Bull. Aust. Math. Soc. **81** (2010), 195–207
doi:10.1017/S0004972709001038

PARTIAL ORDERS ON PARTIAL BAER–LEVI SEMIGROUPS

BOORAPA SINGHA, JINTANA SANWONG and R. P. SULLIVAN ✉

(Received 1 July 2009)

Abstract

Marques-Smith and Sullivan ['Partial orders on transformation semigroups', *Monatsh. Math.* **140** (2003), 103–118] studied various properties of two partial orders on $P(X)$, the semigroup (under composition) consisting of all partial transformations of an arbitrary set X. One partial order was the 'containment order': namely, if $\alpha, \beta \in P(X)$ then $\alpha \subseteq \beta$ means $x\alpha = x\beta$ for all $x \in \operatorname{dom}\alpha$, the domain of α. The other order was the so-called 'natural order' defined by Mitsch ['A natural partial order for semigroups', *Proc. Amer. Math. Soc.* **97**(3) (1986), 384–388] for any semigroup. In this paper, we consider these and other orders defined on the symmetric inverse semigroup $I(X)$ and the partial Baer–Levi semigroup $PS(q)$. We show that there are surprising differences between the orders on these semigroups, concerned with their compatibility with respect to composition and the existence of maximal and minimal elements.

2000 *Mathematics subject classification*: primary 20M20; secondary 04A05, 06A06.

Keywords and phrases: partial transformation semigroup, Baer–Levi semigroup, natural partial order, maximal (minimal) element, compatible.

1. Introduction

In [5] Mitsch defined a partial order on an arbitrary semigroup S by

$$a \leq b \quad \text{if and only if } a = xb = by \text{ and } a = ay \text{ for some } x, y \in S^1,$$

and now this is called the *natural partial order* on S. Later in [3] the authors studied various properties of this order on the semigroup $T(X)$ consisting of all total transformations of an arbitrary nonempty set X. Then in [4] Marques-Smith and Sullivan extended some of the previous work to the semigroup $P(X)$ consisting of all partial transformations of X.

In [4] the authors also considered another 'natural' partial order on $P(X)$: namely, regarding $\alpha, \beta \in P(X)$ as subsets of $X \times X$, it is clear that $\subseteq$ is a partial order on

The first author thanks the Office of the Higher Education Commission, Thailand, for its support by a 'Strategic Scholarships for Frontier Research Network' grant that enabled him to join a Thai PhD program and complete this research for his doctoral degree. The third author thanks the Faculty of Science, Chiangmai University, Thailand, for its financial assistance during his visit in May 2009.

$P(X)$ and that

$$\alpha \subseteq \beta \quad \text{if and only if } x\alpha = x\beta \text{ for all } x \in \operatorname{dom}\alpha,$$

where $\operatorname{dom}\alpha$ denotes the *domain* of $\alpha \in P(X)$. In particular, they characterized the meet and join of $\leq$ and $\subseteq$ in the poset consisting of all partial orders on $P(X)$ (surprisingly, the join always exists and equals $\subseteq \circ \leq$, the composition of the two relations). In this paper, we investigate similar ideas for a subsemigroup of $P(X)$ defined as follows.

For any set X, we let

$$I(X) = \{\alpha \in P(X) : \alpha \text{ is injective}\}$$

denote the *symmetric inverse semigroup* on X (see [1, Section 1.9]). In addition, if $\alpha \in P(X)$, we let $\operatorname{ran}\alpha$ denote the *range* of α and say that the cardinals

$$g(\alpha) = |X \backslash \operatorname{dom}\alpha|, \quad d(\alpha) = |X \backslash \operatorname{ran}\alpha|$$

are the *gap* and *defect* of α, respectively. Next, if $|X| = p \geq q \geq \aleph_0$, we write

$$PS(q) = \{\alpha \in I(X) : d(\alpha) = q\} \quad \text{and} \quad BL(q) = T(X) \cap PS(q),$$

where $BL(q)$ is the *Baer–Levi semigroup* of type (p, q) defined on X (see [1, Section 8.1]). It is well known that this semigroup is right simple, right cancellative and idempotent-free. On the other hand, in [6] the authors showed that $PS(q)$, the *partial Baer–Levi semigroup* on X, never has these properties. Nonetheless, they characterized Green's relations and ideals of $PS(q)$, and in this paper we study some properties of three partial orders on $PS(q)$.

In particular, unlike for $I(X)$, we show that $\leq$ is properly contained in $\subseteq$ (as relations) on $PS(q)$. In addition, $\leq$ is always right compatible on $PS(q)$ but is never left compatible. These and other results differ greatly from those obtained for $P(X)$ in [4].

2. Partial orders

Throughout this paper, $|X| = p \geq q \geq \aleph_0$. Also, $Y = A \dot{\cup} B$ means that Y is a *disjoint* union of A and B. As usual, $\emptyset$ denotes the empty (one-to-one) mapping which acts as a zero for $P(X)$. In particular, $d(\emptyset) = p$, so $\emptyset \in PS(q)$ precisely when $q = p$. For each nonempty $A \subseteq X$, we write id_A for the identity transformation on A: these mappings constitute all the idempotents in $I(X)$ and belong to $PS(q)$ precisely when $|X \backslash A| = q$.

It is well known that, for each nonzero $\alpha \in I(X)$, $\alpha\alpha^{-1} = \mathrm{id}_{\operatorname{dom}\alpha}$ and $\alpha^{-1}\alpha = \mathrm{id}_{\operatorname{ran}\alpha}$. Consequently, this is also true for $PS(q)$ and we use this fact without further comment.

https://doi.org/10.1017/S0004972709001038 Published online by Cambridge University Press

We modify the convention introduced in [1, Vol. 2, p. 241]: namely, if $\alpha \in I(X)$ is nonzero then we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unspecified) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $\operatorname{ran} \alpha = \{x_i\}$, $x_i\alpha^{-1} = \{a_i\}$ and $\operatorname{dom} \alpha = \{a_i : i \in I\}$. For simplicity, we often write $X\alpha$ instead of $\operatorname{ran} \alpha$, in which case $X\alpha^{-1} = \operatorname{ran} \alpha^{-1} = \operatorname{dom} \alpha$.

