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GENERATORS OF THE EISENSTEIN–PICARD MODULAR GROUP

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Abstract

We prove that the Eisenstein–Picard modular group $\mathrm{SU}(2, 1; \mathbb{Z}[\omega_3])$ can be generated by four given transformations.

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1. Introduction

The Picard modular groups $\mathrm{SU}(2, 1; O_d)$ are the subgroups of $\mathrm{SU}(2, 1)$ with entries in O_d. Here O_d is the ring of algebraic integers in the imaginary quadratic number field $\mathbb{Q}(i\sqrt{d})$ for any positive square-free integer d. If $d \equiv 1, 2 \bmod 4$, then $O_d = \mathbb{Z}[i\sqrt{d}]$, and if $d \equiv 3 \bmod 4$, then $O_d = \mathbb{Z}[\frac{1}{2}(1 + i\sqrt{d})]$. It is well known that the ring O_d is Euclidean for positive square-free integers d if and only if $d = 1, 2, 3, 7, 11$.

The Picard modular groups $\mathrm{SU}(2, 1; O_d)$ are the simplest arithmetic lattices in $\mathrm{SU}(2, 1)$. In the case that $d \equiv 3 \bmod 4$, the ring O_d can be described as $O_d = \mathbb{Z}[\frac{1}{2}(-1 + i\sqrt{d})]$. Here the ring $\mathbb{Z}[\frac{1}{2}(-1 + i\sqrt{d})]$ is isomorphic to the ring $\mathbb{Z}[\frac{1}{2}(1 + i\sqrt{d})]$. The Picard modular groups can also be denoted by $\mathrm{SU}(2, 1; \mathbb{Z}[\omega_d])$ if we let $\omega_d = \frac{1}{2}(-1 + i\sqrt{d})$.

In general the presentation of a group can be obtained by constructing an explicit fundamental domain. Falbel and Parker (see [4]) studied the Eisenstein–Picard group $\mathrm{SU}(2, 1; \mathbb{Z}[\omega_3])$ and gave a system of generators and the corresponding presentation for this lattice. They similarly obtained a presentation of the Gauss–Picard modular group $\mathrm{SU}(2, 1; \mathbb{Z}[i])$ in [3].

In [2] the authors used a constructive method to obtain a finite system of generators for the Gauss–Picard modular group $\mathrm{SU}(2, 1; \mathbb{Z}[i])$. More precisely, they

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proved that the Gauss–Picard modular group $SU(2, 1; \mathbb{Z}[i])$ can be generated by four transformations: two Heisenberg translations, a rotation and an involution. Their description was applied to instanton corrections in string theory in [1].

It would be interesting to know whether the method used in [2] can be extended to the Euclidean Picard modular groups $SU(2, 1; \mathcal{O}_d)$ for $d = 2, 3, 7, 11$. In this note we show that the method used in [2] can applied to the Eisenstein–Picard modular group $SU(2, 1; \mathbb{Z}[\omega_3])$ and obtain a simple description of this group in terms of its generators. Recently, using a different method, Zhao found generators of the Euclidean Picard modular groups $SU(2, 1; \mathcal{O}_d)$ for $d = 2, 7, 11$ in [11].

In this paper we find a connection between the generators of the Eisenstein–Picard modular group $SU(2, 1; \mathbb{Z}[\omega_3])$ given in [4] and the generators given in this note. This connection leads to a new presentation of the lattice.

This paper is organized as follows. In Section 2 we introduce some basic general definitions and results from complex hyperbolic geometry and the Picard modular groups. The main result and its proof appear in Section 3.

2. Preliminaries

In this section we recall some basic definitions and results from complex hyperbolic geometry which can be found, for example, in [2, 7–10].

Let $\mathbb{C}^{2,1}$ denote the three-dimensional complex vector space $\mathbb{C}^3$ equipped with the Hermitian form

$$\langle z, w\rangle = w^* J z = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1$$

of signature (2, 1). Here the matrix J is defined by

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and the vectors z and w have the form

$$z = (z_1, z_2, z_3)^t, \quad w = (w_1, w_2, w_3)^t$$

where we denote by x^t the transpose of the vector x.

