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# ON VANISHING CRITERIA THAT CONTROL FINITE GROUP STRUCTURE II

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**Abstract**

The first author [J. Brough, 'On vanishing criteria that control finite group structure', *J. Algebra* **458** (2016), 207–215] has shown that for certain arithmetical results on conjugacy class sizes it is enough to consider only the vanishing conjugacy class sizes. In this paper we further weaken the conditions to consider only vanishing elements of prime power order.

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## 1. Introduction

Many results have been proven which connect the structure of a finite group $G$ to arithmetical data connected to $G$. One type of data that has often been considered is the set of conjugacy class sizes in a group. Recently, instead of considering all conjugacy class sizes, this set has been refined by using the irreducible characters of a group. In particular, the set of vanishing conjugacy class sizes is of interest. (See [1, 3, 4, 6] and also [8] for properties related to vanishing elements.) In [1, 4, 6] and [8], the arithmetical data for conjugacy class sizes is weakened to only the vanishing conjugacy class sizes. An element $x \in G$ is called a vanishing element if there exists some irreducible character $\chi$ of $G$ such that $\chi(x) = 0$; the conjugacy class $x^G$ is called a vanishing conjugacy class.

In [4], the first author showed that the criterion given by Cossey and Wang to determine solubility and supersolubility only required the vanishing conjugacy class sizes. Furthermore, the author weakened the vanishing criterion for $p$-nilpotence given by Dolfi *et al.* [6] to only considering the vanishing $p'$-elements (that is, the elements whose order is not divisible by $p$). We restate here the three main theorems given in [4].

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THEOREM 1.1 [4, Theorem A]. *Let $G$ be a finite group and $p$ a prime divisor of $|G|$ such that, if $q$ is any prime divisor of $G$, then $q$ does not divide $p-1$. Suppose that no vanishing conjugacy class size of $G$ is divisible by $p^2$. Then $G$ is a soluble group.*

THEOREM 1.2 [4, Theorem B]. *Let $G$ be a finite group and suppose that every vanishing conjugacy class size of $G$ is square free. Then $G$ is a supersoluble group.*

THEOREM 1.3 [4, Theorem C]. *Let $G$ be a finite group and suppose that a prime $p$ does not divide the size of any vanishing conjugacy class size $|x^G|$ for $x$ a $p'$-element of prime power order in $G$. Then $G$ has a normal $p$-complement.*

The aim of this paper is to further refine which vanishing conjugacy classes are required. In particular, it is shown that it is sufficient to consider vanishing elements of prime power order. In other words, we shall prove the following results.

THEOREM 1.4. *Let $G$ be a finite group and $p$ a prime divisor of $G$ such that, if $q$ is any prime divisor of $G$, then $q$ does not divide $p-1$. Suppose that no conjugacy class size of a vanishing element of prime power order in $G$ is divisible by $p^2$. Then $G$ is a soluble group.*

THEOREM 1.5. *Let $G$ be a finite group and suppose that no conjugacy class size of a vanishing element of prime power order in $G$ is square free. Then $G$ is a supersoluble group.*

THEOREM 1.6. *Let $G$ be a finite group and suppose that a prime $p$ does not divide the size of any vanishing conjugacy class size $|x^G|$ for $x$ a $p'$-element of prime power order in $G$. Then $G$ has a normal $p$-complement.*

Note that Theorem 1.4 has the following immediate corollary.

COROLLARY 1.7. *Let $G$ be a finite group and suppose that no vanishing conjugacy class size of an element of prime power order in $G$ is divisible by 4. Then $G$ is a soluble group.*

The proofs of these theorems use very similar arguments to those in [4]. Therefore, some of the details will be omitted here and we will instead refer the reader to the previous paper. The key difference is that we now need to ensure that our chosen elements from the nonabelian simple groups without an irreducible character of $q$-defect zero have prime power order. In particular, the adapted version of [4, Lemma 2.4] is given by two lemmas at the end of Section 2, which split the cases for the sporadic and alternating groups into two parts.