For convenience, we begin by quoting [4, Theorems 2 and 3] and [6, Theorem 8].

THEOREM 2.1. *If* $\alpha, \beta \in P(X)$ *then* $\alpha \leq \beta$ *if and only if* $X\alpha \subseteq X\beta$, $\operatorname{dom} \alpha \subseteq \operatorname{dom} \beta$, $\alpha\beta^{-1} \subseteq \alpha\alpha^{-1}$ *and* $\beta\beta^{-1} \cap (\operatorname{dom} \beta \times \operatorname{dom} \alpha) \subseteq \alpha\alpha^{-1}$.

THEOREM 2.2. *If* $\alpha, \beta \in P(X)$ *then the following are equivalent.*

(a) $\alpha \subseteq \beta$.
(b) $X\alpha \subseteq X\beta$ *and* $\alpha\beta^{-1} \subseteq \beta\beta^{-1}$.
(c) $X\alpha \subseteq X\beta$ *and* $\alpha\alpha^{-1} \subseteq \alpha\beta^{-1}$.

THEOREM 2.3. *If* $\alpha, \beta \in PS(q)$ *then* $\alpha = \lambda\beta$ *for some* $\lambda \in PS(q)$ *if and only if* $X\alpha \subseteq X\beta$ *and*

$$q \leq \max(g(\beta), |X\beta \backslash X\alpha|) \leq \max(g(\alpha), q). \tag{2.1}$$

Hence, $\alpha \mathcal{L} \beta$ *in* $PS(q)$ *if and only if*

$$(X\alpha = X\beta \text{ and } g(\alpha) = g(\beta) \geq q) \quad \text{or} \quad (\alpha = \beta \text{ and } g(\alpha) < q).$$

Clearly, Theorem 2.2 holds for $PS(q)$ but the same is not true for Theorem 2.1. In order to characterize $\leq$ on $PS(q)$, note that the relation $\mathbb{L}$ defined on $PS(q)$ by

$$(\alpha, \beta) \in \mathbb{L} \quad \text{if and only if } PS(q)^1\alpha \subseteq PS(q)^1\beta$$

is reflexive and transitive. However, in general, it is not anti-symmetric. For example, let $X = A \mathbin{\dot\cup} B \mathbin{\dot\cup} \{c, d, e\}$ where $|A| = p$ and $|B| = q$, and define $\alpha, \beta, \lambda, \mu \in PS(q)$ by

$$\alpha = \mathrm{id}_A \cup \begin{pmatrix} d \\ c \end{pmatrix}, \quad \beta = \mathrm{id}_A \cup \begin{pmatrix} e \\ c \end{pmatrix}, \quad \lambda = \mathrm{id}_A \cup \begin{pmatrix} d \\ e \end{pmatrix}, \quad \mu = \mathrm{id}_A \cup \begin{pmatrix} e \\ d \end{pmatrix}.$$

Then $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, so $(\alpha, \beta) \in \mathbb{L}$ and $(\beta, \alpha) \in \mathbb{L}$, but $\alpha \neq \beta$.

Nonetheless, if ρ is any partial order on $PS(q)$, then $\rho \cap \mathbb{L}$ is also a partial order on $PS(q)$. This idea leads to a simple description of $\leq$ on $PS(q)$.

THEOREM 2.4. *When restricted to* $PS(q)$, $\leq$ *equals* $\subseteq \cap \mathbb{L}$.

PROOF. Suppose that $\alpha, \beta \in PS(q)$ are distinct and $\alpha \leq \beta$ in $PS(q)$. Then $\alpha = \lambda\beta = \beta\mu$ and $\alpha = \alpha\mu$ for some $\lambda, \mu \in PS(q)$, and so $(\alpha, \beta) \in \mathbb{L}$. We also have $X\alpha \subseteq X\beta$ and $\operatorname{ran} \alpha \subseteq \operatorname{dom} \mu$. Hence

$$\alpha\alpha^{-1} = \alpha\mu(\beta\mu)^{-1} = \alpha(\mu\mu^{-1})\beta^{-1} = \alpha\beta^{-1},$$

and so $\alpha \subseteq \beta$ by Theorem 2.2. Therefore, $\leq$ is a subset of $\subseteq \cap \mathbb{L}$.

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Conversely, suppose that $(\alpha, \beta) \in \subseteq \cap \mathbb{L}$ and $\alpha \neq \beta$. Then $\alpha = \lambda\beta$ for some $\lambda \in PS(q)$. Moreover, since $\alpha \subseteq \beta$, we can write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \quad \beta = \begin{pmatrix} a_i & a_j \\ x_i & x_j \end{pmatrix}, \quad \mu = \begin{pmatrix} x_i \\ x_i \end{pmatrix},$$

where $d(\mu) = d(\alpha) = q$. Hence $\mu \in PS(q)$ and clearly $\alpha = \beta\mu$ and $\alpha = \alpha\mu$. Therefore, $\alpha \leq \beta$ in $PS(q)$. □

In [5, p. 384 and Lemma 1(x)], Mitsch observed that, if S is an inverse semigroup, then the natural partial order on S equals the order $\preceq$ defined on S by

$$a \preceq b \quad \text{if and only if } a = eb \text{ for some idempotent } e \in S.$$

Moreover, from [2, Proposition V.2.3], we know that $\preceq$ equals $\subseteq$ on $I(X)$, and thus $\leq = \subseteq$ on $I(X)$. On the other hand, from Theorem 2.4, we deduce that $\leq$ is a subset of $\subseteq$ on $PS(q)$ and we assert that this containment is always proper on $PS(q)$. For, suppose that $X = A \dot{\cup} B \dot{\cup} \{c\}$ where $|A| = p$ and $|B| = q$, and let $\alpha : A \cup B \to A$ be a bijection. Then $d(\alpha) = |B \cup \{c\}| = q$ and so $\alpha \in PS(q)$. Likewise, if $\beta \in T(X)$ equals α on $A \cup B$ and satisfies $c\beta = c$, then $\beta \in PS(q)$ and $\alpha \subseteq \beta$. But $g(\beta) = 0 < q$ and $|X\beta \setminus X\alpha| = 1 < q$, hence $(\alpha, \beta) \notin \mathbb{L}$ by Theorem 2.3 and so $\alpha \not\leq \beta$ by Theorem 2.4.

