Canad. Math. Bull. Vol. 13 (4), 1970

MAGIC VALUATIONS OF FINITE GRAPHS

BY

ANTON KOTZIG AND ALEXANDER ROSA(1)

The purpose of this paper is to investigate for graphs the existence of certain valuations which have some "magic" property. The question about the existence of such valuations arises from the investigation of another kind of valuations which are introduced in [1] and are related to cyclic decompositions of complete graphs into isomorphic subgraphs.

Throughout this paper the word graph will mean a finite undirected graph without loops or multiple edges having at least one edge. By $G(m, n)$ we denote a graph having m vertices and n edges, by $V(G)$ and $E(G)$ the vertex-set and the edge-set of G, respectively. Both vertices and edges are called the elements of the graph.

DEFINITION 1. A graph $G(m, n)$ is said to have a magic valuation (M-valuation) with the constant C if there exists a one-to-one mapping f: $V(G) \cup E(G) \to \{1, 2, \ldots, m+n\}$ such that $f(a)+f(b)+f([a, b])=C$ for all $[a, b] \in E(G)$.

DEFINITION 2. (a) Two M-valuations f, f' of a graph G are equal, $f=f'$, if there exists an automorphism α of G such that $f(\alpha x)=f'(x)$ for all elements $x \in G$.

(b) Given an M-valuation f of a graph $G(m, n)$, the valuation $\bar{f}$ such that $\bar{f}(x) = m+n+1-f(x)$ for all elements $x \in G$ is said to be complementary to f.

(c) Two M-valuations f_1, f_2 of G are equivalent if $f_1=f_2$ or $f_1=\bar{f}_2$.

(d) An M-valuation f of G is said to be self-complementary (SCM-valuation) if $f=\bar{f}$.

LEMMA 1. *If a graph $G(m, n)$ without isolated vertices has an SCM-valuation then m is even and n is odd.*

Proof. It follows immediately from definitions that if $G(m, n)$ has an SCM-valuation then m and n are of different parity, and the constant C is $3(m+n+1)/2$. In an SCM-valuation f of $G(m, n)$ there is an element $x \in G$ such that $f(x)=\frac{1}{2}(m+n+1)$. Suppose x to be a vertex. Since G has no isolated vertices, there is a vertex y adjacent to x and $f(x)+f(y)+f([x, y])=3(m+n+1)/2$, i.e., $f(y)+f([x, y])=m+n+1$. On the other hand, $\bar{f}(y)=m+n+1-f(y)=f([x, y])$ which is a contradiction, whence x must be an edge. It follows easily that the number of edges n is odd and hence m is even.

Let us remark that there exist graphs $G(m, n)$ with isolated vertices having an SCM-valuation such that m is odd and n is even (for an example see Fig. 1).

Received by the editors November 12, 1969.

(1) Author was supported by the National Research Council of Canada.

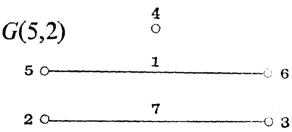

Figure 1.

THEOREM 1. (a) *If G is an m-gon then there exists no SCM-valuation of G.*
(b) *A complete graph K_m has an SCM-valuation if and only if $m=2$.*

Proof. (a) follows immediately from Lemma 1.

(b) An SCM-valuation of K_2 evidently exists; K_3 is a triangle. Let $m \geq 4$, and let f be an SCM-valuation of K_m. According to Lemma 1 there is an edge $h=[u, v]$ such that $f(h)=\frac{1}{2}(m+n+1)$. Pick another vertex w, and let $f(u)=a, f(w)=b$. According to the definition of an SCM-valuation we have $f(v)=m+n+1-a$, and there must be another vertex z such that $f(z)=m+n+1-b$. But then we have $f([w, z])=\frac{1}{2}(m+n+1)$ which is a contradiction.

We do not consider here the existence of an M-valuation for complete graphs; to this case a separate paper of the first author will be devoted. Let us state here only without proof:

A complete graph K_m has an M-valuation if and only if $n=2, 3, 5$, or 6.

