2). A similar definition applies to BVP (1.1), (1.3). We note that, in our definition, a positive solution $u(t)$ satisfies $u(t) > 0$ for all $t \in [0,T]_{\mathbb{T}}$. In (1.1), the function $a(t)$ may be singular at $t = 0$ or $t = T$, and our nonlinear term $f(x)$ may be singular at $x = 0$. One special case of (1.1) takes the form

$$(\phi(u^{\Delta^{n-1}}))^{\nabla} + \lambda a(t)(\eta u^{\mu} + u^{-\nu}) = 0, \quad t \in (0,T)_{\mathbb{T}}, \tag{1.4}$$

where $\eta \geqslant 0$ and μ, $\nu > 0$.

We note that, in the case where $n = 2$, the first equation in BC (1.3) vanishes, and BVPs (1.1), (1.2) and (1.1), (1.3) now reduce to the second-order BVPs consisting of the equation

$$(\phi(u^{\Delta}))^{\nabla} + \lambda a(t) f(u) = 0, \quad t \in (0,T)_{\mathbb{T}}, \tag{1.5}$$

one of the BCs

$$u(0) = \sum_{j=1}^{m} \alpha_j u(\xi_j), \phi(u^{\Delta}(T)) = \sum_{j=1}^{m} \beta_j \phi(u^{\Delta}(\xi_j)) \tag{1.6}$$

and

$$\phi(u^{\Delta}(0)) = \sum_{j=1}^{m} \alpha_j \phi(u^{\Delta}(\xi_j)), u(T) = \sum_{j=1}^{m} \beta_j u(\xi_j). \tag{1.7}$$

In recent years, the existence of positive solutions of the second-order BVPs (1.5), (1.6) and (1.5), (1.7), or some of their variations, has been extensively investigated by many researchers. We refer the reader to [**1**,**6**,**8**,**9**,**13**,**14**,**17**] for a small sample of some recent work on these problems. Here, we note that all of these cited works study the case when the nonlinearities involved in the associated problems are regular in the phase variable, and none of them consider the uniqueness of solutions. To the best of our knowledge, for these BVPs, results on the uniqueness of positive solutions are rare in the literature. In this paper we study the higher-order singular BVPs (1.1), (1.2) and (1.1), (1.3). We not only investigate the existence and uniqueness of positive solutions but we also discuss the dependence of positive solutions on the parameter λ. Moreover, as a simple application of our theory, we present some uniqueness and dependence results for BVPs (1.4), (1.2) and (1.4), (1.3).

We wish to point out that, even in the case of second-order ($n = 2$) differential ($\mathbb{T} = \mathbb{R}$) and difference ($\mathbb{T} = \mathbb{N}$) equations, our results are significantly new. In our proofs, the analysis mainly relies on some results from mixed monotone operator theory. Mixed monotone operators were introduced by Guo and Lakshmikantham [**5**]. Since then, many authors have investigated such operators and their related applications (see, for example, [**4**, **10**–**12**, **15**, **16**] and the references therein).

The remainder of the paper is organized as follows. In § 2 we present our main results. The proofs of the main results, together with several technical lemmas, are given in § 3.

2. Main results

We need the following assumptions:

(H_1) $0 < \sum_{j=1}^{m} \alpha_j < 1$ and $0 < \sum_{j=1}^{m} \beta_j < 1$;

(H_2) $\int_0^T a(\tau)\,\nabla\tau < \infty$ and $\sum_{j=1}^{m} \alpha_j \int_0^{\xi_j} \phi^{-1}\Big(\int_r^T a(\tau)\,\nabla\tau + \sum_{j=1}^{m} \beta_i \int_{\xi_j}^T a(\tau)\,\nabla\tau\Big)\,\Delta r > 0$;

(H_3) f can be written as $f(x) = g(x) + h(x)$, where $g\colon [0,\infty) \to [0,\infty)$ is continuous and non-decreasing and $h\colon (0,\infty) \to [0,\infty)$ is continuous and non-increasing;

(H_4) there exists $\delta \in (0, 1)$ such that

$$g(\kappa x) \geqslant \phi(\kappa^{\delta})g(x) \tag{2.1}$$

and

$$h(\kappa^{-1}x) \geqslant \phi(\kappa^{\delta})h(x) \tag{2.2}$$

for $\kappa \in (0, 1)$ and $x > 0$;

(H_5) $\delta \in (0, \frac{1}{2})$ and $\phi^{-1}(x)$ is differentiable in a neighbourhood of 1.

We now state the main results of this paper. Here, for any $u \in C_{\mathrm{ld}}[0,T]_{\mathbb{T}}$, we write $\|u\| = \sup_{t\in[0,T]_{\mathbb{T}}} |u(t)|$.

Theorem 2.1. *Assume that* (H_1)–(H_4) *hold. Then, for any* $\lambda > 0$, *BVP (1.1), (1.2) has a unique positive solution* $u_\lambda(t)$. *If, in addition,* (H_5) *holds, then the unique solution* $u_\lambda(t)$ *satisfies the following properties:*

(i) $u_\lambda(t)$ *is strictly increasing in* λ, *i.e. if* $\lambda_1 > \lambda_2 > 0$, *then* $u_{\lambda_1}(t) > u_{\lambda_2}(t)$ *on* $[0,T]_{\mathbb{T}}$;

(ii) $\lim_{\lambda\to 0^+}\|u_\lambda\| = 0$ *and* $\lim_{\lambda\to\infty}\|u_\lambda\| = \infty$;

(iii) $u_\lambda(t)$ *is continuous in* λ, *i.e. if* $\lambda \to \lambda_0 > 0$, *then* $\|u_\lambda - u_{\lambda_0}\| \to 0$.

Corollary 2.2. *Assume that* (H_1) *and* (H_2) *hold, and* $\max\{\mu,\nu\} < 1$. *Then, for any* $\lambda > 0$, *BVP (1.4), (1.2) has a unique positive solution* $u_\lambda(t)$. *If, in addition,* (H_5) *holds, then the unique solution* $u_\lambda(t)$ *satisfies the three properties stated in Theorem 2.1.*

The following condition is a companion to (H_2).

$$(\mathrm{H}_2')\quad \int_0^T a(\tau)\,\nabla\tau < \infty \text{ and } \sum_{j=1}^m \beta_j \int_{\xi_j}^T \phi^{-1}\biggl(\int_0^r a(\tau)\,\nabla\tau + \sum_{j=1}^m \alpha_i \int_0^{\xi_j} a(\tau)\,\nabla\tau\biggr)\,\Delta r > 0.$$

The following results are analogous to to Theorem 2.1 and Corollary 2.2.