## 2. Preliminaries

Given a normal subgroup $N$ in $G$, there is a natural bijection between the set of irreducible characters of $G/N$ and the set of irreducible characters of $G$ with $N$ in their kernel. In particular, this natural bijection implies that if $x$ is an element not in $N$, then $xN$ is vanishing in $G/N$ if and only if $x$ is vanishing in $G$. In addition, recall that for an element $x$ in $G$, both $|x^N|$ and $|xN^{G/N}|$ divide $|x^G|$.

Let $q$ be a prime number and $\chi$ an irreducible character of $G$. The character $\chi$ is said to have $q$-defect zero if $q$ does not divide $|G|/\chi(1)$. A result of Brauer highlights the significance $q$-defect zero has for vanishing elements. If $\chi$ is an irreducible character of $G$ with $q$-defect zero, then $\chi(g) = 0$ for every $g \in G$ such that $q$ divides the order of $g$ [10, Theorem 8.17].

Corollary 2.1 [9, Corollary 2]. *Let $S$ be a nonabelian simple group and assume that there exists a prime $q$ such that $S$ does not have an irreducible character of $q$-defect zero. Then $q = 2$ or $3$ and $S$ is isomorphic either to one of the following sporadic simple groups $M_{12}, M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3, BM$, or some alternating group $\mathrm{Alt}(n)$ with $n \geq 7$.*

In the particular case that $M$ is a minimal normal subgroup, we shall use the preceding corollary together with the following lemma; this result forms a generalisation of a comment made during the proof of [6, Theorem A].

Lemma 2.2 [4, Lemma 2.2]. *Let $G$ be a group and $N$ a normal subgroup of $G$. If $N$ has an irreducible character of $q$-defect zero, then every element of $N$ of order divisible by $q$ is a vanishing element in $G$.*

It still remains to consider those simple groups which have no character of $q$-defect zero for some prime $q$. The next result provides a condition for an irreducible character of a minimal normal subgroup $M$ of $G$ to extend to an irreducible character of $G$.

Proposition 2.3 [2, Lemma 5]. *Let $G$ be a group and $M = S_1 \times \cdots \times S_k$ a minimal normal subgroup of $G$, where every $S_i$ is isomorphic to a nonabelian simple group $S$. If $\theta \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\theta \times \cdots \times \theta \in \mathrm{Irr}(M)$ extends to $G$.*

We want to obtain a version of [4, Lemma 2.4] for elements of prime power order; however, it is not straightforward to construct an element $x$ of prime power order in $\mathrm{Sym}(n)$ such that 8 and every prime which divides $|\mathrm{Sym}(n)|$ also divides $|x^{\mathrm{Sym}}(n)|$. Fortunately, for sporadic simple groups we do have the analogous result.

Lemma 2.4. *Let $S$ be a nonabelian sporadic simple group and assume that there exists a prime $q$ such that $S$ does not have an irreducible character of $q$-defect zero.*

(1) *There exists a prime power element $x$ whose conjugacy class $x^S$ is of size divisible by every prime dividing $S$ and by 4, and there exists $\theta \in \mathrm{Irr}(S)$ which extends to $\mathrm{Aut}(S)$ such that $\theta$ vanishes on $x^S$.*

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TABLE 1. Pairs $\{x_1, \theta_1\}$ and $\{x_2, \theta_2\}$ for the sporadic groups.

| Group | Character $\theta_1$ | Class $x_1$ | Character $\theta_2$ | Class $x_2$ |
|---|---|---|---|---|
| $M_{12}$ | $\chi_7$ | $3B$ | $\chi_7$ | $8A$ |
| $M_{22}$ | $\chi_7$ | $8A$ | $\chi_2$ | $7A$ |
| $M_{24}$ | $\chi_7$ | $4C$ | $\chi_5$ | $7A$ |
| $J_2$ | $\chi_6$ | $3B$ | $\chi_{10}$ | $4B$ |
| $HS$ | $\chi_7$ | $5C$ | $\chi_{16}$ | $4C$ |
| $Suz$ | $\chi_3$ | $8B$ | $\chi_9$ | $3C$ |
| $Ru$ | $\chi_{11}$ | $4D$ | $\chi_9$ | $5B$ |
| $Co_1$ | $\chi_2$ | $4F$ | $\chi_2$ | $9B$ |
| $Co_3$ | $\chi_6$ | $4B$ | $\chi_{10}$ | $5B$ |
| $BM$ | $\chi_{20}$ | $4J$ | $\chi_{27}$ | $9B$ |