THEOREM 2. *An M-valuation of the complete bipartite graph $K_{p,q}$ exists for all $p, q \geq 1$.*

Proof. We can easily construct an M-valuation of $K_{p,q}$. Denote the p blue vertices by $a_1, a_2, \ldots, a_p$ and the q red vertices by $b_1, b_2, \ldots, b_q$, and define f in the following way:

$$f(a_i) = i, \qquad f(b_i) = ip+i+p, \qquad f([a_i, b_j]) = (p+1)\cdot(q-j+2)-i-1.$$

Evidently, f is an M-valuation of $K_{p,q}$ with the constant $C=pq+3p+q+1$ since all verifications are trivial.

As an example, an M-valuation of $K_{3,4}$ is shown in Fig. 2.

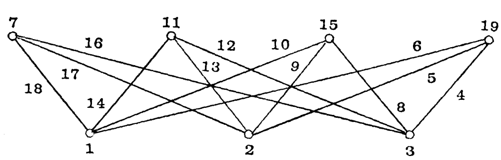

Figure 2.

THEOREM 3. *An M-valuation of the n-gon exists for all $n \geq 3$.*

Proof. Let $a_1, a_2, \ldots, a_n$ be the vertices and $[a_1, a_2], [a_2, a_3], \ldots, [a_n, a_1]$ the edges of the n-gon. Consider three cases:

https://doi.org/10.4153/CMB-1970-084-1 Published online by Cambridge University Press

Case 1. n is odd, $n=2k+1$. Define

$$f(a_i) = \begin{cases} i & \text{for } i \text{ odd} \\ i+n & \text{for } i \text{ even} \end{cases}$$

$$f([a_i, a_{i+1}]) = 2n-2i \quad \text{for } i = 1, 2, \dots, n-1, \mathrm{f}([a_n, a_1]) = 2n.$$

Since we have

$$\bigcup_{V(G)} \{f(a_i)\} = \{1, 3, \dots, 2n-1\},$$

$$\bigcup_{E(G)} \{f([a_i, a_{i+1}])\} = \{2, 4, \dots, 2n\},$$

$$f(a_n)+f(a_1)+f([a_n, a_1]) = n+1+2n = 3n+1,$$

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = 2i+n+1+2n-2i = 3n+1 \quad \text{for } i = 1, 2, \dots, n-1,$$

f is an M-valuation of the n-gon with the constant $C=3n+1$.

Case 2. $n \equiv 0 \pmod 4$, $n=4k$. Define

$$f(a_i) = \begin{cases} i & \text{for } i = 1, 3, \dots, 2k-1 \\ i+1 & \text{for } i = 2k, 2k+2, \dots, 4k-2 \\ n+1 & \text{for } i = 2k+1, 2k+3, \dots, 4k-3, \quad k \geq 2 \\ n+i+1 & \text{for } i = 2, 4, \dots, 2k-2, \qquad k \geq 2 \\ 2 & \text{for } i = 4k-1 \\ 2n-2 & \text{for } i = 4k \end{cases}$$

$$f([a_i, a_{i+1}]) = \begin{cases} 2n-2i-2 & \text{for } i = 1, 2, \dots, 2k-2, 2k, 2k+1, \dots, 4k-3, \quad k \geq 2 \\ 2n & \text{for } i = 2k-1 \\ 2n-1 & \text{for } i = 4k-2 \\ n & \text{for } i = 4k-1 \\ n+1 & \text{for } i = 4k \end{cases}$$

We have

$$\bigcup_{V(G)} \{f(a_i)\} = \{1, 3, 5, \dots, n-1, n+3, n+5, \dots, 2n-3\} \cup \{2, 2n-2\}$$

$$\bigcup_{E(G)} \{f([a_i, a_{i+1}])\} = \{4, 6, 8, \dots, 2n-4, 2n\} \cup \{n+1, 2n-1\}.$$

Now we show that

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = C \quad \text{for all } i = 1, 2, \dots, 4k.$$

(1) Let $i \in \{1, 3, \dots, 2k-3\}$; then

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = i+(n+i+2)+(2n-2i-2) = 3n;$$

(2) Let $i = 2k-1$; then

$$f(a_{2k-1})+f(a_{2k})+f([a_{2k-1}, a_{2k}]) = (2k-1)+(2k+1)+2n = 3n;$$

(3) Let $i \in \{2k, 2k+2, \ldots, 4k-4\}$; then

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = (i+1)+(n+i+1)+(2n-2i-2) = 3n;$$

(4) Let $i = 4k-2$; then

$$f(a_{4k-2})+f(a_{4k-1})+f([a_{4k-2}, a_{4k-1}]) = (4k-1)+2+(2n-1) = 3n;$$

(5) Let $i \in \{2k+1, 2k+3, \ldots, 4k-3\}$; then

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = (n+i)+(i+2)+(2n-2i-2) = 3n;$$