In [4], the authors defined partial orders Ω' and Ω on $P(X)$ as follows.

$(\alpha, \beta) \in \Omega'$ if and only if

$$X\alpha \subseteq X\beta, \text{ dom } \alpha \subseteq \text{dom } \beta \text{ and } \alpha\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1},$$

$(\alpha, \beta) \in \Omega$ if and only if $(\alpha, \beta) \in \Omega'$ and $\beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) \subseteq \alpha\alpha^{-1}$.

They showed that Ω' is an upper bound for $\leq$ and $\subseteq$, and that $\Omega = \leq \vee \subseteq = \subseteq \circ \leq$ on $P(X)$. Clearly $\Omega \subseteq \Omega'$ and these are also partial orders on $I(X)$, a semigroup in which $\leq = \subseteq$. Therefore, the next result is not surprising.

THEOREM 2.5. *$\Omega = \Omega'$ on $I(X)$.*

PROOF. Suppose that $\alpha, \beta \in I(X)$ and $(\alpha, \beta) \in \Omega'$. Then $\text{dom } \alpha \subseteq \text{dom } \beta$ and $\beta\beta^{-1} = \text{id}_{\text{dom } \beta}$, so

$$\beta\beta^{-1} \cap (\text{dom } \alpha \times \text{dom } \alpha) = \text{id}_{\text{dom } \alpha} = \alpha\alpha^{-1}.$$

Hence $(\alpha, \beta) \in \Omega$, and thus $\Omega' \subseteq \Omega$ as required. □

Given that $\leq = \subseteq$ and $\Omega = \Omega'$ on $I(X)$, it is natural to ask whether all four orders are equal on $I(X)$. In fact, $\Omega = \subseteq$ on $I(X)$ precisely when $|X| = 1$. For example, if $|X| > 1$, we can choose distinct $x, y \in X$ and define $\alpha, \beta \in I(X)$ by

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix}, \quad \beta = \begin{pmatrix} x & y \\ y & x \end{pmatrix}.$$

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Then $X\alpha \subseteq X\beta$, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) = \emptyset \subseteq \alpha\alpha^{-1}.$$

Hence $(\alpha, \beta) \in \Omega' = \Omega$ but $\alpha \not\subseteq \beta$, so $\subseteq$ is properly contained in Ω on $I(X)$ for $|X| > 1$. It is easy to see that $\Omega = \subseteq$ when $|X| = 1$, so we omit the details.

From Theorem 2.5 and the definition of Ω and Ω', we also know that $\Omega = \Omega'$ on $PS(q)$. As we show in Example 2.6 below, $\subseteq$ is always properly contained in Ω, hence on $PS(q)$ we always have:

$$\leq \, = \, \subseteq \cap \, \mathbb{L} \quad \subsetneq \quad \subseteq \quad \subsetneq \quad \Omega.$$

EXAMPLE 2.6. Suppose that $X = A \dot\cup B \dot\cup \{x\} \dot\cup \{y\}$ where $|A| = p$ and $|B| = q$, and let $\theta : A \cup B \to A$ be a bijection. Define $\alpha, \beta \in PS(q)$ by

$$\alpha = \begin{pmatrix} A \cup B & x \\ A & x \end{pmatrix}, \quad \beta = \begin{pmatrix} A \cup B & x & y \\ A & y & x \end{pmatrix}$$

where $\alpha|(A \cup B) = \theta = \beta|(A \cup B)$. Then $(\alpha, \beta) \in \Omega$ since $y \notin \operatorname{dom}\alpha$ and so

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) = \mathrm{id}_{A\cup B} \subseteq \mathrm{id}_{\operatorname{dom}\alpha} = \alpha\alpha^{-1}.$$

But $\alpha \not\subseteq \beta$ since $x\alpha \neq x\beta$, and so $\subseteq$ is always properly contained in Ω. Moreover, $\Omega \neq \subseteq \circ \leq$ on $PS(q)$: otherwise $\subseteq \subsetneq \Omega$ and Ω is contained in $\subseteq \circ \subseteq$ (since $\leq$ is contained in $\subseteq$), so Ω is contained in $\subseteq$, which is a contradiction.

It is well known that if $\alpha, \beta \in I(X)$, then $\alpha = \beta\mu$ for some $\mu \in I(X)$ if and only if $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ (see [2, Exercise V.2]). This helps to characterize the $\mathcal{R}$-relation on $I(X)$, and the same is true for $PS(q)$ (see [6, Theorem 7]). Clearly, the relation $\mathbb{D}$ defined on $I(X)$ by

$$(\alpha, \beta) \in \mathbb{D} \iff \alpha = \beta \quad \text{or} \quad \operatorname{dom}\alpha \subsetneq \operatorname{dom}\beta$$

is a partial order on $I(X)$. Moreover, $\Omega \subseteq \mathbb{D}$. For, suppose that $(\alpha, \beta) \in \Omega$ and $\operatorname{dom}\alpha = \operatorname{dom}\beta$. In this event, $x\alpha = y\beta$ for some $y \in \operatorname{dom}\alpha$, and so $(x, y) \in \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha)$. Hence $x = y$ and we deduce that $\alpha = \beta$. That is, if $(\alpha, \beta) \in \Omega$ then $\alpha = \beta$ or $\operatorname{dom}\alpha \subsetneq \operatorname{dom}\beta$, and thus $\Omega \subseteq \mathbb{D}$. In fact, the containment is proper. For example, if $1, 2, 3 \in X$ and

$$\delta = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \delta\varepsilon^{-1} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

then $(\delta, \varepsilon) \in \mathbb{D}$ but $(\delta, \varepsilon) \notin \Omega$. And it is easy to see that also $\Omega \subsetneq \mathbb{D}$ on $PS(q)$.

