Appendix D
Polarization and spin-1/2 fermions

It is essential to know the technique for dealing with the spin and polarization of any spin-1/2 fermion, electron or nucleon, entering into a scattering process. Consider first the case of a massless fermion, for example a relativistic electron. The positive energy, stationary state, momentum eigenstate of the Dirac equation in this case satisfies[1]

$$\begin{aligned} \boldsymbol{\alpha}\cdot\mathbf{p}\,\psi &= E_p\psi \\ E_p &= |\mathbf{p}| \end{aligned} \tag{D.1}$$

Introduce the Dirac matrix γ_5 with the properties

$$\begin{aligned} \gamma_5 &\equiv \gamma_1\gamma_2\gamma_3\gamma_4 \\ \gamma_5\gamma_\mu + \gamma_\mu\gamma_5 &= 0 \\ \gamma_5^2 &= 1 \end{aligned} \tag{D.2}$$

In the standard representation, γ_5 and $\gamma_5\boldsymbol{\alpha}$ take the form

$$\gamma_5 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad ; \; \gamma_5\boldsymbol{\alpha} = \boldsymbol{\alpha}\gamma_5 = \begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix} \equiv -\boldsymbol{\sigma} \tag{D.3}$$

Introduce the projection operators defined by

$$P_\downarrow = \frac{1}{2}(1+\gamma_5) \qquad\qquad P_\uparrow = \frac{1}{2}(1-\gamma_5) \tag{D.4}$$

These satisfy

$$\begin{aligned} P_\downarrow^2 &= P_\downarrow \\ P_\uparrow^2 &= P_\uparrow \\ P_\downarrow P_\uparrow &= 0 \end{aligned} \tag{D.5}$$

[1] Recall $\hbar = c = 1$ here. The reader can extend the arguments to the negative energy solutions.

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Define

$$\psi_\downarrow = P_\downarrow \psi \qquad\qquad \psi_\uparrow = P_\uparrow \psi \tag{D.6}$$

Now multiply Eq. (D.1) on the left by, for example, $P_\downarrow$. Since γ_5 and $\boldsymbol{\alpha}$ commute, this gives

$$\boldsymbol{\alpha} \cdot \mathbf{p}\, \psi_\downarrow = |\mathbf{p}|\, \psi_\downarrow \tag{D.7}$$

Multiply this equation on the left by γ_5 and make use of the above relations

$$\begin{aligned} -\boldsymbol{\sigma} \cdot \mathbf{p}\, \psi_\downarrow &= |\mathbf{p}|\, \psi_\downarrow \\ \boldsymbol{\sigma} \cdot \left(\frac{\mathbf{p}}{|\mathbf{p}|}\right) \psi_\downarrow &= -\psi_\downarrow \end{aligned} \tag{D.8}$$

One concludes that $P_\downarrow$ projects out of the Dirac spinors that part with *negative helicity*. $P_\uparrow$ does just the opposite.

One can now compute the cross section for massless fermions of any helicity by inserting either $P_\uparrow$ or $P_\downarrow$ before the appropriate Dirac spinors and then summing over *all* helicities. This converts the required expressions to traces, and only the appropriate helicity will contribute to the answer.

Suppose the fermion has a non-zero rest mass m. One can then go to the rest frame of the particle. In this frame, the four-vector $p_\mu = (\mathbf{0}, im)$. The Dirac spinors for a particle at rest reduce to simple Pauli spinors, and the spin operator in this frame is just $\boldsymbol{\sigma}/2$. The spin can be quantized along *any* convenient z-axis in this rest frame. Introduce a spin vector which points along this z-direction

$$\mathbf{S} \equiv \frac{\mathbf{z}}{|\mathbf{z}|} \qquad ; \text{ rest frame} \tag{D.9}$$

Evidently

$$\boldsymbol{\sigma} \cdot \mathbf{S}\, \psi = \pm\, \psi \tag{D.10}$$

One can readily construct projection operators for spin up or down along this z-axis in the rest frame

$$P_\uparrow = \frac{1}{2}(1 + \boldsymbol{\sigma} \cdot \mathbf{S}) \qquad\qquad P_\downarrow = \frac{1}{2}(1 - \boldsymbol{\sigma} \cdot \mathbf{S}) \tag{D.11}$$

Now define a four-vector S_μ to be the result obtained by Lorentz transforming $S_\mu \equiv (\mathbf{S}, 0)$ from the rest frame. One evidently has the Lorentz invariant relations

$$p \cdot S = 0 \qquad\qquad S^2 = 1 \tag{D.12}$$

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The projection operators can be put into covariant form by using Eqs. (D.3), with the result that for positive energy spinors (for which, in the rest frame, $\beta\psi = \psi$)

$$\begin{aligned} P_{\uparrow} &= \frac{1}{2}(1 - \gamma_5\, \boldsymbol{\alpha} \cdot \mathbf{S}) \\ &= \frac{1}{2}(1 + i\gamma_5\, \boldsymbol{\gamma} \cdot \mathbf{S}\, \beta) \\ &= \frac{1}{2}(1 + i\gamma_5\, \gamma_\mu S_\mu) \end{aligned} \tag{D.13}$$

This result can now be readily transformed from the rest frame to any other Lorentz frame. A similar result is obtained for $P_{\downarrow}$, and the reader can verify that Eqs. (D.5) are again satisfied.