(6) Let $i \in \{2, 4, \ldots, 2k-2\}$; then

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = (n+i+1)+(i+1)+(2n-2i-2) = 3n;$$

(7) Let $i = 4k-1$; then

$$f(a_{4k-1})+f(a_{4k})+f([a_{4k-1}, a_{4k}]) = 2+(2n-2)+n = 3n;$$

(8) Let $i = 4k$; then

$$f(a_{4k})+f(a_1)+f([a_{4k}, a_1]) = (2n-2)+1+(n+1) = 3n,$$

thus f is an M-valuation of the n-gon with the constant $C=3n$.

Case 3. $n \equiv 2 \pmod 4$, $n=4k+2$. Define

$$f(a_i) = \begin{cases} \frac{1}{2}(i+1) & \text{for } i = 1, 3, \ldots, 2k+1 \\ \frac{1}{2}(i+3) & \text{for } i = 2k+3, 2k+5, \ldots, 4k+1 \\ 6k+3 & \text{for } i = 2 \\ k+2 & \text{for } i = 2k+2 \\ 2k+3 & \text{for } i = 4k+2 \\ \frac{1}{2}i+2k+2 & \text{for } i = 4, 6, \ldots, 2k, \quad k \geq 2 \\ \frac{1}{2}i+2k+1 & \text{for } i = 2k+4, 2k+6, \ldots, 4k, \quad k \geq 2 \end{cases}$$

$$f([a_i, a_{i+1}]) = \begin{cases} 4k+3 & \text{for } i = 1 \\ 4k+2 & \text{for } i = 2 \\ 8k+4 & \text{for } i = 2k+1 \\ 8k+2 & \text{for } i = 2k+2 \\ 6k+2 & \text{for } i = 4k+1 \\ 8k+3 & \text{for } i = 4k+2 \\ 8k+4-i & \text{otherwise, } k \geq 2. \end{cases}$$

https://doi.org/10.4153/CMB-1970-084-1 Published online by Cambridge University Press

Since

$$\bigcup_{V(G)} \{f(a_i)\} = \{1, 2, \ldots, 4k, 4k+1, 6k+3\},$$

$$\bigcup_{E(G)} \{f([a_i, a_{i+1}])\} = \{4k+2, 4k+3, \ldots, 6k+1, 6k+2, 6k+4, 6k+5, \ldots, 8k+4\},$$

we have to verify only that

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = C \quad \text{for all } i = 1, 2, \ldots, 4k+2.$$

(1) Let $i = 1$; then

$$f(a_1)+f(a_2)+f([a_1, a_2]) = 1+(6k+3)+(4k+3) = 10k+7;$$

(2) Let $i \in \{3, 5, \ldots, 2k-1\}$; then

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = \tfrac{1}{2}(i+1)+(\tfrac{1}{2}(i+1)+2k+2)+(8k+4-i) = 10k+7;$$

(3) Let $i = 2k+1$; then

$$f(a_{2k+1})+f(a_{2k+2})+f([a_{2k+1}, a_{2k+2}]) = \tfrac{1}{2}(2k+2)+(k+2)+(8k+4) = 10k+7;$$

(4) Let $i \in \{2k+3, 2k+5, \ldots, 4k-1\}$; then

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = \tfrac{1}{2}(i+3)+(\tfrac{1}{2}(i+1)+2k+1)+(8k+4-i) = 10k+7;$$

(5) Let $i = 4k+1$; then

$$f(a_{4k+1})+f(a_{4k+2})+f([a_{4k+1}, a_{4k+2}]) = \tfrac{1}{2}(4k+4)+(2k+3)+(6k+2) = 10k+7;$$

(6) Let $i = 2$; then

$$f(a_2)+f(a_3)+f([a_2, a_3]) = (6k+3)+2+(4k+2) = 10k+7;$$

(7) Let $i = 2k+2$; then

$$f(a_{2k+2})+f(a_{2k+3})+f([a_{2k+2}, a_{2k+3}]) = (k+2)+\tfrac{1}{2}(2k+6)+(8k+2) = 10k+7;$$