To prove a result for Ω which is similar to Theorem 2.4 for $\leq$, we define another relation on $PS(q)$ by

$$(\alpha, \beta) \in \Delta \iff X\alpha \subseteq X\beta \quad \text{and} \quad \alpha\beta^{-1} \subseteq \beta\beta^{-1} \cup \operatorname{dom}\alpha \times (\operatorname{dom}\beta \backslash \operatorname{dom}\alpha).$$

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Note that if $(\alpha, \beta) \in \Delta$ then, post-multiplying the above containment by β, we obtain

$$\alpha \subseteq \beta \cup [\operatorname{dom} \alpha \times (\operatorname{dom} \beta \backslash \operatorname{dom} \alpha)] \circ \beta$$

which highlights the difference between $\subseteq$ and Δ. In fact, we assert that $\Omega \subseteq \Delta$.

To see this, suppose that $(\alpha, \beta) \in \Omega$ and let $(x, y) \in \alpha\beta^{-1}$. If $y \in \operatorname{dom} \alpha$, then $(x, y) \in \alpha\beta^{-1} \cap (\operatorname{dom} \alpha \times \operatorname{dom} \alpha)$, so $x = y \in \operatorname{dom} \beta$ and hence $(x, y) \in \beta\beta^{-1}$. On the other hand, if $y \notin \operatorname{dom} \alpha$, then $x \in \operatorname{dom} \alpha$ and $y \in \operatorname{dom} \beta \backslash \operatorname{dom} \alpha$, so $(x, y) \in \operatorname{dom} \alpha \times (\operatorname{dom} \beta \backslash \operatorname{dom} \alpha)$. That is, $(\alpha, \beta) \in \Delta$ and this proves the assertion. Although Δ is not a partial order (see Example 2.9 below), we have the following result.

THEOREM 2.7. *When restricted to $PS(q)$, Ω equals $\Delta \cap \mathbb{D}$.*

PROOF. We have shown that $\Omega \subseteq \Delta \cap \mathbb{D}$. Therefore, suppose that $(\alpha, \beta) \in \Delta \cap \mathbb{D}$ and $\alpha \neq \beta$. Then $X\alpha \subseteq X\beta$ and $\operatorname{dom} \alpha \subsetneq \operatorname{dom} \beta$. Also $\alpha\beta^{-1} \subseteq \beta\beta^{-1} \cup \operatorname{dom} \alpha \times (\operatorname{dom} \beta \backslash \operatorname{dom} \alpha)$ and, by intersecting this containment with $\operatorname{dom} \alpha \times \operatorname{dom} \alpha$, we obtain

$$\alpha\beta^{-1} \cap (\operatorname{dom} \alpha \times \operatorname{dom} \alpha) \subseteq \beta\beta^{-1} \cap (\operatorname{dom} \alpha \times \operatorname{dom} \alpha) = \alpha\alpha^{-1},$$

and so $(\alpha, \beta) \in \Omega$. □

EXAMPLE 2.8. Let $X = A \mathbin{\dot\cup} B \mathbin{\dot\cup} \{c, d, e\}$ where $|A| = p$ and $|B| = q$, and define $\alpha, \beta, \gamma \in PS(q)$ by

$$\alpha = \mathrm{id}_A \cup \begin{pmatrix} c \\ e \end{pmatrix}, \quad \beta = \mathrm{id}_A \cup \begin{pmatrix} d \\ e \end{pmatrix}, \quad \gamma = \mathrm{id}_A \cup \begin{pmatrix} c & d \\ c & d \end{pmatrix}. \tag{2.2}$$

Then $\alpha \neq \beta$ and $\operatorname{dom} \alpha \not\subseteq \operatorname{dom} \beta$, so $(\alpha, \beta) \notin \mathbb{D}$. But $X\alpha = X\beta$ and $\alpha\beta^{-1} = \mathrm{id}_A \cup \{(c, d)\}$, $\beta\beta^{-1} = \mathrm{id}_{A \cup \{d\}}$ and $\operatorname{dom} \alpha \times (\operatorname{dom} \beta \backslash \operatorname{dom} \alpha) = A \times \{d\} \cup \{(c, d)\}$. Therefore $(\alpha, \beta) \in \Delta$. In addition, $(\alpha, \beta) \notin \Omega$ simply because $\operatorname{dom} \alpha \not\subseteq \operatorname{dom} \beta$, hence Ω is properly contained in Δ. On the other hand, $\operatorname{dom} \alpha \subsetneq \operatorname{dom} \gamma$, so $(\alpha, \gamma) \in \mathbb{D}$, but $(\alpha, \gamma) \notin \Delta$ since $X\alpha \not\subseteq X\gamma$. That is, $\mathbb{D}$ and Δ are noncomparable relations on $PS(q)$.

EXAMPLE 2.9. Clearly Δ is reflexive. However, if α and β are defined as in (2.2), then $(\alpha, \beta) \in \Delta$ and $(\beta, \alpha) \in \Delta$ but $\alpha \neq \beta$, so Δ is not anti-symmetric. Also, suppose that $X = A \mathbin{\dot\cup} B \mathbin{\dot\cup} \{c, d, e, f, g\}$ where $|A| = p$ and $|B| = q$, and define $\alpha, \beta, \mu \in PS(q)$ by

$$\alpha = \mathrm{id}_A \cup \begin{pmatrix} c & d \\ e & d \end{pmatrix}, \quad \beta = \mathrm{id}_A \cup \begin{pmatrix} e & f \\ d & e \end{pmatrix}, \quad \mu = \mathrm{id}_A \cup \begin{pmatrix} d & g \\ e & d \end{pmatrix}.$$

Then $X\alpha = X\beta = X\mu$ and

$$\alpha\beta^{-1} = \mathrm{id}_A \cup \begin{pmatrix} c & d \\ f & e \end{pmatrix} \subseteq \beta\beta^{-1} \cup \operatorname{dom} \alpha \times \{e, f\},$$

$$\beta\mu^{-1} = \mathrm{id}_A \cup \begin{pmatrix} e & f \\ g & d \end{pmatrix} \subseteq \mu\mu^{-1} \cup \operatorname{dom} \beta \times \{d, g\}.$$

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So, $(\alpha, \beta) \in \Delta$ and $(\beta, \mu) \in \Delta$. But

$$\alpha\mu^{-1} = \mathrm{id}_A \cup \begin{pmatrix} c & d \\ d & g \end{pmatrix} \not\subseteq \mu\mu^{-1} \cup \operatorname{dom}\alpha \times \{g\},$$

hence $(\alpha, \mu) \notin \Delta$ and so Δ is not transitive.

