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Acute Triangulation of a Triangle in a General Setting

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Abstract. We prove that, in ordered plane geometries endowed with a very weak notion of orthogonality, one can always triangulate any triangle into seven acute triangles, and, in case the given triangle is not acute, into no fewer than seven.

1 Introduction

The subject of acute triangulations goes back to 1960 with [3, 7–9], where it was shown that an obtuse triangle in the Euclidean plane admits a triangulation with 7 acute triangles and that none with fewer acute triangles is possible. Other results obtained so far in this area deal with acute triangulations of polygons in the Euclidean plane or of surfaces (see [10–14, 18, 19, 29–31]). One particular paper, [15], deals with acute triangulations of polygons in the real Euclidean, hyperbolic, and elliptic planes, whereas [12] deals with acute triangulations of spherical triangles, where 10 and 18 acute triangles are needed, depending on the type of the triangle to be triangulated. Our aim is to show that the original statement, that a triangle admits a triangulation with 7 acute triangles, remains valid under much weaker hypotheses than those satisfied by the Euclidean plane.

What we will need is a notion of betweenness and one of orthogonality, the former satisfying all the linear order axioms as well as the Pasch axiom, and the latter satisfying very general requirements, which, however, exclude the elliptic or spherical geometry case.

2 The Axiom System

The axiom system will be expressed in a first-order language, with one sort of individuals, to be interpreted as *points*, containing two ternary predicates: B, with $B(abc)$ to be read as 'b lies between a and c', and $\perp$, with $\perp(abc)$ to be read as 'ab is orthogonal to ac' (or 'triangle abc has a right angle in a'). To improve the readability of the axioms, we introduce the following abbreviations (defined notions): L, with $L(abc)$ to be read as 'a, b, and c are collinear points', Z, with $Z(abc)$ to be read as 'b lies strictly between a and c', ι, with $\iota_x(abc)$ to be read as 'x belongs to the interior of the angle formed by the rays $\overrightarrow{ab}$ and $\overrightarrow{ac}$' (as it will be used only when the points a, b, and c are not collinear), and α, with $\alpha(abc)$ to be read as 'the angle $\widehat{bac}$ formed by the rays $\overrightarrow{ab}$

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and $\overrightarrow{ac}$ is acute'.

$$L(abc) :\Leftrightarrow B(abc) \vee B(bca) \vee B(cab),$$

$$Z(abc) :\Leftrightarrow B(abc) \wedge a \neq b \wedge b \neq c,$$

$$\iota_x(abc) :\Leftrightarrow (\exists u)\, Z(buc) \wedge (B(aux) \vee B(axu) \vee u = x),$$

$$\alpha(abc) :\Leftrightarrow (\exists v)\ \perp(abv) \wedge \iota_c(abv).$$

Although there are no lines in our formal language, we shall informally refer to the *line ab* determined by two different points a and b, the *incidence* of a point c with a line ab meaning the validity of $L(abc)$. In a similar manner, we will refer to perpendicular lines, to segments (open or closed), to sides of a triangle, and to angles.

The axioms are (we omit the universal quantifiers whenever the axioms are universal sentences):

A1 $B(aab)$,

A2 $B(abc) \rightarrow B(cba)$,

A3 $B(aba) \rightarrow a = b$,

A4 $B(abc) \wedge B(acd) \rightarrow B(bcd)$,

A5 $a \neq b \wedge L(abc) \wedge L(abd) \rightarrow L(acd)$,

A6 $(\forall abcde)(\exists f)\, \neg L(abc) \wedge Z(adb) \wedge \neg L(abe)$
$\rightarrow (B(afc) \vee B(bfc)) \wedge L(edf)$,

A7 $(\forall ab)(\exists c)\, a \neq b \rightarrow B(abc) \wedge c \neq b$,

A8 $(\forall abc)(\exists d)\, \neg L(abc) \wedge \neg \perp(acb) \rightarrow L(abd) \wedge \perp(dca)$,

A9 $(\forall ab)(\exists u)\, a \neq b \rightarrow \perp(abu)$,

A10 $\perp(abc) \rightarrow \neg L(abc)$,

A11 $\perp(abc) \rightarrow \perp(acb)$,

A12 $\perp(abc) \rightarrow \neg \perp(bac)$,

A13 $\perp(abc) \wedge L(abu) \wedge u \neq a \rightarrow \perp(auc)$,

A14 $\perp(abc) \wedge \perp(ab'c) \rightarrow L(abb')$,

A15 $\neg L(abc) \wedge \alpha(abc) \rightarrow \alpha(acb)$,

A16 $(\forall abcdd't)(\exists u)(\forall v)\ \perp(bac) \wedge \perp(cbd) \wedge Z(dcd') \wedge Z(atd') \wedge Z(btc)$
$\rightarrow [Z(buc) \wedge (B(bvu) \rightarrow \alpha(vad))]$.