Theorem 2.3. *Assume that* (H_1), (H_2'), (H_3) *and* (H_4) *hold. Then, for any* $\lambda > 0$, *BVP (1.1), (1.3) has a unique positive solution* $u_\lambda(t)$. *If, in addition,* (H_5) *holds, then the unique solution* $u_\lambda(t)$ *satisfies the three properties stated in Theorem 2.1.*

Corollary 2.4. *Assume that* (H_1) *and* (H_2') *hold and* $\max\{\mu,\nu\} < 1$. *Then, for any* $\lambda > 0$, *BVP (1.4), (1.3) has a unique positive solution* $u_\lambda(t)$. *If, in addition,* (H_5) *holds, then the unique solution* $u_\lambda(t)$ *satisfies the three properties stated in Theorem 2.1.*

3. Proofs of the main results

The major tool used to obtain our results is the mixed monotone fixed-point theorem, which is stated as Lemma 3.8. Difficulties occur in ensuring that our problem satisfies the hypotheses of this theorem. This is accomplished using a sequence of lemmas (Lemmas 3.1–3.6). Again, we wish to point out that we are able to obtain the uniqueness of the positive solution, not just its existence.

Throughout this section we assume that (H_1)–(H_4) hold. Recall that if $I \subseteq \mathbb{R}$ is an interval, then the characteristic function χ on I is given by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & t \notin I. \end{cases}$$

Let

$$H(t,s) = \frac{1}{1-\sum_{j=1}^m \alpha_j}\sum_{j=1}^m \alpha_j \chi_{[0,\xi_j]}(s) + \chi_{[0,t]}(s). \tag{3.1}$$

For $i = 1,\dots,n-2$, recursively define $G_i(t,s)$ by

$$G_1(t,s) = H(t,s) \quad\text{and}\quad G_i(t,s) = \int_0^T H(t,\tau)G_{i-1}(\tau,s)\,\Delta\tau, \quad i = 2,\dots,n-2. \tag{3.2}$$

We now present some lemmas. The first lemma gives some useful lower and upper estimates for $G_i(t,s)$.

Lemma 3.1. *For $i = 1, \dots, n-2$, the function $G_i(t,s)$ satisfies*

$$c_i(s) \leqslant G_i(t,s) \leqslant d_i(s) \quad \text{for } t, s \in [0,T]_{\mathbb{T}}, \tag{3.3}$$

where

$$c_i(s) = c(s)\biggl(\int_0^T c(\tau)\,\Delta\tau\biggr)^{i-1} \quad \text{and} \quad d_i(s) = d(s)\biggl(\int_0^T d(\tau)\,\Delta\tau\biggr)^{i-1} \tag{3.4}$$

with

$$c(s) = \frac{\sum_{j=1}^m \alpha_j \chi_{[0,\xi_j]}(s)}{1-\sum_{j=1}^m \alpha_j} \qquad \text{and} \quad d(s) = \frac{\sum_{j=1}^m \alpha_j \chi_{[0,\xi_j]}(s)}{1-\sum_{j=1}^m \alpha_j} + 1. \tag{3.5}$$

Proof. For $t, s \in [0,T]_{\mathbb{T}}$, from (3.1), we see that

$$\frac{\sum_{j=1}^m \alpha_j \chi_{[0,\xi_j]}(s)}{1-\sum_{j=1}^m \alpha_j} \leqslant H(t,s) \leqslant \frac{\sum_{j=1}^m \alpha_j \chi_{[0,\xi_j]}(s)}{1-\sum_{j=1}^m \alpha_j} + 1,$$

i.e.

$$c(s) \leqslant G_1(t,s) = H(t,s) \leqslant d(s). \tag{3.6}$$

Note that $c_1(s) = c(s)$ and $d(s) = d_1(s)$. Thus, (3.3) holds for $i = 1$. By (3.2) and (3.6), we have

$$c(s)\int_0^T c(\tau)\,\Delta\tau \leqslant G_2(t,s) = \int_0^T H(t,\tau)G_1(\tau,s)\,\Delta\tau \leqslant d(s)\int_0^T d(\tau)\,\Delta\tau.$$

Again from (3.2),

$$c(s)\biggl(\int_0^T c(\tau)\,\Delta\tau\biggr)^2 \leqslant G_3(t,s) = \int_0^T H(t,\tau)G_2(\tau,s)\,\Delta\tau \leqslant d(s)\biggl(\int_0^T d(\tau)\,\Delta\tau\biggr)^2.$$

For $i = 1, \dots, n-2$, by (3.2) and induction, we can obtain that

$$\begin{aligned} c(s)\biggl(\int_0^T c(\tau)\,\Delta\tau\biggr)^{i-1} &\leqslant G_i(t,s) \\ &= \int_0^T H(t,\tau)G_{i-1}(\tau,s)\,\Delta\tau \\ &\leqslant d(s)\biggl(\int_0^T d(\tau)\,\Delta\tau\biggr)^{i-1}, \end{aligned}$$

i.e. (3.3) holds. This completes the proof of the lemma. □

The next five lemmas provide the equivalent integral forms for some BVPs.

Lemma 3.2. *Let $k \in C_{\mathrm{rd}}[0,T]_{\mathbb{T}}$. Then we have the following.*

(i) *The function $u(t)$ is a solution of the BVP*

$$u^{\Delta} = k(t) \text{ on } (0,T)_{\mathbb{T}}, \qquad u(0) = \sum_{j=1}^{m} \alpha_j u(\xi_j) \tag{3.7}$$

if and only if

$$u(t) = \int_0^T H(t,s)k(s)\,\Delta s. \tag{3.8}$$

(ii) *For $n \geqslant 3$, the function $u(t)$ is a solution of the BVP*

$$u^{\Delta^{n-2}} = k(t), \qquad t \in (0,T)_{\mathbb{T}}, \tag{3.9}$$

$$u^{\Delta^i}(0) = \sum_{j=1}^{m} \alpha_j u^{\Delta^i}(\xi_j), \quad i = 0,\dots,n-3, \tag{3.10}$$

if and only if

$$u(t) = \int_0^T G_{n-2}(t,s)k(s)\,\Delta s. \tag{3.11}$$

Proof. We first prove part (i). Assume that $u(t)$ is a solution of BVP (3.7). Delta integrating $u^{\Delta}(t) = k(t)$ from 0 to t yields

$$u(t) = u(0) + \int_0^t k(s)\,\Delta s = u(0) + \int_0^T \chi_{[0,t]}(s)k(s)\,\Delta s.$$