3. Compatible partial orders

As in [4, Section 3], if ρ is a partial order on a transformation semigroup S, we say that $\gamma \in S$ is *left compatible* with ρ if $(\gamma\alpha, \gamma\beta) \in \rho$ for all $(\alpha, \beta) \in \rho$; *right compatibility* with ρ is defined dually. For comparison with our results below, we first quote [4, Theorems 9 and 11].

THEOREM 3.1. *Suppose that $\gamma \in P(X)$ is nonzero and $|X| \geq 3$.*

(a) *γ is left compatible with $\leq$ on $P(X)$ if and only if γ is surjective.*
(b) *γ is right compatible with $\leq$ on $P(X)$ if and only if $\gamma \in T(X)$ and γ is injective.*

THEOREM 3.2. *Suppose that $\gamma \in P(X)$ is nonzero and $|X| \geq 3$.*

(1) *γ is left compatible with Ω on $P(X)$ if and only if γ is surjective.*
(2) *γ is right compatible with Ω on $P(X)$ if and only if $\gamma \in T(X)$ and either γ is injective or γ is constant.*

By contrast with Theorem 3.1 above, the next result is surprising.

THEOREM 3.3. *Suppose that $\gamma \in PS(q)$.*

(a) *γ is left compatible with $\leq$ on $PS(q)$ if and only if $q \leq g(\gamma)$.*
(b) *$\leq$ is right compatible on $PS(q)$.*

PROOF. To prove (a), suppose that γ is left compatible with $\leq$. If $\gamma = \emptyset$ (in the case where $p = q$), then $g(\gamma) = p = q$. If $\gamma \neq \emptyset$, we choose $x \in \operatorname{ran}\gamma$ and let $\alpha = \mathrm{id}_{\operatorname{ran}\gamma \setminus \{x\}}$ and $\beta = \mathrm{id}_{\operatorname{ran}\gamma}$. Then $\alpha, \beta \in PS(q)$ and $\alpha \subseteq \beta$. Also $g(\beta) = d(\gamma) = q$ and so $g(\alpha) = g(\beta) = q$ (since $q \geq \aleph_0$). Hence

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) = q = \max(g(\alpha), q).$$

Therefore, $(\alpha, \beta) \in \mathbb{L}$ by Theorem 2.3 and hence $\alpha \leq \beta$ by Theorem 2.4. Since γ is left compatible, we have $\gamma\alpha \leq \gamma\beta$ where $\gamma\alpha \neq \gamma\beta = \gamma$, and then Theorem 2.3 implies that

$$q \leq \max(g(\gamma\beta), |X\gamma\beta \setminus X\gamma\alpha|).$$

But, since $|X\gamma\beta \setminus X\gamma\alpha| = 1 < q$, this implies that $q \leq g(\gamma\beta) = g(\gamma)$.

Conversely, suppose that $q \leq g(\gamma)$. If $\alpha, \beta \in PS(q)$ and $\alpha \leq \beta$, then $\alpha \subseteq \beta$ and $(\alpha, \beta) \in \mathbb{L}$ by Theorem 2.4. Since $\subseteq$ is left compatible on $P(X)$, then $\gamma\alpha \subseteq \gamma\beta$. Also, $\operatorname{dom}\gamma\beta \subseteq \operatorname{dom}\gamma$ implies that $q \leq g(\gamma) \leq g(\gamma\beta)$; and, since $\alpha = \beta\mu$ for some

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$\mu \in PS(q)^1$ (by the definition of $\leq$), we know that $\gamma\alpha = (\gamma\beta)\mu$ and hence $g(\gamma\beta) \leq g(\gamma\alpha)$. Moreover, since $\gamma\alpha \in PS(q)$,

$$|X\gamma\beta \backslash X\gamma\alpha| = |X\gamma\beta \cap (X\backslash X\gamma\alpha)| \leq q$$

and so

$$q \leq g(\gamma\beta) = \max(g(\gamma\beta), |X\gamma\beta \backslash X\gamma\alpha|) \leq g(\gamma\alpha) = \max(g(\gamma\alpha), q).$$

That is, $(\gamma\alpha, \gamma\beta) \in \mathbb{L}$ as required. Finally, note that $\subseteq$ is right compatible, and clearly the same is true for $\mathbb{L}$, so (b) follows from Theorem 2.4. □

The next two results for the compatibility of Ω differ greatly from Theorem 3.2 above. Here, for simplicity, we write x_y for the $\alpha \in I(X)$ with domain $\{x\}$ and range $\{y\}$.

THEOREM 3.4. *Suppose that $p = q$ and let $\gamma \in PS(q)$. Then:*

(a) *$\emptyset$ is the only element of $PS(q)$ which is left compatible with Ω;*
(b) *γ is right compatible with Ω if and only if $\gamma = \emptyset$ or* $\operatorname{dom}\gamma = X$.

PROOF. Clearly $\emptyset \in PS(q)$ and it is left compatible with Ω. Let γ be a nonzero element in $PS(q)$. If we choose $x \in \operatorname{ran}\gamma$, $y \in X\backslash\operatorname{ran}\gamma$ and define

$$\alpha = \begin{pmatrix} x \\ x \end{pmatrix}, \quad \beta = \begin{pmatrix} x & y \\ y & x \end{pmatrix},$$

then $\alpha, \beta \in PS(q)$ and it is easy to check that $(\alpha, \beta) \in \Omega$. However, since $X\gamma\alpha = \{x\} \nsubseteq \{y\} = X\gamma\beta$, then $(\gamma\alpha, \gamma\beta) \notin \Omega$ (by definition) and so γ is not left compatible with Ω.