Here is an informal explanation of the statements the axioms make. A1–A7 are the axioms of ordered geometry, but omitting the lower-dimension axiom D, stating the existence of three non-collinear points. The theory obtained by adding D is called *two-dimensional, unending, linear geometry* in [4] (for the dimension-free version, see [28]) ensuring that the basic notions of convex geometry can be defined and have the expected properties (which we will freely use), as well as the fact that lines do not have endpoints (A7). If D holds, then A1–A7 imply that the order is dense, *i.e.*, that

$$(\forall ab)(\exists c)\, a \neq b \rightarrow B(acb) \wedge c \neq a \wedge c \neq b$$

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holds. A6 is a form of the Pasch axiom. If D holds, then on the basis of A1–A7, one can define the two *sides* a given line ab (determined by two distinct points a and b) divides all the points not on line ab into, by saying that two points u and v, none of which lie on ab, lie on the *same side* of ab if there is no point t, such that $B(utv)$ and $L(abt)$ holds, and on different sides of ab if such a point t does exist. A8 states the existence of a perpendicular from any point c not on a line ab to that line, A9 the existence of a perpendicular raised in a point a on the line ab, A10 that there are no self-orthogonal (isotropic) lines, A11 that orthogonality is a symmetric relation (if line ab is orthogonal to line ac, the ac is orthogonal to ab as well), A12 the uniqueness of the perpendicular dropped from a point c not on ab to ab, A13 that $\perp$ is, in essence, a relation between *lines*, and A14 the uniqueness of the perpendicular raised in a to ac. A15 is a special case, in which the angles involved are right, of Hilbert's Axiom III 7 [11], which states that no angle lies inside a congruent angle with the same vertex. A16 states that, if ab and dc are two perpendiculars to bc, with a and d on the same side of bc, then there exists a point v in the open segment (b, c), arbitrarily close to b such that the angle $\widehat{avd}$ is acute (the plausibility of this axiom can be seen by noticing that if $v = b$, then $\widehat{avd}$ is acute by definition, and that, *by continuity considerations*, the angle $\widehat{avd}$ will stay acute for a while, as v glides along bc towards c).

3 The Main Result

We first prove three lemmas. The first one states that a triangle cannot have more than one nonacute angle, the second one that in a triangle abc with acute angles at b and c, the foot of the altitude from a lies between b and c, and the third one that an angle $\widehat{acb}$ must be acute if it contains 'inside' it an acute angle $\widehat{adb}$ with dc perpendicular to ab.

Lemma 3.1 $\neg L(abc) \wedge \neg\alpha(bca) \to \alpha(cba)$.

Proof Suppose $\neg L(abc)$, $\neg\alpha(bca)$, and $\neg\alpha(cba)$ would hold. Let u and v be such that $\perp(bcu)$ and $\perp(cbv)$ and u and v lie on the same side of bc as a (the existence of u and v is ensured by A9 and the order axioms). By $\neg\alpha(bca)$, we have one of the following:

(i) $\perp(bca)$, and thus we cannot have $\neg\alpha(cba)$, as this would imply that the ray $\overrightarrow{cv}$ is either $\overrightarrow{ca}$ (and that would contradict A12) or else $\iota_v(cba)$, and thus $\overrightarrow{cv}$ would have to intersect the segment ab (again contradicting A12);

(ii) $\perp(bca)$ does not hold, but $\perp(cba)$ does hold, from which we derive, as in (i), a contradiction;

(iii) neither $\perp(bca)$ nor $\perp(cba)$ hold, and thus a and c lie on different sides of the line bu, and a and b lie on different sides of cv.

Thus, in case (iii) holds, there are points x and y such that $Z(bya) \wedge L(cvy)$ and $Z(cxa) \wedge L(bux)$. Applying the Pasch axiom to triangle abx with secant cy, we deduce that cy and bx must intersect, *i.e.*, the lines bu and cv must have a point in common, which is impossible by A12. ■

Lemma 3.2 $\neg L(abc) \wedge \alpha(bca) \wedge \alpha(cba) \wedge \perp(uab) \wedge L(bcu) \to Z(buc)$.

