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The numerical ranges of unbounded operators

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On a Banach space the numerical range of an unbounded operator has a certain density property in the scalar field. Consequently all hermitian and dissipative operators are bounded. For a smooth or separable or reflexive Banach space the numerical range of an unbounded operator is dense in the scalar field.

Given a normed linear space X with dual X^*, for each $x \in X$, $\|x\| = 1$, choose an $f_x \in X^*$, $\|f_x\| = 1$ such that $f_x(x) = 1$; (the Hahn-Banach Theorem guarantees the possibility of such a choice). Write

$$\Pi_s \equiv \{(x, f_x) \in X \times X^* : \|x\| = 1\} .$$

X associated with Π_s is called *a normed linear space with s.i.p. representation*. For a linear operator T on X with a given s.i.p. representation, the set of scalars

$$W(T) \equiv \{f_x(Tx) : (x, f_x) \in \Pi_s\}$$

is called the *Lumer numerical range* of T, and the set of scalars

$$V(T) \equiv \cup\{W(T) : \text{all possible s.i.p. representations associated with } X\}$$

is called the *Bonsall numerical range* of T, [2, pp. 81, 85]. When the normed linear space X is smooth, as is Hilbert space, there exists only one s.i.p. representation for X and then both numerical ranges are identical. We use the Lumer numerical range throughout our argument as it produces the stronger results which will clearly also hold for the Bonsall

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numerical range.

On a Banach space a linear functional maps the unit sphere S into a bounded set if it is bounded and here we see that it maps S onto a dense subset of the scalar field if it is unbounded.

PROPOSITION. *For a linear functional* f *on a Banach space* X *, if* f *is unbounded then* $\overline{f(S)}$ *is the whole scalar field.*

Proof. Consider any scalar λ . Since f is unbounded it follows by the Closed Graph Theorem that there exists a sequence $\{x_n\}$ in X , $x_n \neq 0$, such that $x_n \to 0$ and $f(x_n) \to \lambda$. Consider $x \in S$ such that $f(x) = 0$. Then $f(x_n - x) \to \lambda$ and $\|x_n - x\| \to \|x\| = 1$ so it follows that

$$f\left(\frac{x_n - x}{\|x_n - x\|}\right) \to \lambda$$

and therefore $\lambda \in \overline{f(S)}$.

We have a comparable result for linear operators in that a bounded linear operator maps any Π_s into a bounded set and we here prove that an unbounded linear operator maps Π_s onto a subset which has a certain density property in the scalar field.

THEOREM 1. *For an unbounded linear operator* T *on a Banach space* X *there exists an* $r > 0$ *such that every disc in the scalar field of radius* r *contains a point of* $W(T)$.

Proof. From the Closed Graph Theorem we deduce that T is not closed, so there exists a sequence $\{x_n\}$ in X such that $x_n \to 0$ and $Tx_n \to y$ where $\|y\| = 1$. For any given scalar λ and each n choose, in accordance with the s.i.p. representation, $f_n \in X^*$ such that $\|f_n\| = 1$ and

$$f_n(y + \lambda x_n) = \|y + \lambda x_n\| .$$

By the Banach-Alaoglu Theorem the set $\{f_n\}$ has a weak* cluster point f_λ and $\|f_\lambda\| \le 1$. But since

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$$|f_\lambda(y)-1| \le |f_\lambda(y)-f_n(y)| + |f_n(y)-f_n(y+\lambda x_n)| + |\|y+\lambda x_n\|-1|$$
$$\le |(f_\lambda-f_n)(y)| + 2|\lambda|\|x_n\| ,$$

we deduce that $f_\lambda(y) = 1$ and $\|f_\lambda\| = 1$. Now

$$|f_\lambda(Ty+\lambda y)-f_n(Ty+\lambda Tx_n)| \le |f_\lambda(Ty+\lambda y)-f_n(Ty+\lambda y)| + |f_n(Ty+\lambda y)-f_n(Ty+\lambda Tx_n)|$$
$$\le |(f_\lambda-f_n)(Ty+\lambda y)| + |\lambda|\|Tx_n-y\| .$$

We deduce from this inequality that $f_\lambda(Ty) + \lambda$ is a cluster point of $W(T)$. So for every scalar λ the set $\{f(Ty) : f(y) = \|f\| = 1\} + \lambda$ contains a cluster point of $W(T)$. Since $\{f(Ty) : f(y) = \|f\| = 1\}$ is a bounded convex set, for

$$r > \tfrac{1}{2} \text{ diameter } \{f(Ty) : f(y) = \|f\| = 1\}$$

we have our result.

We have the following significant corollaries.

COROLLARY 1. *For an unbounded linear operator* T *on a smooth Banach space* X, $W(T)$ *is dense in the scalar field.*

Proof. Since X is smooth, $\{f(Ty) : f(y) = \|f\| = 1\}$ is a single point set and so the Theorem holds for all $r > 0$. But this implies that $W(T)$ is dense in the scalar field.

COROLLARY 2. *For an unbounded linear operator* T *on a Banach space* X, *with any given s.i.p. representation,* $\mathrm{co}W(T)$ *is the whole scalar field.*

Proof. Consider X over the complex field. For any complex λ, the disc radius r and centre at $\lambda \pm r$, $\lambda \pm ir$ each contains points of $W(T)$ so $\lambda \in \mathrm{co}W(T)$.

Now the Toeplitz-Hausdorff Theorem actually proves that on a Hilbert space, for any linear operator T bounded or unbounded, $W(T)$ is convex, [3]. So from Corollary 2 we can make the following deduction for Hilbert space.