Then, from the condition $u(0) = \sum_{j=1}^{m} \alpha_j u(\xi_j)$, we have

$$\begin{aligned} u(0) &= u(0)\sum_{j=1}^{m} \alpha_j + \sum_{j=1}^{m} \alpha_j \int_0^T \chi_{[0,\xi_j]}(s)k(s)\,\Delta s \\ &= u(0)\sum_{j=1}^{m} \alpha_j + \int_0^T \sum_{j=1}^{m} \alpha_j \chi_{[0,\xi_j]}(s)k(s)\,\Delta s. \end{aligned}$$

Thus,

$$u(0) = \int_0^T \biggl(\frac{1}{1-\sum_{j=1}^{m}\alpha_j}\sum_{j=1}^{m} \alpha_j \chi_{[0,\xi_j]}(s)\biggr)k(s)\,\Delta s.$$

As a result, we have

$$\begin{aligned} u(t) &= \int_0^T \biggl(\frac{1}{1-\sum_{j=1}^{m}\alpha_j}\sum_{j=1}^{m} \alpha_j \chi_{[0,\xi_j]}(s) + \chi_{[0,t]}(s)\biggr)k(s)\,\Delta s \\ &= \int_0^T H(t,s)k(s)\,\Delta s, \end{aligned}$$

i.e. (3.8) holds. On the other hand, it can be directly verified that $u(t)$ defined by (3.8) satisfies (3.7). This proves part (i).

Next, we show part (ii). Assume that $u(t)$ is a solution of BVP (3.9), (3.10). Then, from part (i), we have

$$u^{\Delta^{n-3}}(t) = \int_0^T H(t,s)k(s)\,\Delta s = \int_0^T G_1(t,s)k(s)\,\Delta s.$$

Applying part (i) again, we obtain that

$$\begin{aligned} u^{\Delta^{n-4}}(t) &= \int_0^T H(t,\tau)\bigg(\int_0^T G_1(\tau,s)k(s)\,\Delta s\bigg)\,\Delta\tau \\ &= \int_0^T \bigg(\int_0^T H(t,\tau)G_1(\tau,s)\,\Delta\tau\bigg)k(s)\,\Delta s \\ &= \int_0^T G_2(t,s)k(s)\,\Delta s, \end{aligned}$$

By induction, we see that

$$u^{\Delta^{n-k}}(t) = \int_0^T G_{k-2}(t,s)k(s)\,\Delta s, \quad k = 3,\dots,n.$$

Hence,

$$u(t) = \int_0^T G_{n-2}(t,s)k(s)\,\Delta s,$$

i.e. (3.11) holds. On the other hand, we can verify directly that $u(t)$ defined by (3.11) satisfies (3.9) and (3.10). This proves part (ii) and completes the proof of the lemma. □

The following lemma is an immediate consequence of [**6**, Lemma 2.1].

Lemma 3.3. *Let $k \in C_{\mathrm{ld}}[0,T]_{\mathbb{T}}$. Then the function $u(t)$ is a solution of the BVP*

$$(\phi(u^{\Delta}))^{\nabla} + k(t) = 0, \quad t \in (0,T)_{\mathbb{T}},$$
$$u(0) = \sum_{j=1}^{m} \alpha_j u(\xi_j), \qquad \phi(u^{\Delta}(T)) = \sum_{j=1}^{m} \beta_j \phi(u^{\Delta}(\xi_j))$$

if and only if

$$u(t) = \int_0^t \phi^{-1}(A(s))\,\Delta s + \frac{\sum_{j=1}^m \alpha_j \int_0^{\xi_j} \phi^{-1}(A(s))\,\Delta s}{1 - \sum_{j=1}^m \alpha_j},$$

where

$$A(s) = \int_s^T k(\tau)\,\nabla\tau + \frac{\sum_{j=1}^m \beta_j \int_{\xi_j}^T k(\tau)\,\nabla\tau}{1 - \sum_{j=1}^m \beta_j}.$$

https://doi.org/10.1017/S0013091509001643 Published online by Cambridge University Press

Our next lemma is analogous to the previous one, except that it is for the higher-order case.

Lemma 3.4. *For $n \geqslant 3$, the function $u(t)$ is a solution of BVP (1.1), (1.2) if and only if*

$$u(t) = \int_0^T G_{n-2}(t,s) w_u(s)\, \Delta s, \tag{3.12}$$

where

$$w_u(t) = \int_0^t \phi^{-1}(A_u(r))\, \Delta r + \frac{\sum_{j=1}^m \alpha_j \int_0^{\xi_j} \phi^{-1}(A_u(r))\, \Delta r}{1 - \sum_{j=1}^m \alpha_j} \tag{3.13}$$

with

$$A_u(r) = \int_r^T \lambda a(\tau) f(u(\tau))\, \nabla\tau + \frac{\sum_{j=1}^m \beta_j \int_{\xi_j}^T \lambda a(\tau) f(u(\tau))\, \nabla\tau}{1 - \sum_{j=1}^m \beta_j}.$$

Proof. Let $u(t)$ be a solution of BVP (1.1), (1.2) and define $w_u(t) = u^{\Delta^{n-2}}(t)$. Note that

$$u^{\Delta^i}(0) = \sum_{j=1}^m \alpha_j u^{\Delta^i}(\xi_j), \quad i = 0, \dots, n-3,$$
$$w_u \in C_{\mathrm{rd}}[0,T]_{\mathbb{T}}.$$

Then, by Lemma 3.2 (ii), $u(t)$ satisfies (3.12). Moreover, we have

$$(\phi(w_u^{\Delta}))^{\nabla} + \lambda a(t) f(u) = 0, \quad t \in (0,T)_{\mathbb{T}}, \tag{3.14}$$

$$w_u(0) = \sum_{j=1}^m \alpha_j w_u(\xi_j), \qquad \phi(w_u^{\Delta}(T)) = \sum_{j=1}^m \beta_j \phi(w_u^{\Delta}(\xi_j)). \tag{3.15}$$

Lemma 3.3 then implies that (3.13) holds.

Now let $u(t)$ be given by (3.12). Then, by Lemma 3.2 (ii),

$$u^{\Delta^{n-2}} = w_u(t), \qquad t \in (0,T)_{\mathbb{T}},$$
$$u^{\Delta^i}(0) = \sum_{j=1}^m \alpha_j u^{\Delta^i}(\xi_j), \quad i = 0, \dots, n-3,$$

and, from Lemma 3.3, $w_u(t)$ satisfies (3.14), (3.15). Therefore, $u(t)$ is a solution of BVP (1.1), (1.2). This completes the proof of this lemma. □

The next lemma follows from [**13**, Lemma 2.1].