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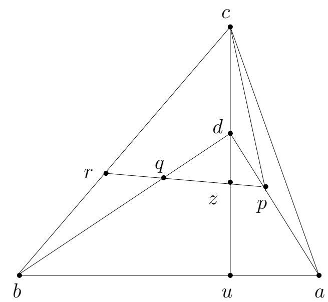

Figure 1: If angle $\widehat{adb}$ is acute, then so is $\widehat{acb}$.

Proof Since u can be neither b nor c (given that $\alpha(bca)$ and $\alpha(cba)$ hold), we have $Z(ubc) \vee Z(buc) \vee Z(bcu)$. Suppose $Z(buc)$ does not hold. We can assume without loss of generality that $Z(ubc)$ holds. By A9 and the order axioms, there is v on the same side of bc as a and such that $\perp(bcv)$. By $\alpha(bca)$, the points a and c must lie on the same side of bv. Since c and u lie on different sides of bv, we deduce that a and u lie on different sides of bv as well, thus, $(\exists t)\, Z(atu) \wedge L(bvt)$. Thus, there exist two different perpendiculars, tu and tb from t to bc, contradicting A12. ■

For a, b, c with $\neg L(abc)$, we will denote by $F(bca)$ the foot of the perpendicular from a to bc, which exists by A8. We are now ready to prove the following.

Lemma 3.3 $\neg L(abc) \wedge Z(aub) \wedge \perp(uca) \wedge Z(cdu) \wedge \alpha(dab) \to \alpha(cab)$.

Proof Suppose $\neg\alpha(cab)$, and let v denote a point, on the same side of bc as a, for which $\perp(cvb)$ (the existence of v is ensured by A9 and the order axioms). We distinguish two cases:

(i) If the lines cv and ca coincide, *i.e.*, if $\perp(cab)$, the perpendicular raised in d on da (which exists by A9) intersects the open segment (a, c) (since, by $\alpha(dab)$, it cannot intersect the side au of $\triangle auc$, so, by the Pasch axiom, it has to intersect the side ac), *i.e.*, $(\exists w)\ \perp(daw) \wedge Z(awc)$. Let $z = F(acd)$. By Lemma 3.1, we must have $\alpha(wda)$ and $\alpha(awd)$, and thus, by Lemma 3.2, we have $Z(wza)$. Since $\alpha(dab)$, the line dw cannot intersect side bu (as closed segment) of $\triangle buc$; thus, by the Pasch axiom, it must intersect side bc at some point t, *i.e.*, we have $L(dwt) \wedge Z(btc)$. The Pasch axiom, applied to $\triangle ctw$ with secant dz, provides a point of intersection of line dz with side ct of $\triangle ctw$, *i.e.*, $(\exists x)\, L(dzx) \wedge Z(txc)$. Thus, there are two perpendiculars from x to the line ac, namely xc and xz, contradicting A12.

(ii) (See Figure 1.) Suppose now that $\perp(cab)$ does not hold. Given $\neg\alpha(cab)$ and $\alpha(cub)$ (by Lemma 3.1), we must have $\iota_v(cau)$, and thus the ray $\overrightarrow{cv}$ must intersect segment ad in a point p (*i.e.*, $(\exists p)\ \perp(cpb) \wedge Z(apd)$). By A8, $(\exists q)\ \perp(qpd) \wedge L(dbq)$. Given $\alpha(dab)$, we cannot have $Z(bdq)$ (by Lemma 3.1). We cannot have $Z(dbq)$ either, or segment pq would have to intersect side ab of $\triangle dab$ (by Pasch's axiom), *i.e.*, $(\exists w)\, Z(pwq) \wedge Z(awb)$, and, given $\alpha(bud)$ (by Lemma 3.1), and thus $\alpha(bwd)$, we must have $\neg\alpha(bwq)$, contradicting Lemma 3.1 for $\triangle bwq$. Also, $q \neq d$, for if $q = d$, then $\neg\alpha(dab)$, contradicting our assumption. Finally, $q \neq b$, for, if $q = b$, then

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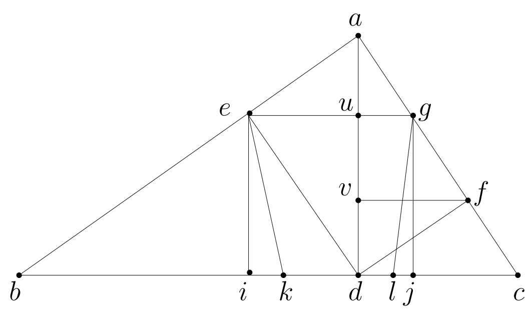

Figure 2: Calibrating the positions of the points of triangulation.