These projection operators can now be inserted in front of the appropriate Dirac spinors and sums then taken over *all* spins, which converts spin sums to traces. Only the appropriate spin states will contribute. The result will be expressed in terms of Lorentz invariant expressions involving the four-vector S_μ which has a simple interpretation in the rest frame of the particle in terms of the direction of its spin.

Let us illustrate these developments with a simple exercise. Consider the scattering of longitudinally polarized, relativistic (massless) electrons from point Dirac nucleons with one-photon exchange. Let $h = \pm 1$ represent the helicity of the incident beam with $P_h = (1 - h\,\gamma_5)/2$. Calculate the polarization of the final nucleon defined by

$$P_S \equiv \frac{N_{\uparrow} - N_{\downarrow}}{N_{\uparrow} + N_{\downarrow}} \equiv \frac{\mathscr{N}}{\mathscr{D}} \tag{D.14}$$

Here the arrows refer to the direction $\mathbf{S}$ in the rest frame. Since all common factors cancel in the ratio, one only needs to consider the Dirac traces obtained upon insertion of the appropriate projection operators. One needs the contraction of

$$\begin{aligned} \tilde{\eta}_{\mu\nu} &= \text{trace}\,\{\gamma_\mu(1 - h\gamma_5)(-ik_\rho\gamma_\rho)\gamma_\nu(-ik'_\sigma\gamma_\sigma)\} \\ \tilde{W}_{\mu\nu} &= \text{trace}\,\{\gamma_\mu(1 + i\gamma_5\gamma_\lambda S_\lambda)(M - ip_\alpha\gamma_\alpha)\gamma_\nu(M - ip'_\beta\gamma_\beta)\} \end{aligned} \tag{D.15}$$

At least four gamma matrices must be paired off with the γ_5 to get a non-zero result, and

$$\text{trace}\,\{\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma_5\} = 4\varepsilon_{\mu\nu\rho\sigma} \tag{D.16}$$

Hence

$$\begin{aligned} \tilde{\eta}_{\mu\nu} &= -4(k_\mu k'_\nu + k_\nu k'_\mu - \delta_{\mu\nu}\, k \cdot k') - 4h\, \varepsilon_{\mu\rho\nu\sigma} k_\rho k'_\sigma \\ \tilde{W}_{\mu\nu} &= 4M^2\delta_{\mu\nu} - 4(p_\mu p'_\nu + p_\nu p'_\mu - \delta_{\mu\nu}\, p \cdot p') \\ &\quad -4M(\varepsilon_{\mu\lambda\nu\beta} S_\lambda p'_\beta + \varepsilon_{\mu\lambda\alpha\nu} S_\lambda p_\alpha) \end{aligned} \tag{D.17}$$

In the contraction of the two tensors, both terms must either be even or odd in the interchange of μ and ν to get a non-zero result. For the contraction of the antisymmetric terms use

$$
\begin{aligned}
\varepsilon_{\mu\nu\rho\sigma}\,\varepsilon_{\mu\nu\alpha\beta} &= 2(\delta_{\rho\alpha}\delta_{\sigma\beta} - \delta_{\rho\beta}\delta_{\sigma\alpha}) \\
\varepsilon_{\mu\nu\rho\sigma}\,\varepsilon_{\mu\nu\alpha\beta}\,a_\rho b_\sigma c_\alpha d_\beta &= 2\,a\cdot c\,b\cdot d - 2\,a\cdot d\,b\cdot c
\end{aligned}
\tag{D.18}
$$

Hence for the polarization P_S in Eq. (D.14) one has for massless electrons

$$
\begin{aligned}
\mathscr{N} &\doteq +2hM(S\cdot k\,p'\cdot k' - S\cdot k'\,p'\cdot k - S\cdot k\,p\cdot k' + S\cdot k'\,p\cdot k) \\
&= -hM\,q^2\,S\cdot(k+k') \\
\mathscr{D} &\doteq 2M^2\,k\cdot k' + 2\,k\cdot p\,k'\cdot p' + 2\,k\cdot p'\,k'\cdot p
\end{aligned}
\tag{D.19}
$$

In the second line $q \equiv k' - k = p - p'$ has been used. One only has a non-zero P_S in this case if h is non-zero and there is a polarization *transfer* (see [Ar81]).

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