Lemma 3.5. *Let $k \in C_{\mathrm{ld}}[0,T]_{\mathbb{T}}$. Then the function $u(t)$ is a solution of the BVP*

$$(\phi(u^{\Delta}))^{\nabla} + k(t) = 0, \quad t \in (0,T)_{\mathbb{T}},$$
$$\phi(u^{\Delta}(0)) = \sum_{j=1}^m \alpha_j \phi(u^{\Delta}(\xi_j)), \qquad u(T) = \sum_{j=1}^m \beta_j u(\xi_j)$$

if and only if

$$u(t)=\int_t^T \phi^{-1}(B(s))\,\Delta s+\frac{\sum_{j=1}^m \beta_j \int_{\xi_j}^T \phi^{-1}(B(s))\,\Delta s}{1-\sum_{j=1}^m \beta_j},$$

where

$$B(s)=\int_0^s k(\tau)\,\nabla\tau+\frac{\sum_{j=1}^m \alpha_j \int_0^{\xi_j} k(\tau)\,\nabla\tau}{1-\sum_{j=1}^m \alpha_j}.$$

Lemma 3.6 concerns the problem (1.1), (1.3). It can be proved in a similar way to Lemma 3.4, so we omit the details.

Lemma 3.6. *For $n \geqslant 3$, the function $u(t)$ is a solution of BVP (1.1), (1.3) if and only if*

$$u(t)=\int_0^T G_{n-2}(t,s)\bar{w}_u(s)\,\Delta s,$$

where

$$\bar{w}_u(t)=\int_t^T \phi^{-1}(B_u(r))\,\Delta r+\frac{\sum_{j=1}^m \beta_j \int_{\xi_j}^T \phi^{-1}(B_u(r))\,\Delta r}{1-\sum_{j=1}^m \beta_j}$$

with

$$B_u(r)=\int_0^r \lambda a(\tau)f(u(\tau))\,\nabla\tau+\frac{\sum_{j=1}^m \alpha_j \int_0^{\xi_j} \lambda a(\tau)f(u(\tau))\,\nabla\tau}{1-\sum_{j=1}^m \alpha_j}.$$

Let X be a real Banach space. Choose $C>1$ and define

$$P_C=\{u\in X\colon C^{-1}\leqslant u(t)\leqslant C \text{ on } [0,T]_{\mathbb{T}}\}. \tag{3.16}$$

To prove our theorems, we need some results from monotone operator theory. The following definition and lemma are well known. For instance, Definition 3.7 can be found in [**4**, **5**, **10**, **11**, **15**, **16**] and Lemma 3.4 is a special case of [**11**, Theorem 2.1]. In what follows, let P_C be defined by (3.16).

Definition 3.7. Assume that $\mathcal{T}\colon P_C\times P_C\to P_C$. Then, $\mathcal{T}$ is called mixed monotone if $\mathcal{T}(x,y)$ is non-decreasing in x and non-increasing in y, i.e. for $x_1,x_2,y_1,y_2\in P_C$, we have

$$x_1\leqslant x_2,\ y_1\geqslant y_2 \Longrightarrow \mathcal{T}(x_1,y_1)\leqslant \mathcal{T}(x_2,y_2).$$

Moreover, an element $u\in P_C$ is said to be a fixed point of $\mathcal{T}$ if $\mathcal{T}(u,u)=u$.

Lemma 3.8. *Assume that $\mathcal{T}\colon P_C\times P_C\to P_C$ is a mixed monotone operator and that there exists $\delta\in(0,1)$ such that*

$$\mathcal{T}(\kappa u,\kappa^{-1}v)\geqslant \kappa^{\delta}\mathcal{T}(u,v) \quad \textit{for } u,v\in P_C \textit{ and } \kappa\in(0,1).$$

Then $\mathcal{T}$ has a unique fixed point in P_C.

We are now in a position to prove Theorem 2.1.

Proof of Theorem 2.1. We will prove the theorem in three steps.

Step 1. Let the Banach space $X := C_{\mathrm{ld}}[0,T]_{\mathbb{T}}$ be endowed with the norm

$$\|u\| = \sup_{t\in[0,T]_{\mathbb{T}}} |u(t)|$$

and let P_C be defined by (3.16) with this X. In this step, we show the claim that BVP (1.1), (1.2) has a unique solution in P_C for any $\lambda > 0$ if C is sufficiently large.

We first prove the claim of this step when $n \geqslant 3$. By letting $\kappa = 1/x$, $x > 1$ and $x = 1$, respectively, in (2.1), we obtain

$$g(x) \leqslant \phi(x^\delta)g(1), \quad x \geqslant 1, \tag{3.17}$$

and

$$g(\kappa) \geqslant \phi(\kappa^\delta)g(1), \quad \kappa \in (0,1). \tag{3.18}$$

From (2.2) with $x = 1$ and $\kappa^{-1}x = y$, respectively, we have

$$h(\kappa^{-1}) \geqslant \phi(\kappa^\delta)h(1), \qquad \kappa \in (0,1), \tag{3.19}$$

and

$$h(\kappa y) \leqslant \phi(\kappa^{-\delta})h(y), \quad \kappa \in (0,1),\ y > 0. \tag{3.20}$$

Choosing $y = 1$ in (3.20) yields

$$h(\kappa) \leqslant \phi(\kappa^{-\delta})h(1), \quad \kappa \in (0,1). \tag{3.21}$$

For any fixed $\lambda > 0$, choose $C = C(\lambda) > 1$ large enough that

$$\begin{aligned} C > \max\Biggl\{ &\biggl[\frac{T\int_0^T d_{n-2}(s)\,\Delta s}{1-\sum_{j=1}^m \alpha_j}\phi^{-1}\biggl(\frac{\lambda(g(1)+h(1))\int_0^T a(\tau)\,\nabla\tau}{1-\sum_{j=1}^m \beta_j}\biggr)\biggr]^{1/(1-\delta)}, \\ &\biggl[\frac{\phi^{-1}(\lambda(g(1)+h(1)))\int_0^T c_{n-2}(s)\,\Delta s}{1-\sum_{j=1}^m \alpha_j} \\ &\quad\times\sum_{j=1}^m \alpha_j \int_0^{\xi_j} \phi^{-1}\biggl(\int_r^T a(\tau)\,\nabla\tau + \sum_{j=1}^m \beta_i \int_{\xi_j}^T a(\tau)\,\nabla\tau\biggr)\Delta r\biggr]^{-1/(1-\delta)}\Biggr\}, \end{aligned} \tag{3.22}$$

where $c_{n-2}(s)$ and $d_{n-2}(s)$ are defined in Lemma 3.1. Let P_C be defined with the above C and define an operator $\mathcal{T}_\lambda : P_C \times P_C \to X$ by

$$\mathcal{T}_\lambda(u,v)(t) = \int_0^T G_{n-2}(t,s)w_{u,v,\lambda}(s)\,\Delta s, \tag{3.23}$$

where

$$w_{u,v,\lambda}(s) = \int_0^s \phi^{-1}(A_{u,v,\lambda}(r))\,\Delta r + \frac{\sum_{j=1}^m \alpha_j \int_0^{\xi_j} \phi^{-1}(A_{u,v,\lambda}(r))\,\Delta r}{1-\sum_{j=1}^m \alpha_j} \tag{3.24}$$

with

$$A_{u,v,\lambda}(r) = \int_r^T \lambda a(\tau)[g(u(\tau)) + h(v(\tau))]\,\nabla\tau + \frac{\sum_{j=1}^m \beta_j \int_{\xi_j}^T \lambda a(\tau)[g(u(\tau)) + h(v(\tau))]\,\nabla\tau}{1 - \sum_{j=1}^m \beta_j}.$$