COROLLARY 3. *For an unbounded linear operator* T *on a Hilbert space,* $W(T)$ *is the whole scalar field.*

On a complex linear space X with given s.i.p. representation, a linear operator T is said to be *hermitian* if $W(T)$ is a subset of the

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real line. It can be shown quite simply for Hilbert space that an hermitian operator is symmetric and it is known that a symmetric operator is bounded, [4, p. 207]. We see from our Theorem 1 that we can make the more general statement for Banach space.

COROLLARY 4. *On a complex Banach space with any given s.i.p. representation, hermitian operators are bounded.*

On a Banach space X with given s.i.p. representation, a linear operator T is said to be *dissipative* if $\operatorname{Re}W(T) \leq 0$. Our Theorem 1 is an adaptation of that given by Lumer and Phillips for dissipative operators [6, p. 693]. However, one of the consequences of their lemma can be deduced directly from our theorem.

COROLLARY 5. *On a Banach space with any given s.i.p. representation, dissipative operators are bounded.*

We may well ask for what other Banach spaces is the numerical range of an unbounded operator dense in the scalar field. The following theorem provides sufficiency conditions for such density.

THEOREM 2. *Let X be a Banach space where every open ball of unit radius with centre on the unit sphere contains a smooth point of the unit sphere. For an unbounded linear operator T on X, with any given s.i.p. representation, $W(T)$ is dense in the scalar field.*

Proof. As in the proof of Theorem 1 there exists a sequence $\{x_n\}$ in X such that $x_n \to 0$ and $Tx_n \to y$ where $\|y\| = 1$. Let z be a smooth point of the unit sphere such that $\|y-z\| < 1$. For any given scalar λ and each n choose, in accordance with the s.i.p. representation, $f_n \in X^*$ such that

$$f_n(z+\lambda x_n) = \|z+\lambda x_n\| .$$

Now the set $\{f_n\}$ has a weak* cluster point f_λ and $f_\lambda(z) = \|f_\lambda\| = 1$. As in the proof of Theorem 1 we deduce that $f_\lambda(Tz) + \lambda f_\lambda(y)$ is a cluster point of $W(T)$. Since z is a smooth point of the unit sphere, there exists a unique $f \in X^*$ such that $f(z) = \|f\| = 1$. So for every scalar λ, $f(Tz) + \lambda f(y)$ is a cluster point of $W(T)$. Since $\|y-z\| < 1$ we have

$$|1-f(y)| = |f(z-y)| \le \|z-y\| < 1$$

so $f(y) \neq 0$ and we conclude that $W(T)$ is dense in the scalar field.

Now Mazur has proved for separable Banach spaces [7, p. 78] and Lindenstrauss has proved for reflexive Banach spaces [5, p. 967] that the set of smooth points on the unit sphere is dense in the unit sphere. We can therefore deduce the following result.

COROLLARY 6. *For an unbounded linear operator* T *on a separable or reflexive Banach space* X, *with any given s.i.p. representation,* $W(T)$ *is dense in the scalar field.*

It is of interest to note how much our results are dependent on the completeness of the space. It is possible to have an unbounded linear operator on an incomplete space with s.i.p. representation such that the numerical range is a single point set.

EXAMPLE. Given the complex linear space $C^{\infty}[0, 1]$ with the supremum norm, consider the normed linear subspace X of mappings f where $f^{(n)}(0) = f^{(n)}(1) = 0$ for all n. For each f, $\|f\| = 1$, choose $x_f \in [0, 1]$ such that $f(x_f) = \lambda$ where $|\lambda| = 1$. Consider the linear functional $\hat{x}_f$ defined by $\hat{x}_f(g) = \frac{1}{\lambda} g(x_f)$ for all $g \in X$. Now $\hat{x}_f(f) = 1$ and $\|\hat{x}_f\| = 1$. In this way we set up a s.i.p. representation for X. Consider the differentiation operator D on X. For each $f \in X$ where $\|f\| = 1$, $Df(x_f) = 0$ and $\hat{x}_f(Df) = 0$. Therefore, $W(D) = \{0\}$.

It should be noted that Phelps has recently proved that for any linear operator T bounded or unbounded, on a complex normed linear space, $V(T)$ is connected, [8, p. 335].

Note added in proof, 10 July 1974. We would like to thank Dr E.N. Dancer for drawing our attention to Asplund's result that for weakly compactly generated Banach spaces the set of smooth points on the unit sphere is dense in the unit sphere, [1, p. 32]. So then Corollary 6 will apply more generally to weakly compactly generated Banach spaces.

References

[1] Edgar Asplund, "Fréchet differentiability of convex functions", *Acta Math.* 121 (1968), 31-47.

[2] F.F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras* (London Mathematical Society Lecture Note Series, 2. Cambridge University Press, Cambridge, 1971).

[3] Karl Gustafson, "The Toeplitz-Hausdorff Theorem for linear operators", *Proc. Amer. Math. Soc.* 25 (1970), 203-204.

[4] Paul R. Halmos, *A Hilbert space problem book* (Van Nostrand, Princeton, New Jersey; Toronto, Ontario; London; 1967).

[5] Joram Lindenstrauss, "On nonseparable reflexive Banach spaces", *Bull. Amer. Math. Soc.* 72 (1966), 967-970.

[6] G. Lumer and R.S. Phillips, "Dissipative operators in a Banach space", *Pacific J. Math.* 11 (1961), 679-698.

[7] S. Mazur, "Über konvexe Mengen in linearen normierten Räumen", *Studia Math.* 4 (1933), 70-84.

[8] R.R. Phelps, "Some topological properties of support points of convex sets", *Israel J. Math.* 13 (1972), 327-336.

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