Suppose that $\gamma \in PS(q)$ is nonempty and right compatible with Ω. If $a \in \operatorname{dom}\gamma$, $x \in X\backslash\operatorname{dom}\gamma$ and $Y = \{a, x\}$ then $x_a, \operatorname{id}_Y \in PS(q)$ and $(x_a, \operatorname{id}_Y) \in \Omega$ (note that $x_a.\operatorname{id}_Y^{-1} \cap \{(x, x)\} = \emptyset$). Hence $(x_a.\gamma, \operatorname{id}_Y.\gamma) \in \Omega$ and so $\operatorname{dom}(x_a.\gamma) = \{x\} \subseteq \operatorname{dom}(\operatorname{id}_Y.\gamma) = \{a\}$, a contradiction. Thus, we have shown that $\operatorname{dom}\gamma = X$. Therefore, to prove (b), it remains to show that, if $\operatorname{dom}\gamma = X$, then γ is right compatible with Ω. To do this, let $\alpha, \beta \in PS(q)$ and $(\alpha, \beta) \in \Omega$. Then, since $\Omega = \Omega'$, we have $X\alpha \subseteq X\beta$, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and

$$\alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha) \subseteq \alpha\alpha^{-1} = \operatorname{id}_{\operatorname{dom}\alpha}.$$

Clearly $X\alpha\gamma \subseteq X\beta\gamma$ and, since $\operatorname{dom}\gamma = X$, $\operatorname{dom}\alpha\gamma = \operatorname{dom}\alpha \subseteq \operatorname{dom}\beta = \operatorname{dom}\beta\gamma$. Also $\gamma\gamma^{-1} = \operatorname{id}_X$ (but note that $\operatorname{id}_X \notin PS(q)$), and hence

$$\alpha\gamma(\beta\gamma)^{-1} \cap (\operatorname{dom}\alpha\gamma \times \operatorname{dom}\alpha\gamma) = \alpha\beta^{-1} \cap (\operatorname{dom}\alpha \times \operatorname{dom}\alpha),$$

from which it follows that $(\alpha\gamma, \beta\gamma) \in \Omega$. □

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THEOREM 3.5. *Suppose that $p > q$ and let $\gamma \in PS(q)$. Then:*

(a) *no element of $PS(q)$ is left compatible with Ω;*

(b) *γ is right compatible with Ω if and only if* $\operatorname{dom}\gamma = X$.

PROOF. To prove (a), let $\theta \in PS(q)$, choose $x \in \operatorname{ran}\theta$, $y \in X \backslash \operatorname{ran}\theta$ and define

$$\alpha = \operatorname{id}_{\operatorname{ran}\theta}, \quad \beta = \begin{pmatrix} \operatorname{ran}\theta\backslash\{x\} & x & y \\ \operatorname{ran}\theta\backslash\{x\} & y & x \end{pmatrix},$$

where $z\beta = z$ for all $z \in \operatorname{ran}\theta\backslash\{x\}$. Then $\alpha, \beta \in PS(q)$ and $(\alpha, \beta) \in \Omega$. Since $x \in X\theta\alpha\backslash X\theta\beta$, $(\theta\alpha, \theta\beta) \notin \Omega$ (by definition). That is, θ is not left compatible with Ω. The proof of (b) is the same as that for Theorem 3.4(b), except that now $\emptyset \notin PS(q)$. □

For completeness, we note the following result for Ω on $I(X)$.

THEOREM 3.6. *If $\gamma \in I(X)$ is nonzero then:*

(a) *γ is left compatible with Ω on $I(X)$ if and only if* $\operatorname{ran}\gamma = X$;

(b) *γ is right compatible with Ω on $I(X)$ if and only if* $\operatorname{dom}\gamma = X$.

PROOF. As shown in [4, pp. 113–114], if γ is surjective then it is left compatible with Ω on $P(X)$, and so the same is true for $I(X)$. For the converse of (a), suppose that $\operatorname{ran}\gamma \neq X$. Then, as in the proof of Theorem 3.5(a), there exists $(\alpha, \beta) \in \Omega$ on $I(X)$ but $(\gamma\alpha, \gamma\beta) \notin \Omega$. The proof of (b) follows that of Theorem 3.5(b). □

4. Minimal and maximal elements

As usual, if $\preceq$ is an order on a set S, then $a \in S$ is *maximal* with respect to $\preceq$ if $a \preceq x$ and $x \in S$ imply that $x = a$; and $a \in S$ is a *maximum* if $x \preceq a$ for all $x \in S$. The notions of *minimal* and *minimum* are defined dually. In this section, we consider the existence of minimal (maximal) elements in $PS(q)$ with respect to each of the orders $\leq$, $\subseteq$ and Ω.

First, recall that, if $\preceq$ is any partial order on a set T, and if $x \in S \subseteq T$ is minimal (maximal) in T, then x is minimal (maximal) in S. Similarly, suppose that $<_1$ and $<_2$ are partial orders on a set S such that $<_2$ contains $<_1$. Clearly, if $x \in S$ is minimal (maximal) with respect to $<_2$, then x is minimal (maximal) with respect to $<_1$. On the other hand, under the same supposition, if x is a minimum (maximum) with respect to $<_1$, then x is a minimum (maximum) with respect to $<_2$.

THEOREM 4.1. *$PS(q)$ has no maximum element with respect to $\leq$, $\subseteq$ or Ω.*

PROOF. Write $X = A \,\dot\cup\, B \,\dot\cup\, C$ where $|A| = p$ and $|B| = q = |C|$. Clearly, if $\alpha = \operatorname{id}_{A\cup B}$ and $\beta = \operatorname{id}_{A\cup C}$, then $\alpha, \beta \in PS(q)$. If $\gamma \in PS(q)$ is a maximum with respect to Ω, then $(\alpha, \gamma) \in \Omega$ and $(\beta, \gamma) \in \Omega$. Consequently $X\alpha \subseteq X\gamma$ and $X\beta \subseteq X\gamma$, hence $X\alpha \cup X\beta \subseteq X\gamma$ and so $\operatorname{ran}\gamma = X$, which contradicts $d(\gamma) = q$. Therefore $PS(q)$ has no maximum element with respect to Ω. Next recall that $\leq$ is properly contained in $\subseteq$ which is properly contained in Ω on $PS(q)$. So, if α is a maximum under $\subseteq$, then it is also a maximum under Ω, a contradiction. Likewise, there is no maximum under $\leq$. □

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THEOREM 4.2. *The following are equivalent for* $\alpha \in PS(q)$.

(a) α *is maximal with respect to* Ω.
(b) α *is maximal with respect to* $\subseteq$.
(c) $\operatorname{dom}\alpha = X$.