From the monotonicity of g and h assumed in (H_3), it is easy to verify that $\mathcal{T}_\lambda$ is mixed monotone.

We now show that $\mathcal{T}\colon P_C \times P_C \to P_C$. Let $u, v \in P_C$ and $t \in [0,T]_{\mathbb{T}}$. Clearly, we have

$$\begin{aligned} A_{u,v,\lambda}(r) &\leqslant \left(1 + \frac{\sum_{j=1}^m \beta_j}{1 - \sum_{j=1}^m \beta_j}\right) \int_0^T \lambda a(\tau)[g(u(\tau)) + h(v(\tau))]\,\nabla\tau \\ &= \frac{\int_0^T \lambda a(\tau)[g(u(\tau)) + h(v(\tau))]\,\nabla\tau}{1 - \sum_{j=1}^m \beta_j}. \end{aligned} \tag{3.25}$$

Note that, from (3.17),

$$g(u(t)) \leqslant g(C) \leqslant \phi(C^\delta) g(1),$$

and from (3.21),

$$h(v(t)) \leqslant h(C^{-1}) \leqslant \phi(C^\delta) h(1).$$

Then

$$g(u(t)) + h(v(t)) \leqslant \phi(C^\delta)(g(1) + h(1)).$$

Combining this with (3.25) yields

$$A_{u,v,\lambda}(r) \leqslant \frac{\lambda\phi(C^\delta)(g(1) + h(1)) \int_0^T a(\tau)\,\nabla\tau}{1 - \sum_{j=1}^m \beta_j}.$$

From (3.24), we see that

$$\begin{aligned} w_{u,v,\lambda}(s) &\leqslant \left(1 + \frac{\sum_{j=1}^m \alpha_j}{1 - \sum_{j=1}^m \alpha_j}\right) T \phi^{-1}\left(\frac{\lambda\phi(C^\delta)(g(1) + h(1)) \int_0^T a(\tau)\,\nabla\tau}{1 - \sum_{j=1}^m \beta_j}\right) \\ &= \frac{C^\delta T}{1 - \sum_{j=1}^m \alpha_j} \phi^{-1}\left(\frac{\lambda(g(1) + h(1)) \int_0^T a(\tau)\,\nabla\tau}{1 - \sum_{j=1}^m \beta_j}\right) \end{aligned}$$

for $s \in [0,T]_{\mathbb{T}}$. Then, from (3.3) with $i = n-2$, (3.22) and (3.23), we have

$$\begin{aligned} \mathcal{T}_\lambda(u,v)(t) &\leqslant \frac{C^\delta T \int_0^T d_{n-2}(s)\,\Delta s}{1 - \sum_{j=1}^m \alpha_j} \phi^{-1}\left(\frac{\lambda(g(1) + h(1)) \int_0^T a(\tau)\,\nabla\tau}{1 - \sum_{j=1}^m \beta_j}\right) \\ &\leqslant C. \end{aligned} \tag{3.26}$$

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On the other hand, from (3.18),

$$g(u(t)) \geqslant g(C^{-1}) \geqslant \phi(C^{-\delta})g(1),$$

and from (3.19),

$$h(v(t)) \geqslant h(C) = h(1/C^{-1}) \geqslant \phi(C^{-\delta})h(1).$$

Then

$$g(u(t)) + h(v(t)) \geqslant \phi(C^{-\delta})(g(1) + h(1)).$$

This further implies that

$$\begin{aligned} A_{u,v,\lambda}(r) &\geqslant \lambda\phi(C^{-\delta})(g(1)+h(1))\biggl(\int_r^T a(\tau)\,\nabla\tau + \frac{\sum_{j=1}^m \beta_i \int_{\xi_j}^T a(\tau)\,\nabla\tau}{1-\sum_{j=1}^m \beta_j}\biggr) \\ &\geqslant \lambda\phi(C^{-\delta})(g(1)+h(1))\biggl(\int_r^T a(\tau)\,\nabla\tau + \sum_{j=1}^m \beta_i \int_{\xi_j}^T a(\tau)\,\nabla\tau\biggr). \end{aligned} \tag{3.27}$$

Thus, in view of (3.24) and (3.27), we have

$$\begin{aligned} w_{u,v,\lambda}(s) &\geqslant \frac{\sum_{j=1}^m \alpha_j \int_0^{\xi_j} \phi^{-1}(A_{u,v,\lambda}(r))\,\Delta r}{1-\sum_{j=1}^m \alpha_j} \\ &\geqslant \frac{C^{-\delta}\phi^{-1}(\lambda(g(1)+h(1)))}{1-\sum_{j=1}^m \alpha_j} \\ &\qquad \times \sum_{j=1}^m \alpha_j \int_0^{\xi_j} \phi^{-1}\biggl(\int_r^T a(\tau)\,\nabla\tau + \sum_{j=1}^m \beta_i \int_{\xi_j}^T a(\tau)\,\nabla\tau\biggr)\,\Delta r. \end{aligned}$$

Hence, from (3.3) with $i = n-2$, (3.22) and (3.23), it follows that

$$\begin{aligned} \mathcal{T}_\lambda(u,v)(t) &\geqslant \frac{C^{-\delta}\phi^{-1}(\lambda(g(1)+h(1)))\int_0^T c_{n-2}(s)\,\Delta s}{1-\sum_{j=1}^m \alpha_j} \\ &\qquad \times \sum_{j=1}^m \alpha_j \int_0^{\xi_j} \phi^{-1}\biggl(\int_r^T a(\tau)\,\nabla\tau + \sum_{j=1}^m \beta_i \int_{\xi_j}^T a(\tau)\,\nabla\tau\biggr)\,\Delta r \\ &\geqslant C^{-1}. \end{aligned} \tag{3.28}$$

From (3.26) and (3.28) we see that $\mathcal{T}(P_C \times P_C) \subseteq P_C$.