(2) *Let $p$ be a prime dividing the order of $S$. Then there exists a $p'$-element $x$ of prime power order whose conjugacy class $x^S$ is of size divisible by $p$, and there exists $\theta \in \mathrm{Irr}(S)$ which extends to $\mathrm{Aut}(S)$ such that $\theta$ vanishes on $x^S$.*

PROOF. To prove (1), Table 1 gives a pair $\{x_1, \theta_1\}$ satisfying the required conditions.

If a pair $\{x_1, \theta_1\}$ satisfies the conditions required for (1), then it also satisfies the conditions required for (2), unless $x_1$ turns out to have order divisible by $p$. Thus, to establish (2) from (1), it is enough to provide an additional pair $\{x_2, \theta_2\}$ such that if $x_1$ has order divisible by $p$, then $x_2$ has order not divisible by $p$. We cannot take exactly the same list as in either [4, Lemma 2.4] or [6, Lemma 2.2], as the given elements were not all prime power elements.

Table 1 provides pairs $\{x_1, \theta_1\}$ and $\{x_2, \theta_2\}$ taken from [5], for the required sporadic groups. □

It remains to study the alternating groups. We study $\mathrm{Alt}(n)$ for all $n \geq 7$, although in fact [9, Corollary 2] yields some additional restrictions on $n$. For $n \geq 7$, recall that $\mathrm{Aut}(\mathrm{Alt}(n)) \cong \mathrm{Sym}(n)$. As we are considering elements of prime power order, it is enough to show the existence of such an element for a prime $l$ not equal to 2 or 3. Then the simple group has a character of $l$-defect zero.

LEMMA 2.5. *Let $S$ be isomorphic to $\mathrm{Alt}(n)$ for $n \geq 7$ and assume that there exists a prime $q$ such that $S$ does not have an irreducible character of $q$-defect zero.*

(1) *There exists an $l$-element $x$ whose conjugacy class $x^S$ is of size divisible by 4 for some prime $l \neq 2$ or 3.*

(2) *Let $p$ be a prime dividing the order of $S$. Then there exists an $l$-element $x$ whose conjugacy class $x^S$ is of size divisible by $p$ for some prime $l \neq 2, 3$ or $p$.*

PROOF. To prove this statement, we first produce an $l$-element $x$ whose conjugacy class size is divisible by 4 and every prime dividing $\mathrm{Alt}(n)$ except for $l$. In order to then obtain the second statement, we can assume that the given prime $p$ is equal to $l$ for the example given to the first statement. In this case it is then enough to produce another $l'$-element of prime power order with conjugacy class size divisible by $l$.

Let $l$ be the largest prime less than $n$ (that is, $l$ is the largest prime dividing the order of $\mathrm{Alt}(n)$). Since $n \geq 7$, it is clear that $l \geq 5$. If $x$ is an $l$-cycle in $\mathrm{Alt}(n)$, then the size of its conjugacy class in $\mathrm{Sym}(n)$ is given by

$$\frac{n!}{l \cdot (n-l)!} = \frac{n(n-1)\cdots(n-l+1)}{l}.$$

As $l$ was chosen to be the largest prime less than $n$, it follows that both 4 and every other prime divisor of $\mathrm{Alt}(n)$ not equal to $l$ divides the conjugacy class size of $x$ in $\mathrm{Alt}(n)$. This completes the proof of the first claim.