PROOF. (a) implies (b) since $\subseteq$ is contained in Ω. To show that (b) implies (c), suppose that (b) holds and assume that $\operatorname{dom}\alpha \subsetneq X$. Choose $x \in X\backslash\operatorname{dom}\alpha$ and $y \in X\backslash\operatorname{ran}\alpha$ (recall that $d(\alpha) = q$) and let β be the mapping such that $\operatorname{dom}\beta = \operatorname{dom}\alpha \cup \{x\}$, $\beta|\operatorname{dom}\alpha = \alpha$ and $x\beta = y$. Then $\beta \in PS(q)$ and $\alpha \subseteq \beta$ with $\alpha \neq \beta$, contradicting our supposition.

Finally, to show that (c) implies (a), suppose that $\operatorname{dom}\alpha = X$ and let $\beta \in PS(q)$ satisfy $(\alpha, \beta) \in \Omega$. Then, by Theorem 2.5, $\operatorname{dom}\alpha \subseteq \operatorname{dom}\beta$ and $X\alpha \subseteq X\beta$. So $\operatorname{dom}\beta = X$. Moreover, if $x, x' \in X$ and $x\alpha = x'\beta$, then $(x, x') \in \alpha\beta^{-1} \subseteq \operatorname{id}_X$ and it follows that $x = x'$. That is, $\alpha = \beta$ and we have shown that (a) holds. □

The corresponding result for $\leq$ is substantially different.

THEOREM 4.3. *Let* $\alpha \in PS(q)$. *Then* α *is maximal with respect to* $\leq$ *if and only if* $g(\alpha) < q$.

PROOF. Suppose that $g(\alpha) \geq q$. By defining $\beta \in PS(q)$ as in the first paragraph of the proof of Theorem 4.2, we obtain $\alpha \subseteq \beta$, $X\beta = X\alpha \,\dot{\cup}\, \{y\}$ and $g(\alpha) = g(\beta)$. Hence (1) in Theorem 2.3 is satisfied and thus $\alpha \leq \beta$ but $\alpha \neq \beta$, so α is not maximal. Conversely, suppose that $g(\alpha) < q$ and assume that $\alpha < \beta$ for some $\beta \in PS(q)$. Thus, by Theorem 2.4, $\alpha \subsetneq \beta$ and

$$q \leq \max(g(\beta), |X\beta\backslash X\alpha|) \leq \max(g(\alpha), q) = q.$$

Therefore, $g(\beta) \leq g(\alpha) < q$ and so $|X\beta\backslash X\alpha| = q$. Consequently, since $X\alpha \subseteq X\beta$, then

$$q = |(X\beta\backslash X\alpha)\beta^{-1}| = |\operatorname{dom}\beta\backslash\operatorname{dom}\alpha| \leq g(\alpha) < q,$$

a contradiction. □

REMARK 4.4. By [6, Theorem 4(b)], the above result means that the elements of $PS(q)$ which are maximal under $\leq$ are precisely the nonregular elements of $PS(q)$. In fact, they form a subsemigroup of $PS(q)$ since, for each $\alpha, \beta \in PS(q)$, $\operatorname{dom}\alpha\beta = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1}$ and so

$$g(\alpha\beta) = |X\backslash X\alpha^{-1}| + |(X\backslash\operatorname{dom}\beta)\alpha^{-1}|.$$

As in many algebraic settings, it is interesting to know when $\alpha \in PS(q)$ lies below some maximal element of $PS(q)$.

THEOREM 4.5. *The following are equivalent for* $\alpha \in PS(q)$.

(a) $g(\alpha) \leq q$.
(b) $\alpha \leq \beta$ *for some* $\beta \in PS(q)$ *maximal with respect to* $\leq$.

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(c) *$\alpha \subseteq \beta$ for some $\beta \in PS(q)$ maximal with respect to $\subseteq$.*
(d) *$(\alpha, \beta) \in \Omega$ for some $\beta \in PS(q)$ maximal with respect to Ω.*

PROOF. Suppose that (a) holds. If $g(\alpha) < q$ then $\alpha \leq \alpha$ and α is maximal under $\leq$ by Theorem 4.3. Therefore, suppose that $g(\alpha) = q$. Since $d(\alpha) = q$, we can write $X \backslash \operatorname{ran} \alpha = A \,\dot\cup\, B$ where $|A| = |B| = q$. Let $\theta : X \backslash \operatorname{dom} \alpha \to A$ be any bijection and define $\beta \in PS(q)$ by letting $\operatorname{dom} \beta = X$, $\beta | \operatorname{dom} \alpha = \alpha$ and $\beta | (X \backslash \operatorname{dom} \alpha) = \theta$. Then $g(\beta) = 0$ and $X\beta = X\alpha \,\dot\cup\, A$, so

$$q = |A| = \max(g(\beta), |X\beta \backslash X\alpha|) = \max(g(\alpha), q).$$

That is, $(\alpha, \beta) \in \mathbb{L}$ and clearly $\alpha \subseteq \beta$. Hence $\alpha < \beta$ where β is maximal with respect to $\leq$.

Now suppose that (b) holds: namely, suppose that $\alpha \leq \beta$ where $g(\beta) = r < q$. Then $\alpha \subseteq \beta$ and $d(\beta) = q$, so we can write $X \backslash \operatorname{ran} \beta = A \,\dot\cup\, B$ where $|A| = r$ and $|B| = q$. Let $\theta : X \backslash \operatorname{dom} \beta \to A$ be any bijection and define $\beta^+ \in PS(q)$ by letting $\operatorname{dom} \beta^+ = X$, $\beta^+ | \operatorname{dom} \beta = \beta$ and $\beta^+ | (X \backslash \operatorname{dom} \beta) = \theta$. Then $\alpha \subseteq \beta \subseteq \beta^+$ where β^+ is maximal with respect to $\subseteq$: that is, (c) holds by Theorem 4.2(b).

Next, suppose that (c) holds. Since $\subseteq$ is contained in Ω, and any element which is maximal under $\subseteq$ is also maximal under Ω, we deduce that (d) also holds.