(8) Let $i = 4k+2$; then

$$f(a_{4k+2})+f(a_1)+f([a_{4k+2}, a_1]) = (2k+3)+1+(8k+3) = 10k+7;$$

(9) Let $i \in \{4, 6, \ldots, 2k\}$; then

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = (\tfrac{1}{2}i+2k+2)+\tfrac{1}{2}(i+2)+(8k+4-i) = 10k+7;$$

(10) Let $i \in \{2k+4, 2k+6, \ldots, 4k\}$; then

$$f(a_i)+f(a_{i+1})+f([a_i, a_{i+1}]) = (\tfrac{1}{2}i+2k+1)+\tfrac{1}{2}(i+4)+(8k+4-i) = 10k+7,$$

thus f is an M-valuation of the $(4k+2)$-gon with the constant $C=10k+7$, which completes the proof of Theorem 3.

In Fig. 3 there are some illustrations of Theorem 3, in Fig. 4 all nonequivalent M-valuations of n-gons for $n=3, 4, 5, 6$ are shown.

https://doi.org/10.4153/CMB-1970-084-1 Published online by Cambridge University Press

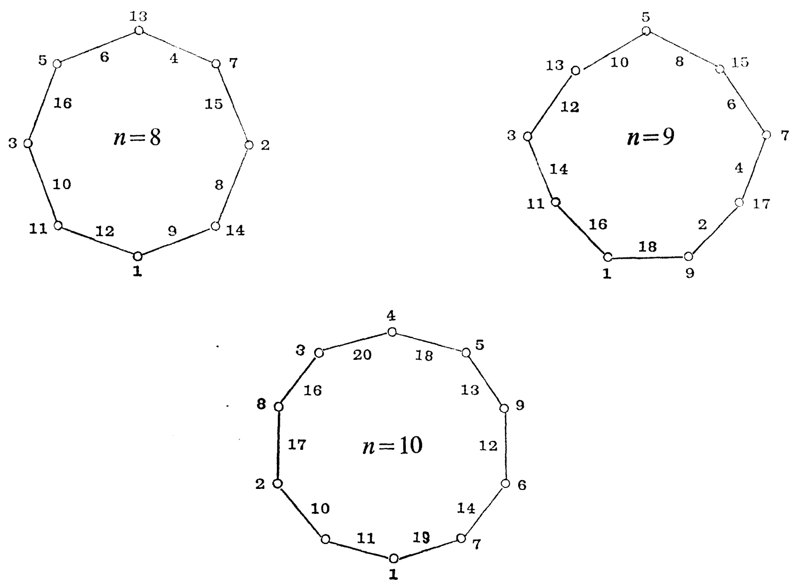

Figure 3.

Denote by C_i the tree with exactly two end-vertices (vertices of degree one) and i edges. Denote by H_i the star—the tree with i edges ($i \geq 3$) and exactly i end-vertices (and consequently, with exactly one vertex of degree i).

Definition 3. The base B_T of a tree T is the subgraph of T formed by all elements of T except all its end-vertices and end-edges.

For any tree T define the number $z(T)$ as follows:

(1) $z(C_i) = 0, \quad z(H_i) = 1$

(2) $z(T) = 1 + z(B_T)$ for $T \not\equiv C_i, T \not\equiv H_i$.

An edge $h = [u, v]$ of a tree T is said to be symmetric if there is an automorphism α of T such that $\alpha u = v$, $\alpha v = u$, $\alpha h = h$. A symmetric tree is a tree having a symmetric edge.

Theorem 4. *If T is a tree such that $z(T) \leq 1$ then there exists an M-valuation of T.*

Proof. Any tree T with $z(T) \leq 1$ can be realized in the plane so that its vertices (points) are displaced in two rows, the edges are segments joining these points (from different rows) and no two segments cross. Let $a_1, a_2, \ldots, a_p$ be the vertices in the first row and let $b_1, b_2, \ldots, b_q$ be the vertices in the second row. Define

$$f(a_i) = i, \quad f(b_i) = p+i, \qquad f([a_i, b_j]) = 2p+2q-i-j+1.$$

We have

$$\bigcup_{i=1}^{p} \{f(a_i)\} = \{1, 2, \ldots, p\}; \qquad \bigcup_{i=1}^{q} \{f(b_i)\} = \{p+1, p+2, \ldots, p+q\};$$

$$\bigcup_{E(T)} \{f([a_i, b_j])\} = \{p+q+1, p+q+2, \ldots, 2(p+q)-1\}$$

https://doi.org/10.4153/CMB-1970-084-1 Published online by Cambridge University Press

and

$$f(a_i)+f(b_j)+f([a_i, b_j]) = i+p+j+2p+2q-i-j+1 = 3p+2q+1,$$

thus f is an M-valuation of T with the constant $C=3p+2q+1$.