Let $z \in \mathbb{C}^{2,1}$. Then $\langle z, z\rangle$ is real. Thus we can define subsets V_0, V_- of $\mathbb{C}^{2,1}$ by

$$\begin{aligned} V_0 &= \{z \in \mathbb{C}^{2,1} - \{0\} \mid \langle z, z\rangle = 0\}, \\ V_- &= \{z \in \mathbb{C}^{2,1} \mid \langle z, z\rangle < 0\}. \end{aligned}$$

The complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ is defined to be the complex projective subspace $\mathbb{P}(V_-)$ equipped with the Bergman metric where

$$\mathbb{P} : \mathbb{C}^{2,1} - \{0\} \to \mathbb{CP}^2$$

is the canonical projection onto the complex projective space. The boundary of the complex hyperbolic space is defined to be $\partial\mathbf{H}^2_{\mathbb{C}} = \mathbb{P}(V_0)$.

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Using nonhomogeneous coordinates, we see that the complex hyperbolic space $\mathbf{H}^2_{\mathbb{C}}$ is equal to the Siegel domain

$$\left\{ \begin{bmatrix} z_1 \\ z_2 \\ 1 \end{bmatrix} \in \mathbb{CP}^2 \mid 2\,\mathrm{Re}(z_1) + |z_2|^2 < 0 \right\}.$$

Let $\mathfrak{N}$ denote the Heisenberg group which is equal to the set $\mathbb{C} \times \mathbb{R}$ with the product

$$(z_1, t_1)(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + 2\,\mathrm{Im}(z_1 \bar{z}_2)).$$

Then $\mathbf{H}^2_{\mathbb{C}}$ can be parameterized in horospherical coordinates by $(z, t, u) \in \mathfrak{N} \times \mathbb{R}^+$ with the connection map

$$(z, t, u) \to \begin{bmatrix} (-|z|^2 - u + it)/2 \\ z \\ 1 \end{bmatrix}.$$

The boundary of the complex hyperbolic space $\partial\mathbf{H}^2_{\mathbb{C}}$ can be identified with the one-point compactification $\bar{\mathfrak{N}} = \mathfrak{N} \bigcup \{q_\infty\}$ by the stereographic projection. Here $q_\infty = (1, 0, 0)^t$ denotes the point at infinity.

The holomorphic isometry group of $\mathbf{H}^2_{\mathbb{C}}$ is $\mathrm{PU}(2, 1)$. Recall that $\mathrm{PU}(2, 1)$ is the projectivization of the special unitary group $\mathrm{SU}(2, 1)$ that preserves the above Hermitian form. The matrix $G = (g_{jk})^3_{j,k=1} \in \mathrm{SU}(2, 1)$ satisfies the condition

$$G^* J G = J.$$

Here G^* denotes the conjugate transpose of the matrix G and the determinant of the matrix G is normalized to be equal to 1. The Picard modular groups $\mathrm{SU}(2, 1; O_d)$ are discrete holomorphic automorphism subgroups of $\mathbf{H}^2_{\mathbb{C}}$. The stabilizer subgroup Γ_∞ of q_∞ in $\mathrm{SU}(2, 1)$ contains three important classes of elements, namely the Heisenberg translations, dilations and rotations.

The Heisenberg translation by $(z, t) \in \partial\mathbf{H}^2_{\mathbb{C}}$ is given by the matrix

$$N_{(z,t)} \equiv \begin{pmatrix} 1 & -\bar{z} & (-|z|^2 + it)/2 \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}.$$

The two Heisenberg translations $N_{(z_1,t_1)}$ and $N_{(z_2,t_2)}$ have product

$$N_{(z_1,t_1)} \circ N_{(z_2,t_2)} = N_{(z_1+z_2, t_1+t_2+2\,\mathrm{Im}(z_1\bar{z}_2))}$$

which is the Heisenberg translation corresponding to the product of the two points (z_1, t_1) and (z_2, t_2) in the Heisenberg group $\mathfrak{N}$.