Consider the second claim. If the largest prime $l \leq n$ is not equal to $p$, then we are done. Thus, assume that $l = p$. From the verified Bertrand's postulate [13, page 67], since $n \geq 7$, it follows that $l \leq n \leq 2l - 1$. Let $q$ be the second largest prime less than $n$, so $3 < q \leq l \leq n$. Let $k$ be a natural number such that $0 \leq n - kq < l$. If $k \geq l$, then $n - kq \leq n - pq \leq n - 2p < 0$, which is a contradiction. Thus, $k < l$. Now let $x$ be a product of $k$ $q$-cycles. It follows that the conjugacy class size of $x$ in $\mathrm{Sym}(n)$ is

$$\frac{n!}{q^k \cdot (k)!(n-kq)!}.$$

However, no term in the denominator of this fraction is divisible by $l$ and so $x$ is a $q$-element (that is, a prime power $p'$-element) such that $p$ divides its conjugacy class size in $\mathrm{Alt}(n)$. □

## 3. The proofs

THEOREM 3.1 (Theorem 1.4). *Let $G$ be a finite group and $p$ a prime divisor of $G$ such that, if $q$ is any prime divisor of $G$, then $q$ does not divide $p-1$. Suppose that no conjugacy class size of a vanishing element of prime power order in $G$ is divisible by $p^2$. Then $G$ is a soluble group.*

PROOF. Suppose that $G$ is chosen of minimal order satisfying the hypothesis of the theorem, but is not soluble. By the same arguments as in [4, Theorem A], it can be assumed that $p = 2$ and, if $G$ has a proper normal subgroup $N$, then $G/N$ is soluble. Moreover, a minimal normal subgroup $M \cong S_1 \times \cdots \times S_n$ is nonabelian. If $S_i$ has a character of $q$-defect zero for all $q$, then, as $S_i$ is nonsoluble, [11, Proposition] implies that there is an element of prime power order with conjugacy class size divisible by 4. On the other hand, if $S_i$ is isomorphic to $\mathrm{Alt}(n)$ with $n \geq 7$, then, by Lemma 2.5, there is an $l$-element for $l > 3$ such that the conjugacy class size is divisible by 4. In both cases applying [4, Lemma 2.2] shows that $G$ has a prime power vanishing element with conjugacy class size divisible by 4. Hence, it can be assumed that $S_i$ is isomorphic to one of the sporadic groups given in Corollary 2.1. In this case the same argument as in [4, Theorem A] now using Lemma 2.4 produces a vanishing element of $G$ with prime power order and 4 dividing its conjugacy class size. □

THEOREM 3.2 (Theorem 1.5). *Let $G$ be a finite group and suppose that no conjugacy class size of a vanishing element of prime power order in $G$ is square free. Then $G$ is a supersoluble group.*

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Proof. This result now follows by combining Theorem 1 with the proof of [4, Theorem B], as the only elements considered are of prime power order. □

Theorem 3.3 (Theorem 1.6). *Let $G$ be a finite group and suppose that a prime $p$ does not divide the size of any vanishing conjugacy class size $|x^G|$ for $x$ a $p'$-element of prime power order in $G$. Then $G$ has a normal $p$-complement.*

Proof. Suppose that $G$ is chosen of minimal order satisfying the hypothesis of the theorem, but does not have a normal $p$-complement. By the same arguments used in the proof of [4, Theorem C], we can conclude that $O_{p'}(G) = 1$. Let $M = S_1 \times \cdots \times S_k$ be a minimal normal subgroup of $G$ with each $S_i \cong S$ a simple group. Then $p$ divides the order of $S$. If $S$ is abelian, the proof of [4, Theorem C] shows that any vanishing $p'$-element of prime power order lies in $O_p(G)$. Hence, $G$ has a normal $p$-complement by [7, Corollary C].

Hence, assume that $S$ is nonabelian. First assume that $S$ is not sporadic. If $S$ has an irreducible character of $q$-defect zero for each prime $q$, then as $S \not\cong O_p(S) \times O_{p'}(S)$ it follows by [12, Theorem 5] that $S$ has a $p'$-element of prime power order such that $p$ divides its conjugacy class size. Moreover, Lemma 2.5 implies for $S$ of alternating type (on at least seven points) that there exists a $p'$-element which has order a power of a prime $l > 3$ and conjugacy class size divisible by $p$. Thus, [4, Lemma 2.2] shows that $G$ has a $p'$-element of prime power order which is vanishing and conjugacy class size divisible by $p$. Finally, assume that $S$ is sporadic. In this case the same argument as in [4, Theorem C] but using Lemma 2.4 produces a vanishing $p'$-element of $G$ with prime power order and $p$ dividing its conjugacy class size. □

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