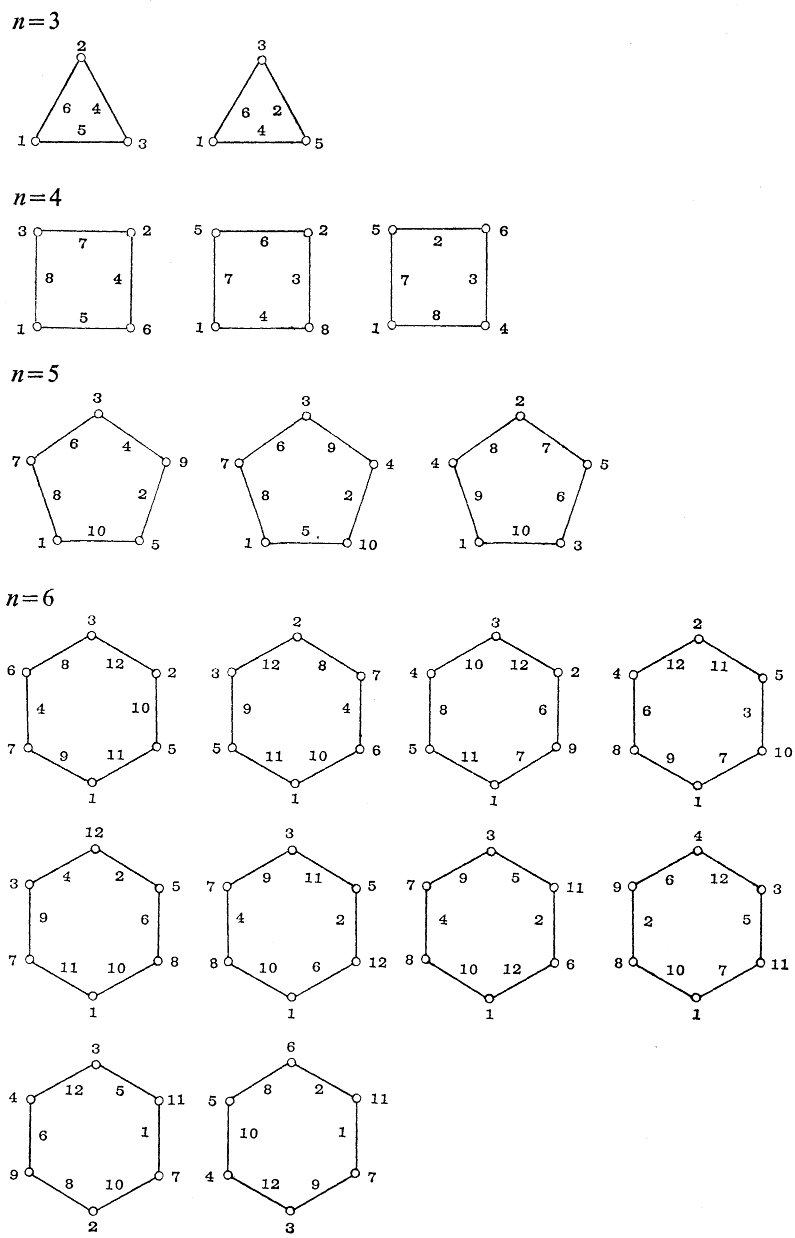

Figure 4.

THEOREM 5. *Let T be a tree and let $z(T) \leq 1$. There exists an SCM-valuation of T if and only if T is symmetric.*

Proof. The necessity is obvious. On the other hand, $p=q$ for any realization of a symmetric tree T as described in the proof of Theorem 4, and a valuation f of T defined by $f(a_i)=i, f(b_i)=3p+i-1, f([a_i, b_j])=3p-i-j+1$ is clearly an SCM-valuation of T with the constant $C=6p$.