The Heisenberg rotation by $\beta \in \mathbb{S}^1$ is given by the matrix

$$M_\beta \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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The Heisenberg dilation by $\lambda \in \mathbb{R}^+$ is given by the matrix

$$A_\lambda \equiv \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}.$$

The holomorphic involution R swaps the point $q_0 = (0, 0) \in \partial \mathbf{H}^2_{\mathbb{C}}$ and the point at infinity q_∞. It is given by the matrix

$$R \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Using the Langlands decomposition, any element $P \in \Gamma_\infty$ can be decomposed into a product of a Heisenberg translation, a dilation and a rotation. Thus all elements of Γ_∞ can be written in the form

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ 0 & p_{22} & p_{23} \\ 0 & 0 & p_{33} \end{pmatrix} = N_{(z,t)} A_\lambda M_\beta = \begin{pmatrix} \lambda & -\beta\bar{z} & \frac{1}{2}(-|z|^2 + it)\lambda \\ 0 & \beta & \lambda^{-1} z \\ 0 & 0 & \lambda^{-1} \end{pmatrix}. \tag{2.1}$$

The parameters satisfy the corresponding conditions.

Equation (2.1) tells us that all elements of Γ_∞ are upper triangular. However, the following lemma gives a more precise characterization of the elements of Γ_∞. We omit the proof since it is similar to that of [2, Lemma 1].

LEMMA 2.1. *Let* $G = (g_{jk})^3_{j,k=1} \in \mathrm{SU}(2, 1)$. *Then* $G \in \Gamma_\infty$ *if and only if* $g_{31} = 0$.

In [5, 6] it is shown that the Langlands decomposition (2.1) can also be used to parameterize a holomorphic automorphism $G = (g_{jk})^3_{j,k=1}$ which is not in the subgroup Γ_∞. Let $N_{G(q_\infty)}$ denote the Heisenberg translation which maps q_0 to $G(q_\infty)$. Then the transformation $P \equiv RN^{-1}_{G(\infty)}G$ belongs to Γ_∞. Hence there are a Heisenberg translation N, a dilation A and a rotation M satisfying the equation

$$G = N_{G(\infty)}RP = N_{G(\infty)}RNAM.$$

The transformations N and P in the decomposition of G are not necessarily in the Picard modular groups $\mathrm{SU}(2, 1; \mathcal{O}_d)$ even if $G \in \mathrm{SU}(2, 1; \mathcal{O}_d)$. It is clear that the entries of N and P are not necessarily integers in the ring $\mathcal{O}_d$.

3. Main result and proof

We use the notation $\mathrm{SU}(2, 1; \mathbb{Z}[\omega_3])$ to denote the Eisenstein–Picard modular group with $\omega_3 = (-1 + i\sqrt{3})/2$. In this section we extend the techniques of [2] to prove the following theorem.

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THEOREM 3.1. *The Picard modular group* $\mathrm{SU}(2,1;\mathbb{Z}[\omega_3])$ *is generated by the Heisenberg translations*

$$N_{(\omega_3,\sqrt{3})} = \begin{pmatrix} 1 & -\overline{\omega}_3 & \omega_3 \\ 0 & 1 & \omega_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_{(1,\sqrt{3})} = \begin{pmatrix} 1 & -1 & \omega_3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

the rotation

$$M_{-\omega_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\omega_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the involution

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In order to prove this theorem we first characterize the elements of the stabilizer subgroup Γ_∞ of q_∞ in the Picard modular group $\mathrm{SU}(2,1;\mathbb{Z}[\omega_3])$.