Next, for $u, v \in P_C$, $\kappa \in (0,1)$ and $t \in [0,T]_{\mathbb{T}}$, from (2.1) and (2.2) we have

$$g(\kappa u(t)) + h(\kappa^{-1}v(t)) \geqslant \phi(\kappa^\delta)(g(u(t)) + h(v(t))).$$

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Then

$$
\begin{aligned}
A_{\kappa u,\kappa^{-1}v,\lambda}(r) &= \int_r^T \lambda a(\tau)[g(\kappa u(\tau)) + h(\kappa^{-1}v(\tau))]\,\nabla\tau \\
&\qquad + \frac{\sum_{j=1}^m \beta_j \int_{\xi_j}^T \lambda a(\tau)[g(\kappa u(\tau)) + h(\kappa^{-1}v(\tau))]\,\nabla\tau}{1-\sum_{j=1}^m \beta_j} \\
&\geqslant \phi(\kappa^\delta)\biggl(\int_r^T \lambda a(\tau)[g(u(\tau)) + h(v(\tau))]\,\nabla\tau \\
&\qquad + \frac{\sum_{j=1}^m \beta_j \int_{\xi_j}^T \lambda a(\tau)[g(u(\tau)) + h(v(\tau))]\,\nabla\tau}{1-\sum_{j=1}^m \beta_j}\biggr) \\
&= \phi(\kappa^\delta)A_{u,v,\lambda}(r).
\end{aligned}
$$

Combining this with (3.24) yields

$$w_{\kappa u,\kappa^{-1}v,\lambda}(s) \geqslant \kappa^\delta w_{u,v,\lambda}(s).$$

Thus, from (3.23), we see that

$$\mathcal{T}_\lambda(\kappa u, \kappa^{-1}v)(t) \geqslant \kappa^\delta \mathcal{T}_\lambda(u,v)(t).$$

Therefore, all the conditions of Lemma 3.8 hold, so there exists a unique $u_\lambda \in P_C$ such that $\mathcal{T}_\lambda(u_\lambda, u_\lambda) = u_\lambda$. We can see from Lemma 3.4 that $u_\lambda(t)$ is the unique solution of BVP (1.1), (1.2) in P_C. This completes the proof of this step if $n \geqslant 3$. If $n = 2$, Lemma 3.3 (instead of Lemma 3.4) is needed to define an operator $\mathcal{T}_\lambda$ whose fixed point is a solution of BVP (1.1), (1.2). The idea of this part of the proof is essentially the same as that for the case where $n \geqslant 3$. In fact, the proof is relatively simpler when $n = 2$. We omit the details here.

Step 2. In this step, we show that BVP (1.1), (1.2) has at most one positive solution for each fixed $\lambda > 0$. Assume that BVP (1.1), (1.2) has two positive solutions $u_1(t)$ and $u_2(t)$ corresponding to the same $\lambda > 0$. Then, there exists $C > 1$ large enough that (3.22) holds and

$$C^{-1} \leqslant u_1(t) \leqslant C \quad \text{and} \quad C^{-1} \leqslant u_2(t) \leqslant C \quad \text{for } t \in [0,T]_{\mathbb{T}},$$

i.e. u_1, $u_2 \in P_C$. By Step 1, we know that $u_1(t) \equiv u_2(t)$ on $[0,T]_{\mathbb{T}}$. Hence, BVP (1.1), (1.2) has at most one positive solution.

Step 3. In this step, we finish the proof of the theorem. Combining Steps 1 and 2, we see that BVP (1.1), (1.2) has a unique positive solution $u_\lambda(t)$ for any $\lambda > 0$. In the remainder of the proof we will show that the three properties hold when $n \geqslant 3$. The proof when $n = 2$ is similar but simpler, and hence is omitted.

For $\lambda_1, \lambda_2 \in (0,\infty)$, define $\tau_i(\lambda_1,\lambda_2)$, $i = 1, 2$, by

$$\tau_1(\lambda_1,\lambda_2) = \begin{cases} \dfrac{\ln(\phi^{-1}(\lambda_1\lambda_2^{-1}))}{(1-\delta)\ln(\lambda_1\lambda_2^{-1})}, & \lambda_1 \neq \lambda_2, \\ (1-\delta)^{-1}(\phi^{-1})'(1), & \lambda_1 = \lambda_2, \end{cases}$$

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and

$$\tau_2(\lambda_1,\lambda_2)=\begin{cases}\dfrac{(1-2\delta)\ln(\phi^{-1}(\lambda_1\lambda_2^{-1}))}{(1-\delta)\ln(\lambda_1\lambda_2^{-1})}, & \lambda_1\neq\lambda_2,\\[2mm] \dfrac{1-2\delta}{1-\delta}(\phi^{-1})'(1), & \lambda_1=\lambda_2.\end{cases}$$

Then, in view of (H_5), it is easy to see that $\tau_i\in C((0,\infty)\times(0,\infty),(0,\infty))$, $i=1,2$,

$$\phi((\lambda_1^{-1}\lambda_2)^{\delta\tau_1(\lambda_1,\lambda_2)})\lambda_1=\phi((\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)})\lambda_2 \tag{3.29}$$

and

$$\phi((\lambda_1^{-1}\lambda_2)^{\delta\tau_1(\lambda_1,\lambda_2)})\lambda_2=\phi((\lambda_1^{-1}\lambda_2)^{\tau_1(\lambda_1,\lambda_2)})\lambda_1. \tag{3.30}$$

In what follows, we assume that $\lambda_1>\lambda_2>0$ are fixed and let

$$\begin{aligned}&B(\lambda_1,\lambda_2)\\&\quad=\{\gamma>0\colon \gamma^{-1}(\lambda_1\lambda_2^{-1})^{\tau_1(\lambda_1,\lambda_2)}u_{\lambda_2}(t)\geqslant u_{\lambda_1}(t)\geqslant\gamma(\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)}u_{\lambda_2}(t)\text{ on }[0,T]_{\mathbb{T}}\},\end{aligned}$$

where u_{λ_1} and u_{λ_2} are the unique solutions corresponding to λ_1 and λ_2, respectively. To see that $B(\lambda_1,\lambda_2)\neq\emptyset$, first note that $u_{\lambda_1}(t)>0$ and $u_{\lambda_2}(t)>0$ for $t\in[0,T]_{\mathbb{T}}$. Set

$$\gamma^*:=\min\left\{\frac{1}{(\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)}}\min_{t\in[0,T]_{\mathbb{T}}}\frac{u_{\lambda_1}(t)}{u_{\lambda_2}(t)},(\lambda_1\lambda_2^{-1})^{\tau_1(\lambda_1,\lambda_2)}\min_{t\in[0,T]_{\mathbb{T}}}\frac{u_{\lambda_2}(t)}{u_{\lambda_1}(t)}\right\}>0.$$

Clearly, any γ satisfying $0<\gamma<\gamma^*$ is in $B(\lambda_1,\lambda_2)$.