THEOREM 6. *Let T be a symmetric tree and let S_1, S_2 be the two (isomorphic) components of the graph T' obtained from T by deleting the symmetric edge. If $z(S_i) \leq 1$ then there exists an SCM-valuation of T.*

Proof. Let $h=[u, v]$ be the symmetric edge of T and let α be an automorphism of T such that $\alpha u=v$, $\alpha v=u$, $\alpha h=h$. Let $u \in S_1$. Take a realization of S_1 as described in the proof of Theorem 4, and let $a_1, a_2, \ldots, a_p$ be the vertices in the first row and $b_1, b_2, \ldots, b_q$ the vertices in the second row, the vertex u being, without loss of generality, in the second row. Denote $a_i'=\alpha a_i$ $(i=1, 2, \ldots, p)$, $b_i'=\alpha b_i (i=1, 2, \ldots, q)$. We can realize T in the plane so that its vertices are displaced in four rows, namely

$$\begin{array}{cccc} a_1 & a_2 & \ldots & a_p \\ b_1 & b_2 & \ldots & b_q \\ b_1' & b_2' & \ldots & b_q' \\ a_1' & a_2' & \ldots & a_p', \end{array}$$

the edges (segments) join the vertices in the first and second row, and in the third and fourth row, respectively (the edges of S_1 and S_2, respectively), there is a single edge joining a vertex in the second row to a vertex in the third row, namely the the edge h, and no two edges cross. Now define

$$f(a_i) = i, \qquad f(b_i) = 3p+3q+i-1, \qquad f(a_i') = 4p+4q-i,$$
$$f(b_i') = p+q+1-i, \qquad f([a_i, b_j]) = 3p+3q-i-j+1,$$
$$f([a_i', b_i']) = p+q+i+j-1, \qquad f([u, v]) = 2p+2q.$$

We have

$$\bigcup_{i=1}^{p} \{f(a_i)\} = \{1, 2, \ldots, p\},$$

$$\bigcup_{i=1}^{q} \{f(b_i)\} = \{3p+3q, 3p+3q+1, \ldots, 3p+4q-1\},$$

$$\bigcup_{i=1}^{p} \{f(a_i')\} = \{3p+4q, 3p+4q+1, \ldots, 4p+4q-1\},$$

$$\bigcup_{i=1}^{q} \{f(b_i')\} = \{p+1, p+2, \ldots, p+q\},$$

$$\bigcup_{E(S_1)} \{f([a_i, b_j])\} = \{2p+2q+1, 2p+2q+2, \ldots, 3p+3q-1\},$$

$$\bigcup_{E(S_2)} \{f([a_i', b_j'])\} = \{p+q+1, p+q+2, \ldots, 2p+2q-1\}$$

and

$$f(a_i)+f(b_j)+f([a_i, b_j]) = i+(3p+3q+j-1)+(3p+3q-i-j+1) = 6p+6q,$$

$$f(a_i')+f(b_j')+f([a_i', b_j']) = (4p+4q-i)+(p+q+1-j)+(p+q+i+j-1)$$
$$= 6p+6q,$$

$$f(u)+f(v)+f([u, v]) = f(b_k)+f(b_k')+f([u, v]) = (3p+3q+k-1)$$
$$+(p+q+1-k)+(2p+2q) = 6p+6q,$$

thus f is an M-valuation of T, and since, as it is easy to see, $f=\bar{f}$, f is an SCM-valuation of T with the constant $C=6p+6q$.

COROLLARY. *For any symmetric tree with less than 14 vertices there exists an SCM-valuation.*

Proof. If T is a symmetric tree with less than 14 vertices then we must have $z(S_i)\le 1$, and the corollary follows.

THEOREM 7. *Let $G(2n, n)$ be the regular graph of degree one. An M-valuation of G exists if and only if n is odd.*

Proof. Since $G(2n, n)$ has $3n$ elements and no two of its n edges are adjacent, $\binom{3n+1}{2}/n$ must be necessarily an integer whence it follows that n must be odd, $n=2k+1$. Denote by a_i, b_i, $[a_i, b_i]$, $i=1, 2, \ldots, n$, the vertices and edges of $G(2n, n)$, respectively. Define f by

$$f(a_i) = i, \qquad f(b_i) = \begin{cases} 5k+2+i & \text{for } i = 1, 2, \ldots, k+1 \\ 3k+1+i & \text{for } i = k+2, k+3, \ldots, 2k+1. \end{cases}$$

$$f([a_i, b_i]) = \begin{cases} 4k+4+2i & \text{for } i = 1, 2, \ldots, k+1 \\ 6k+5-2i & \text{for } i = k+2, k+3, \ldots, 2k+1. \end{cases}$$

Evidently,

$$\bigcup_{i=1} \{f(b_i)\} = \{2n+1, 2n+2, \ldots, 3n\}, \bigcup_{i=1} \{f([a_i, b_i])\}$$
$$= \{n+1, n+2, \ldots, 2n\}$$

and

$$f(a_i)+f(b_i)+f([a_i, b_i]) = 9k+6,$$

thus f is an M-valuation of $G(2n, n)$ with the constant $C=9k+6$.