$\neg\alpha(bud)$, contradicting Lemma 3.1 for $\triangle bud$. Thus $Z(dqb)$. By the order axioms, the open segments (p,q) and (d,u) must intersect, *i.e.*, $(\exists z)\, Z(pzq) \wedge Z(dzu)$. Applying Pasch's axiom to $\triangle cdb$ and secant zq, we get a point r such that $Z(zqr) \wedge Z(crb)$. Applying Lemma 3.1 to $\triangle qrb$ we get $\alpha(rqb)$, thus $\neg\alpha(rqc)$, contradicting Lemma 3.1 for $\triangle cpr$. ■

Theorem *Any triangle allows an acute triangulation consisting of 7 triangles. If the triangle is obtuse, there is no acute triangulation with fewer than 7 triangles.*

Proof (See Figures 2 and 3.) Let abc be a triangle, of which we can assume, by Lemma 3.1, that $\alpha(bca)$ and $\alpha(cba)$. Let $d = F(bca)$, $e = F(abd)$, $f = F(acd)$, $u = F(ade)$, and $v = F(adf)$. By Lemma 3.2, $Z(bdc)$, $Z(aeb)$, $Z(afc)$, $Z(aud)$ and $Z(avd)$. We may assume without loss of generality that $B(auv)$. By the Pasch axiom, line eu must intersect a second side of $\triangle adc$ besides ad. Since the lines eu and dc cannot intersect (or there would be two different perpendiculars from that intersection point to ad, contradicting A12), eu must intersect side ac, *i.e.*, $(\exists g)\, Z(eug) \wedge Z(agc)$. Let $i = F(bde)$, $j = F(dcg)$, and k be the intersection point of the perpendicular in e to eu with side db of $\triangle dba$ (the intersection point must exist by Pasch's axiom and the fact that it cannot intersect side ad by A12). Let l be the intersection point of

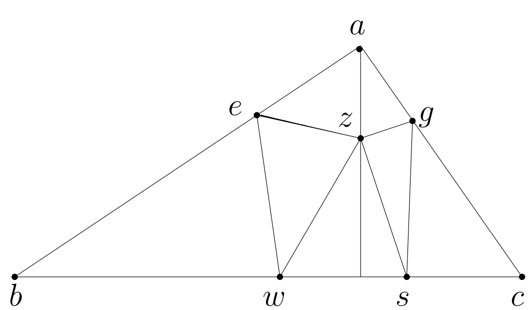

Figure 3: The triangulation.

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the perpendicular in g to gu with side dc of $\triangle dca$ (existence deduced analogously). Let g' be such that $\perp(gg'a)$ (g' exists by A9). Line gg' must, by Pasch, intersect one of the sides ad and dc of $\triangle adc$. It cannot intersect the open segment (d, c); otherwise, if $f \neq g$, it would have to intersect the open segment (d, f) as well (by Pasch's axiom), in a point h, giving two different perpendiculars, hg and hf from h to ac, (contradicting A12) or, in case $f = g$, we would have two perpendiculars raised in f on ac (contradicting A14). Thus, gg' can only intersect the closed segment ad. Let h be that intersection point. Note that h must be such that $B(dhu)$; otherwise, if it were such that $B(uha)$, we would have $\neg\alpha(gua)$, contradicting Lemma 3.1, and that $h \neq u$, for, if $h = u$, then there would be two perpendiculars from a to eg, namely au and ag, contradicting A12. Let t be such that $Z(htu)$ (such points exist, given that the order is dense).