Define $\bar\gamma=\bar\gamma(\lambda_1,\lambda_2)$ by

$$\bar\gamma=\bar\gamma(\lambda_1,\lambda_2)=\sup B(\lambda_1,\lambda_2).$$

Then, we have

$$\bar\gamma^{-1}(\lambda_1\lambda_2^{-1})^{\tau_1(\lambda_1,\lambda_2)}u_{\lambda_2}(t)\geqslant u_{\lambda_1}(t)\geqslant\bar\gamma(\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)}u_{\lambda_2}(t)\quad\text{on }[0,T]_{\mathbb{T}}. \tag{3.31}$$

We claim that $\bar\gamma\geqslant 1$. Assume, to the contrary, that $0<\bar\gamma<1$. Then, from the monotonicity of g and h and (3.31), we obtain

$$\begin{aligned}&\lambda_1a(t)[g(u_{\lambda_1}(t))+h(u_{\lambda_1}(t))]\\&\qquad\geqslant\lambda_1a(t)[g(\bar\gamma(\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)}u_{\lambda_2}(t))+h(\bar\gamma^{-1}(\lambda_1\lambda_2^{-1})^{\tau_1(\lambda_1,\lambda_2)}u_{\lambda_2}(t))]\\&\qquad\geqslant\lambda_1a(t)[g(\bar\gamma u_{\lambda_2}(t))+h(\bar\gamma^{-1}(\lambda_1\lambda_2^{-1})^{\tau_1(\lambda_1,\lambda_2)}u_{\lambda_2}(t))].\end{aligned}$$

From (2.1), (2.2), (3.20) with $\kappa=(\lambda_1^{-1}\lambda_2)^{\tau_1(\lambda_1,\lambda_2)}$ and (3.29), we have

$$\begin{aligned}&\lambda_1a(t)[g(u_{\lambda_1}(t))+h(u_{\lambda_1}(t))]\\&\qquad\geqslant\lambda_1a(t)[\phi(\bar\gamma^{\delta})g(u_{\lambda_2}(t))+\phi(\bar\gamma^{\delta})h((\lambda_1\lambda_2^{-1})^{\tau_1(\lambda_1,\lambda_2)}u_{\lambda_2}(t))]\\&\qquad\geqslant\lambda_1a(t)[\phi(\bar\gamma^{\delta})g(u_{\lambda_2}(t))+\phi(\bar\gamma^{\delta})\phi((\lambda_1^{-1}\lambda_2)^{\delta\tau_1(\lambda_1,\lambda_2)})h(u_{\lambda_2}(t))]\\&\qquad\geqslant\phi(\bar\gamma^{\delta})\phi((\lambda_1^{-1}\lambda_2)^{\delta\tau_1(\lambda_1,\lambda_2)})\lambda_1a(t)[g(u_{\lambda_2}(t))+h(u_{\lambda_2}(t))]\\&\qquad=\phi(\bar\gamma^{\delta})\phi((\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)})\lambda_2a(t)[g(u_{\lambda_2}(t))+h(u_{\lambda_2}(t))].\end{aligned} \tag{3.32}$$

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Then

$$\begin{aligned}
&A_{u_{\lambda_1},u_{\lambda_1},\lambda_1}(r)\\
&\quad=\int_r^T \lambda_1 a(\tau)[g(u_{\lambda_1}(\tau))+h(u_{\lambda_1}(\tau))]\,\nabla\tau\\
&\qquad+\frac{\sum_{j=1}^m\beta_j\int_{\xi_j}^T\lambda_1 a(\tau)[g(u_{\lambda_1}(\tau))+h(u_{\lambda_1}(\tau))]\,\nabla\tau}{1-\sum_{j=1}^m\beta_j}\\
&\quad\geqslant \phi(\bar\gamma^\delta)\phi((\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)})\biggl(\int_r^T \lambda_2 a(\tau)[g(u_{\lambda_2}(\tau))+h(u_{\lambda_2}(\tau))]\,\nabla\tau\\
&\qquad\qquad+\frac{\sum_{j=1}^m\beta_j\int_{\xi_j}^T\lambda_2 a(\tau)[g(u_{\lambda_2}(\tau))+h(u_{\lambda_2}(\tau))]\,\nabla\tau}{1-\sum_{j=1}^m\beta_j}\biggr)\\
&\quad=\phi(\bar\gamma^\delta)\phi((\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)})A_{u_{\lambda_2},u_{\lambda_2},\lambda_2}(r).
\end{aligned}$$

Hence, from (3.24), we see that

$$w_{u_{\lambda_1},u_{\lambda_1},\lambda_1}(s)\geqslant\bar\gamma^\delta(\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)}w_{u_{\lambda_2},u_{\lambda_2},\lambda_2}(s).$$

This, together with (3.23), implies that

$$\begin{aligned}
u_{\lambda_1}(t)&=\mathcal{T}_\lambda(u_{\lambda_1},u_{\lambda_1})(t)\\
&\geqslant\bar\gamma^\delta(\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)}\mathcal{T}_\lambda(u_{\lambda_2},u_{\lambda_2})(t)\\
&=\bar\gamma^\delta(\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)}u_{\lambda_2}(t)\quad\text{on }[0,T]_{\mathbb{T}}.
\end{aligned}\tag{3.33}$$

On the other hand, from the monotonicity of g and h, and from (3.31), we obtain

$$\begin{aligned}
&\lambda_2 a(t)[g(u_{\lambda_2}(t))+h(u_{\lambda_2}(t))]\\
&\qquad\geqslant\lambda_2 a(t)[g(\bar\gamma(\lambda_1^{-1}\lambda_2)^{\tau_1(\lambda_1,\lambda_2)}u_{\lambda_1}(t))+h(\bar\gamma^{-1}(\lambda_1^{-1}\lambda_2)^{\tau_2(\lambda_1,\lambda_2)}u_{\lambda_1}(t))]\\
&\qquad\geqslant\lambda_1 a(t)[g(\bar\gamma(\lambda_1^{-1}\lambda_2)^{\tau_1(\lambda_1,\lambda_2)}u_{\lambda_1}(t))+h(\bar\gamma^{-1}u_{\lambda_1}(t))].
\end{aligned}$$