LEMMA 3.2. *Let* $\Gamma_\infty(2,1;\mathbb{Z}[\omega_3])$ *be the stabilizer subgroup of* q_∞ *in* $\mathrm{SU}(2,1;\mathbb{Z}[\omega_3])$. *Then any element* $P \in \mathrm{SU}(2,1;\mathbb{Z}[\omega_3])$ *lies in* $\Gamma_\infty(2,1;\mathbb{Z}[\omega_3])$ *if and only if the parameters in the Langlands decomposition of* P *satisfy the conditions*

$$\lambda = 1, \quad t \in \sqrt{3}\mathbb{Z}, z \in \mathbb{Z}[\omega_3], \beta = \pm 1, \pm\omega_3, \pm\omega_3^2$$

and the integers $t/\sqrt{3}$ *and* $|z|^2$ *have the same parity.*

PROOF. It is quite easy to see that $\lambda = 1$. Considering the Langlands decomposition when $P \in \Gamma_\infty(2,1;\mathbb{Z}[\omega_3])$ allows us to deduce that $|\beta| = 1$, $z \in \mathbb{Z}[\omega_3]$ and $t \in \sqrt{3}\mathbb{Z}$. Since $\frac{1}{2}(-|z|^2 + it) \in \mathbb{Z}[\omega_3]$, $t/\sqrt{3} \in \mathbb{Z}$ and $|z|^2 \in \mathbb{Z}$, the integers $t/\sqrt{3}$ and $|z|^2$ have the same parity. As ω_3 is a cube root of unity it follows that $\beta = \pm 1, \pm\omega_3$ or $\pm\omega_3^2$. □

PROPOSITION 3.3. *Let* $\Gamma_\infty(2,1;\mathbb{Z}[\omega_3])$ *be the stabilizer subgroup of* q_∞ *in* $\mathrm{SU}(2,1;\mathbb{Z}[\omega_3])$. *Then* $\Gamma_\infty(2,1;\mathbb{Z}[\omega_3])$ *is generated by the Heisenberg translations* $N_{(\omega_3,\sqrt{3})}$, $N_{(1,\sqrt{3})}$ *and the rotation* $M_{-\omega_3}$.

PROOF. We know that any $P \in \Gamma_\infty(2,1;\mathbb{Z}[\omega_3])$ is upper triangular. By Lemma 3.2 there is no dilation component in the Langlands decomposition of P, that is,

$$P = N_{(z,t)} M_\beta = \begin{pmatrix} 1 & -\overline{z} & \frac{1}{2}(-|z|^2 + it) \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\beta^6 = 1$ the rotation component of P is one of $M_{-\omega_3}$, $M_{\omega_3^2} = M_{-\omega_3}^2$, $M_{-1} = M_{-\omega_3}^3$, $M_{\omega_3} = M_{-\omega_3}^4$, $M_{-\omega_3^2} = M_{-\omega_3}^5$ or $I = M_{-\omega_3}^6$. Therefore the rotation component of P in the Langlands decomposition is generated by $M_{-\omega_3}$.

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We now consider the Heisenberg translation component of P, namely $N_{(z,t)}$. Let $z = a + b\omega_3$, where $a, b \in \mathbb{Z}$ since $z \in \mathbb{Z}[\omega_3]$. Then $N_{(z,t)}$ splits as

$$N_{(z,t)} = N_{(a+b\omega_3,t)} = N_{(b\omega_3,\sqrt{3}b)} \circ N_{(a,\sqrt{3}a)} \circ N_{(0,t-\sqrt{3}ab-\sqrt{3}a-\sqrt{3}b)}.$$

Here $N_{(b\omega_3,\sqrt{3}b)}$ can be written in the form $N_{(b\omega_3,\sqrt{3}b)} = N^b_{(\omega_3,\sqrt{3})}$ since $b \in \mathbb{Z}$. The Heisenberg translation $N_{(a,\sqrt{3}a)}$ can be written in the form $N_{(a,\sqrt{3}a)} = N^a_{(1,\sqrt{3})}$ since $a \in \mathbb{Z}$.