By A16, there exists a point p with $Z(ipd)$, such that, for all p' with $B(pp'd)$, we have $\alpha(p'et)$. By the same axiom, there exists a point q with $Z(jqd)$, such that, for all q' with $B(qq'd)$, we have $\alpha(q'gt)$. Let s be a point with $Z(dsq)$ and $Z(dsl)$ (such points exist, given that the order is dense). Let t' be such that $\perp(tst')$ (t' exists by A9). By Pasch's axiom, line tt' must intersect (i) side ab or (ii) side bd of $\triangle abd$. If case (i) holds, let $s' = p$. If case (ii) holds, let s' be the intersection point of line tt' with side bd of $\triangle abd$. Let w be a point such that $Z(pwd)$, $Z(s'wd)$, and $Z(kwd)$ hold (given that the order is dense, such points exist). The choice of w implies that $\alpha(tws)$. By A16, $(\exists x)(\forall x')\, Z(uxd) \wedge (B(xx'u) \rightarrow \alpha(x'ew))$ and $(\exists y)(\forall y')\, Z(uyd) \wedge (B(yy''u) \rightarrow \alpha(y'gs))$. Let z be the point among t, x, y which is closest (in the sense of the order on the open segment (d, u)) to u. Then $\triangle ebw$, $\triangle ewz$, $\triangle zws$, $\triangle eza$, $\triangle gza$, $\triangle szg$, $\triangle sgc$ are the seven acute triangles into which $\triangle abc$ was triangulated. To see that these triangles are indeed acute, note that $\alpha(bwe)$, $\alpha(wbe)$, $\alpha(wzs)$, $\alpha(swz)$, $\alpha(sgc)$, $\alpha(csg)$, $\alpha(zga)$, $\alpha(azg)$, $\alpha(aze)$, $\alpha(zae)$ all follow from Lemma 3.1. Before we move to the other angles, let us point out that, by the very definition of α, $(\alpha(abc) \vee \perp(abc)) \wedge \iota_d(abc) \rightarrow \alpha(abd)$, in other words, an angle γ, sharing a side with an acute angle β, and lying inside β, must be acute. Notice that $\alpha(euw)$, since $\iota_w(euk)$, and thus, since $\iota_z(euw)$, we have $\alpha(ewz)$. Since $\perp(eda)$ and $\iota_z(eda)$, we have $\alpha(eza)$. Since z was chosen to be such that $B(tzu)$, we have $\alpha(gza)$. Given that $\alpha(gdc) \vee \perp(gdc)$ and $\iota_s(gdc)$, we have $\alpha(gsc)$. Since z was chosen such that $B(xzu)$, we have, by the definition of x, $\alpha(zew)$. Since $B(yzu)$, we have, by the definition of y, $\alpha(zgs)$. By Lemma 3.3, $\alpha(tws)$, and $B(dtz)$ (by z's definition), we get $\alpha(zws)$. That $\alpha(ebw)$ holds can be seen from the definition of α by noticing that $Z(bwd)$ and $\perp(ebd)$. That $\alpha(wez)$ holds can be sen by noticing that, by $Z(pwd)$ and the choice of p, we have $\alpha(wet)$, and thus, given that $B(dtz)$, $\widehat{ewz}$ is included in or coincides with $\widehat{ewt}$, so that $\alpha(wez)$ as well. That $\alpha(sgz)$ holds can be seen by noticing that, by the definition of q and the fact that $Z(qsd)$ holds, we have $\alpha(sgt)$, and thus, given that $B(dtz)$, $\widehat{gsz}$ is included in or coincides with $\widehat{gst}$, so that $\alpha(sgz)$ as well. Given $Z(uzd)$, $Z(dsl)$, and $\perp(gul)$, we conclude that $\widehat{sgz}$ is included in $\widehat{lgu}$, which is a right angle, and thus, by A15, that $\alpha(gsz)$.

The proof, given in [10], that there is no acute triangulation with fewer than 7 triangles in the case where $\triangle abc$ is obtuse, carries over to our setting. The key point there is the need for an interior point that is a vertex of the triangulation graph and the fact that at least 5 edges have to emanate from it. This fact remains true, as can easily be seen from the definition of α, in our setting as well, for if we have four or fewer rays emanating from a point o, then one of the angles formed by a pair of consecutive rays must be nonacute. ■

4 Geometries Satisfying the Axiom System

The axiom system {A1-A16} describes a geometry so general that no representation theorem connecting it so some class of algebraic models is imaginable. It is thus worth seeing which geometries that have received attention in the literature on the foundations of geometry satisfy these axioms in order to see for which classes of known geometries our theorem remains valid. All *ordered* geometries with a Euclidean metric, such as those defined in [1, 2, 16, 17, 21, 22, 24–26], satisfy our axioms. The only one that one would need to check is A16, and, given the existence of algebraic descriptions for those geometries, this turns out to be a matter of checking. The Minkowski planes (two dimensional normed spaces, see [27]) in which the unit circles are strictly convex Radon curves (the so-called *strictly convex Radon planes*) also satisfy our axioms. Radon planes were characterized in [5, 6, 20]. Our axioms also hold in the class of all ordered metric planes (those with Euclidean metric were mentioned earlier already, and those with non-Euclidean metric were characterized algebraically in [23], allowing for the checking of A16), which includes the class of hyperbolic planes.

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