Thus, from (2.1), (2.2) and (3.30), we have

$$\begin{aligned}
&\lambda_2 a(t)[g(u_{\lambda_2}(t))+h(u_{\lambda_2}(t))]\\
&\qquad\geqslant\lambda_2 a(t)[\phi(\bar\gamma^\delta)\phi((\lambda_1^{-1}\lambda_2)^{\delta\tau_1(\lambda_1,\lambda_2)})g(u_{\lambda_1}(t))+\phi(\bar\gamma^\delta)h(u_{\lambda_1}(t))]\\
&\qquad\geqslant\phi(\bar\gamma^\delta)\phi((\lambda_1^{-1}\lambda_2)^{\delta\tau_1(\lambda_1,\lambda_2)})\lambda_2 a(t)[g(u_{\lambda_1}(t))+h(u_{\lambda_1}(t))]\\
&\qquad=\phi(\bar\gamma^\delta)\phi((\lambda_1^{-1}\lambda_2)^{\tau_1(\lambda_1,\lambda_2)})\lambda_1 a(t)[g(u_{\lambda_2}(t))+h(u_{\lambda_2}(t))].
\end{aligned}$$

From (3.24), we see that

$$w_{u_{\lambda_2},u_{\lambda_2},\lambda_2}(s)\geqslant\bar\gamma^\delta(\lambda_1^{-1}\lambda_2)^{\tau_1(\lambda_1,\lambda_2)}w_{u_{\lambda_1},u_{\lambda_1},\lambda_1}(s),$$

so (3.23) yields

$$\begin{aligned}
u_{\lambda_2}(t)&=\mathcal{T}_\lambda(u_{\lambda_2},u_{\lambda_2})(t)\\
&\geqslant\bar\gamma^\delta(\lambda_1^{-1}\lambda_2)^{\tau_1(\lambda_1,\lambda_2)}\mathcal{T}_\lambda(u_{\lambda_1},u_{\lambda_1})(t)\\
&=\bar\gamma^\delta(\lambda_1^{-1}\lambda_2)^{\tau_1(\lambda_1,\lambda_2)}u_{\lambda_1}(t)\quad\text{on }[0,T]_{\mathbb{T}}.
\end{aligned}\tag{3.34}$$

Now, (3.33) and (3.34) imply

$$\bar{\gamma}^{-\delta}(\lambda_1\lambda_2^{-1})^{\tau_1(\lambda_1,\lambda_2)}u_{\lambda_2}(t) \geqslant u_{\lambda_1}(t) \geqslant \bar{\gamma}^{\delta}(\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)}u_{\lambda_2}(t) \quad \text{on } [0,T]_{\mathbb{T}},$$

so $\bar{\gamma}^{\delta} \in B(\lambda_1,\lambda_2)$. Since $0<\delta<1$, we see that $\bar{\gamma}^{\delta} > \bar{\gamma}$. But this contradicts the definition of $\bar{\gamma}$. Therefore, $\bar{\gamma} \geqslant 1$. Thus, for any $\lambda_1 > \lambda_2 > 0$, from the second inequality in (3.31), we have

$$u_{\lambda_1}(t) \geqslant (\lambda_1\lambda_2^{-1})^{\tau_2(\lambda_1,\lambda_2)}u_{\lambda_2}(t) \quad \text{on } [0,T]_{\mathbb{T}}. \tag{3.35}$$

Consequently,

$$u_{\lambda_1}(t) > u_{\lambda_2}(t) \quad \text{on } [0,T]_{\mathbb{T}}.$$

This proves part (i).

Next, we prove part (ii). In (3.35), let λ_1 be fixed and write λ_2 as λ. Then

$$u_{\lambda}(t) \leqslant (\lambda\lambda_1^{-1})^{\tau_2(\lambda_1,\lambda)}u_{\lambda_1}(t) \text{ on } [0,T]_{\mathbb{T}},$$

which implies that

$$\|u_{\lambda}\| \leqslant (\lambda\lambda_1^{-1})^{\tau_2(\lambda_1,\lambda)}\|u_{\lambda_1}\|.$$

Thus, $\|u_{\lambda}\| \to 0$ as $\lambda \to 0^+$. Similarly, in (3.35), let λ_2 be fixed and write λ_1 as λ. Then,

$$u_{\lambda}(t) \geqslant (\lambda\lambda_2^{-1})^{\tau_2(\lambda,\lambda_2)}u_{\lambda_2}(t) \quad \text{on } [0,T]_{\mathbb{T}},$$

which implies that

$$\|u_{\lambda}\| \geqslant (\lambda\lambda_2^{-1})^{\tau_2(\lambda,\lambda_2)}\|u_{\lambda_2}\|.$$

Thus, $\|u_{\lambda}\| \to \infty$ as $\lambda \to \infty$.

Finally, we prove part (iii). For any fixed $\lambda_0 > 0$, let $\lambda > \lambda_0$. From the first inequality in (3.31) with $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_0$, we have

$$u_{\lambda}(t) \leqslant (\lambda\lambda_0^{-1})^{\tau_1(\lambda,\lambda_0)}u_{\lambda_0}(t) \quad \text{on } [0,T]_{\mathbb{T}}.$$

From part (i) of Theorem 2.1, $u_{\lambda}(t) > u_{\lambda_0}(t)$, so

$$\|u_{\lambda} - u_{\lambda_0}\| \leqslant ((\lambda\lambda_0^{-1})^{\tau_1(\lambda,\lambda_0)} - 1)\|u_{\lambda_0}\|.$$

As a result, $\|u_{\lambda} - u_{\lambda_0}\| \to 0$ as $\lambda \to \lambda_0^+$. Similarly, we can show that $\|u_{\lambda} - u_{\lambda_0}\| \to 0$ as $\lambda \to \lambda_0^-$. Hence, part (iii) holds. This completes the proof of the theorem. □

Proof of Corollary 2.2. With

$$f(x) = \eta x^{\mu} + x^{-\nu}, \qquad g(x) = \eta x^{\mu}, \qquad h(x) = x^{-\nu}, \qquad \delta = \max\{\mu,\nu\},$$

it is easy to verify that (H_3) and (H_4) hold. The conclusion readily follows from Theorem 2.1. □

Finally, we note that Theorem 2.3 and Corollary 2.4 can be proved by Lemmas 3.5 and 3.6 using essentially the same methods as those used above for Theorem 2.1 and Corollary 2.2. We omit the details here.

In conclusion, as mentioned above, our results are new, even for second-order differential and difference equations.

Acknowledgements. A.D. gratefully acknowledges Grant 2219 of the Science Fellowship Program by the Scientific and Technical Research Foundation of Turkey (TUBITAK). This research was conducted while A.D. was visiting the University of Tennessee at Chattanooga.

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