To obtain the equality

$$N_{(0,t-\sqrt{3}ab-\sqrt{3}a-\sqrt{3}b)} = N_{(0,2\sqrt{3})}^{(t-\sqrt{3}(ab+a+b))/2\sqrt{3}}$$

it suffices to show that the number $(t - \sqrt{3}(ab + a + b))/2\sqrt{3}$ is an integer. By Lemma 3.2 the integers $t/\sqrt{3}$ and $|z|^2 = |a + b\omega_3|^2 = a^2 - ab + b^2$ have the same parity. It is easy to see that

$$a^2 - ab + b^2 + (ab + a + b) = a(a + 1) + b(b + 1) \in 2\mathbb{Z}.$$

Hence $t/\sqrt{3}$ and $ab + a + b$ have the same parity. It follows that $(t - \sqrt{3}(ab + a + b))/2\sqrt{3}$ is an integer.

The Heisenberg translation $N_{(0,2\sqrt{3})}$ can be generated by $N_{(1,\sqrt{3})}$ and M_{-1}, that is,

$$N_{(0,2\sqrt{3})} = (N_{(1,\sqrt{3})} \circ M_{-1})^2.$$

Our proposition has now been established. □

Proof of Theorem 3.1. Let $G = (g_{jk})^3_{j,k=1}$ be an element of the group $\mathrm{SU}(2, 1; \mathbb{Z}[\omega_3])$. Since the result is obviously true when $G \in \Gamma_\infty$, which is the stabilizer subgroup of q_∞, we may assume that G does not belong to the subgroup Γ_∞.

In this case $g_{31} \neq 0$ by Lemma 3.2 and G maps q_∞ to $(g_{11}/g_{31}, g_{21}/g_{31})$. Since $G(q_\infty)$ is an element of $\partial\mathbf{H}^2_{\mathbb{C}}$, we see that

$$2\,\mathrm{Re}\Big(\frac{g_{11}}{g_{31}}\Big) = -\Big|\frac{g_{21}}{g_{31}}\Big|^2. \tag{3.1}$$

Consider the Heisenberg translation $N_{G(q_\infty)}$ that maps q_0 to $G(q_\infty)$. Note that the translation $N_{G(q_\infty)}$ does not necessarily lie in the Picard modular group $\mathrm{SU}(2, 1; \mathbb{Z}[\omega_3])$ except when $|g_{31}| = 1$. However, we know that

$$RN^{-1}_{G(q_\infty)}G = P.$$

It is well known that the ring $\mathcal{O}_3 = \mathbb{Z}[\omega_3]$ is Euclidean. Thus we may successively approximate $N^{-1}_{G(q_\infty)}$ by Heisenberg translations in the Picard modular group and so decrease the value of $|g_{31}|^2 \in \mathbb{Z}$ until it becomes 0. Therefore G belongs to the subgroup Γ_∞ by Lemma 3.2 and can be expressed as a product of generators by Proposition 3.3.

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We calculate the entry in the lower left corner of the product

$$G_1 \equiv RN_{(z,t)}G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & -z \\ 1 & -\bar{z} & (-|z|^2 + it)/2 \end{pmatrix} G.$$

Now the entry $g_{31}^{(1)}$ lying in the lower left corner of $G_1 = (g_{jk}^{(1)})_{j,k=1}^3$ is equal to

$$\begin{aligned} g_{31}^{(1)} &= g_{11} - g_{21}\bar{z} + \frac{1}{2}(-|z|^2 + it)g_{31} \\ &= g_{31}\Big(\frac{g_{11}}{g_{31}} - \frac{g_{21}}{g_{31}}\bar{z} + \frac{1}{2}(-|z|^2 + it)\Big) \\ &= g_{31}\Big[\Big(\mathrm{Re}\Big(\frac{g_{11}}{g_{31}}\Big) - \mathrm{Re}\Big(\frac{g_{21}}{g_{31}}\bar{z}\Big) - \frac{1}{2}|z|^2\Big) + i\Big(\mathrm{Im}\Big(\frac{g_{11}}{g_{31}}\Big) - \mathrm{Im}\Big(\frac{g_{21}}{g_{31}}\bar{z}\Big) + \frac{1}{2}t\Big)\Big] \\ &= g_{31}(I_1 + iI_2). \end{aligned}$$