Observe that the described M-valuation f is even an SCM-valuation, thus we have

COROLLARY. *A regular graph of degree one has an M-valuation if and only if it has an SCM-valuation.*

Given a graph G, denote by $G+\{i\}$ a new graph obtained from G by adjoining to it i isolated vertices. Define also $G+\{0\}=G$.

DEFINITION 4. The magic deficiency (M-deficiency) $\mu(G)$ of a graph G is the least integer i such that the graph $G+\{i\}$ is M-valuable.

Denote by $\mathcal{M}_i$, $i=0, 1, 2, \ldots$ the class of all graphs G such that $\mu(G)=i$, thus $\mathcal{M}_0$ is the class of all M-valuable graphs.

For a graph G with n vertices we can derive a trivial bound

$$\mu(G) \leq F_{n+2}-2-\binom{n+1}{2}$$

where F_n is the nth Fibonacci number. Indeed, we can obtain an M-valuation f of the graph $K_n+\left\{F_{n+2}-2-\binom{n+1}{2}\right\}$ by putting $f(a_i)=F_i$ for all vertices $a_i \in K_n$ and $f([a_i, a_j])=F_{n+2}+1-F_i-F_j$.

All graphs with less than 7 vertices belonging to $\mathcal{M}_1$ are shown in Fig. 5. There is no graph with less than 7 vertices having M-deficiency greater than 1.

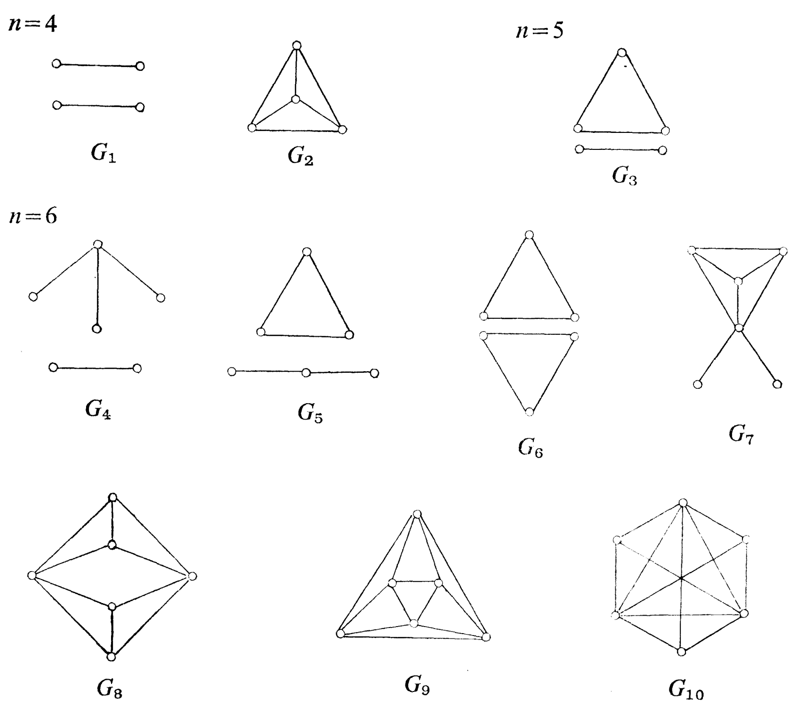

Figure 5.

Let us close with some open problems:

Problem 1. Does there exist an M-valuation for any tree?

Problem 2. What is the necessary and sufficient condition for a regular graph of degree two (three and four, respectively) in order to have an M-valuation?

Reference

1. A. Rosa, *On certain valuations of the vertices of a graph,* Theory of Graphs, Internat. Sympos., ICC Rome 1966, Paris, Dunod (1967), 349–355.

Komenský University,
Bratislava, Czechoslovakia
University of Calgary,
Calgary, Alberta

McMaster University,
Hamilton, Ontario