Finally, suppose that (d) holds: that is, suppose that $(\alpha, \beta) \in \Omega$ where $\operatorname{dom} \beta = X$, and write

$$\begin{aligned} A &= \{x \in \operatorname{dom} \alpha : x\alpha\beta^{-1} = x\}, \\ B &= \{x \in \operatorname{dom} \alpha : x\alpha\beta^{-1} \notin \operatorname{dom} \alpha\}. \end{aligned}$$

By the definition of Ω, if $x \in \operatorname{dom} \alpha$ and $x\alpha = y\beta$ (possible since $X\alpha \subseteq X\beta$) then either $y \in \operatorname{dom} \alpha$ (so $y = x$ and $x \in A$) or $y \notin \operatorname{dom} \alpha$ (so $x \in B$). It follows that $\operatorname{dom} \alpha = A \,\dot\cup\, B$, $A\alpha = A\beta$ and $B\alpha = C\beta$ for some $C \subseteq \operatorname{dom} \beta \backslash \operatorname{dom} \alpha$. Note that $X\alpha = (A \cup C)\beta$ and $(A \cup C) \cap B = \emptyset$. Therefore $X\alpha \cap B\beta = \emptyset$ (since β is injective) and so, since $\operatorname{dom} \beta = X$,

$$|B| = |B\alpha| = |B\beta| \leq |X \backslash X\alpha| = q.$$

Next let $D = X \backslash (A \cup B \cup C)$ and observe that $D\beta \cap X\alpha = D\beta \cap (A \cup C)\beta = \emptyset$. Therefore

$$|D\beta| \leq |X \backslash X\alpha| = q.$$

Now $X\beta = A\beta \,\dot\cup\, B\beta \,\dot\cup\, C\beta \,\dot\cup\, D\beta$ and thus

$$(X \backslash \operatorname{dom} \alpha)\beta = (X \backslash (A \cup B))\beta = X\beta \backslash (A \cup B)\beta = C\beta \cup D\beta.$$

Consequently $g(\alpha) = |(X \backslash \operatorname{dom} \alpha)| \leq |B\alpha| + q = q$, and so (a) holds. □

Observe that if $p = q$, then $g(\alpha) \leq q$ for all $\alpha \in PS(q)$. Hence, in this case, every $\alpha \in PS(q)$ is contained in some maximal element.

THEOREM 4.6. *If $p > q$, then $PS(q)$ has no minimal element with respect to $\leq$, $\subseteq$ or Ω, and hence also no minimum element.*

PROOF. Suppose that $p > q$ and let $\alpha \in PS(q)$. Then $|\mathrm{dom}\,\alpha| = p$ and we can write $\mathrm{dom}\,\alpha = A \dot\cup B$ where $|A| = p$ and $|B| = q$. If $\gamma = \alpha|A$, then $d(\gamma) = |B\alpha| + d(\alpha) = q$, thus $\gamma \in PS(q)$ and clearly $\gamma \subsetneq \alpha$. Also, if $X = A \dot\cup B \dot\cup C$ and $\lambda = \mathrm{id}_{A\cup C}$, then $d(\lambda) = |B| = q$, so $\lambda \in PS(q)$ and $\gamma = \lambda\alpha$ (since $C = X \setminus \mathrm{dom}\,\alpha$). Consequently, $(\gamma, \alpha) \in \mathbb{L}$ and so $\gamma < \alpha$ by Theorem 2.4. Therefore, there is no minimal element under $\leq$, and hence none for $\subseteq$ and Ω (due to their containing $\leq$). Hence, there is also no minimum element under each of these orders. □

When $p = q$, it is easy to see that $\emptyset$ is the minimum under $\leq$, $\subseteq$ and Ω. In this case, we say that $\alpha \in PS(q)$ is *nonzero minimal* with respect to an order $\preceq$ on $PS(q)$ if α is minimal among the nonzero elements of $PS(q)$ under $\preceq$.

THEOREM 4.7. *If $p = q$, then the following are equivalent for $\alpha \in PS(q)$.*

(a) *α is nonzero minimal with respect to Ω.*
(b) *α is nonzero minimal with respect to $\subseteq$.*
(c) *α is nonzero minimal with respect to $\leq$.*
(d) $|\mathrm{dom}\,\alpha| = 1$.

PROOF. Since Ω contains $\subseteq$, and $\subseteq$ contains $\leq$, then (a) implies (b), and (b) implies (c). To show that (c) implies (d), suppose that (c) holds and assume that $|\mathrm{dom}\,\alpha| > 1$. Now, as in the proof of Theorem 4.6, if $|\mathrm{dom}\,\alpha| = p$, then there exists $\gamma \in PS(q)$ such that $\emptyset < \gamma < \alpha$, contradicting (c). On the other hand, if $|\mathrm{dom}\,\alpha| < p$ then $g(\alpha) = p$. In this case, choose $a \in \mathrm{dom}\,\alpha$ and write $C = \mathrm{dom}\,\alpha \setminus \{a\}$ (which is nonempty by assumption). If $\beta = \alpha|C$ and $\lambda = \mathrm{id}_C$ then $\beta, \lambda \in PS(q)$ and $\beta = \lambda\alpha$. Therefore, $(\beta, \alpha) \in \mathbb{L}$ and clearly $\beta \subsetneq \alpha$. That is, $\emptyset < \beta < \alpha$, contradicting (c) again.

Finally, to show that (d) implies (a), suppose that $|\mathrm{dom}\,\alpha| = 1$, say $\mathrm{dom}\,\alpha = \{x\}$. Since $\Omega = \Omega'$ and by the definition of Ω', if there exists $\beta \neq \emptyset$ such that $(\beta, \alpha) \in \Omega$, then $\mathrm{dom}\,\beta = \{x\}$ and $\mathrm{ran}\,\beta = \{x\alpha\}$. Hence $\alpha = \beta$ and so α is nonzero minimal under Ω. □

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https://doi.org/10.1017/S0004972709001038 Published online by Cambridge University Press

BOORAPA SINGHA, Department of Mathematics, Chiang Mai University,
Chiangmai 50200, Thailand
e-mail: boorapas@yahoo.com

JINTANA SANWONG, Department of Mathematics, Chiang Mai University,
Chiangmai 50200, Thailand
e-mail: scmti004@chiangmai.ac.th

R. P. SULLIVAN, School of Mathematics and Statistics,
University of Western Australia, Nedlands 6009, Australia
e-mail: bob@maths.uwa.edu.au

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