We can use (3.1) to simplify I_1 to

$$I_1 = -\frac{1}{2}\Big|\frac{g_{21}}{g_{31}} + z\Big|^2.$$

Let $(g_{21}/g_{31}) = x + iy$. Since

$$z = a + b\omega_3 = (a - \tfrac{1}{2}b) + \tfrac{1}{2}b\sqrt{3}i$$

we can select two appropriate integers a and b satisfying the conditions $|x + (a - \frac{1}{2}b)| \le \frac{1}{2}$ and $|y + \frac{1}{2}b\sqrt{3}i| \le \frac{\sqrt{3}}{4}$. Hence we obtain the upper bound

$$|I_1| \le \frac{1}{2}\Big(\Big(\frac{1}{2}\Big)^2 + \Big(\frac{\sqrt{3}}{4}\Big)^2\Big) = \frac{7}{32}.$$

Choosing some t in I_2, we calculate the inequality

$$|I_2| = \Big|\mathrm{Im}\Big(\frac{g_{11}}{g_{31}}\Big) - \mathrm{Im}\Big(\frac{g_{21}}{g_{31}}\bar{z}\Big) + \frac{1}{2}t\Big| \le \frac{\sqrt{3}}{4}$$

since $t \in \sqrt{3}\mathbb{Z}$. Therefore we have the following estimate for $g_{31}^{(1)}$:

$$|g_{31}^{(1)}|^2 = |g_{31}|^2|I_1 + iI_2|^2 = |g_{31}|^2(I_1^2 + I_2^2) \le |g_{31}|^2\Big[\Big(\frac{7}{32}\Big)^2 + \Big(\frac{\sqrt{3}}{4}\Big)^2\Big] < \frac{1}{4}|g_{31}|^2.$$

The preceding inequality tells us that we can reduce the matrix of the transformation G to the matrix of a transformation G_n with $g_{31}^{(n)} = 0$ by repeating this approximation procedure finitely many times. Moreover, by Lemma 3.2, this condition implies that the transformation G_n belongs to the subgroup $\Gamma_\infty(2, 1; \mathbb{Z}[\omega_3])$. As we showed in Proposition 3.3, the subgroup $\Gamma_\infty(2, 1; \mathbb{Z}[\omega_3])$ can be generated by the Heisenberg translations $N_{(\omega_3,\sqrt{3})}$, $N_{(1,\sqrt{3})}$ and the Heisenberg rotation $M_{-\omega_3}$. Since the approximation procedure just contains the transformations in $\Gamma_\infty(2, 1; \mathbb{Z}[\omega_3])$ and the transformation R the proof of Theorem 3.1 is now complete. □

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REMARK 3.4. In [4] Falbel and Parker gave the following presentation for the Eisenstein–Picard modular group $PU(2, 1; \mathbb{Z}[\omega_3])$:

$$\langle P, Q, R : R^2 = (QP^{-1})^6 = PQ^{-1}RQP^{-1}R = P^3Q^{-2} = (RP)^3 = 1\rangle.$$

Moreover, the stabilizer subgroup of infinity q_∞ has the presentation $\Gamma_\infty = \langle P, Q\rangle$. Here

$$P = \begin{pmatrix} 1 & 1 & \omega_3 \\ 0 & \omega_3 & -\omega_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 & \omega_3 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

By Proposition 3.3 it is clear that $PQ^{-1} = M_{-\omega_3}$, $Q = N_{(1,\sqrt{3})} \circ M^3_{-\omega_3}$ and

$$P = M_{-\omega_3} \circ Q = M_{-\omega_3} \circ N_{(1,\sqrt{3})} \circ M^3_{-\omega_3}.$$

This means that the subgroup Γ_∞ of $PU(2, 1; \mathbb{Z}[\omega_3])$ can be generated by a Heisenberg translation $N_{(1,\sqrt{3})}$ and a rotation $M_{-\omega_3}$. Hence the Picard modular group $PU(2, 1; \mathbb{Z}[\omega_3])$ is generated by $N_{(1,\sqrt{3})}$, $M_{-\omega_3}$ and R.

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