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RESTRICTED LIE ALGEBRAS OF MAXIMAL CLASS

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Let L be a possibly infinite-dimensional Lie algebra of maximal class. We show that if L admits the structure of a Lie p-algebra then the dimension of L can be at most $p+1$. Furthermore, this bound is best possible.

1. INTRODUCTION

A nilpotent Lie algebra L with finite dimension n and nilpotency class $n-1$ is said to be of maximal class. This definition extends naturally to infinite dimensional Lie algebras: L has maximal class if L is residually nilpotent and

$$\dim L/\gamma_n(L) = n$$

for all $n > 1$, where $\gamma_n(L)$ denotes the nth term of the lower central series of L.

The analogous notion defined for finite p-groups and pro-p groups has been studied extensively by many authors (see [**6**] for an overview). In fact, Blackburn's original study of finite p-groups of maximal class [**2**] predates Vergne's seminal work on Lie algebras of maximal class [**9, 10, 11**].

We single out now a few relevant results about maximal class. First, Alperin [**1**] proved every pro-p group with maximal class has an open Abelian subgroup. Second, and along this same vein, Shalev and Zelmanov proved in [**8**] that every graded (that is, Z^+-graded and generated by its first homogeneous component) Lie algebra of maximal class in characteristic zero is virtually Abelian. Actually, Vergne proved a similar result long ago, but Shalev and Zelmanov's theorem was proved more generally under the weaker hypothesis of finite coclass. Nowadays, pro p-groups and Lie algebras of finite coclass are actively studied. See [**6**], for example. We note here only that having maximal class is equivalent to having coclass 1.

In contrast to the aforementioned characteristic zero result, Shalev constructed in [**7**] examples of modular graded Lie algebras of maximal class that are not virtually soluble. Moreover, by a recent result of Caranti, Mattarei and Newman [**3**], the number of isomorphism types of such Lie algebras is 2^ω. It might be surprising then that the structure of restricted Lie algebras of maximal class cannot be nearly so complicated.

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Indeed, it follows from a result of Semple and the present author in [4] that the dimension of a restricted Lie algebra of maximal class is finite and bounded above by $2p+2$ if the characteristic p is odd, and 14 if $p=2$. A similar statement also holds under the weaker hypothesis of finite coclass. The aim of this note is to sharpen this bound to $p+1$ for all $p>0$. It will transpire that this new bound is best possible.

Theorem B. *Suppose that L is a restricted Lie algebra over a field with prime characteristic $p>0$ and that L is of maximal class. Then L is nilpotent of class at most p and* $\dim L \leqslant p+1$.

Let us now illustrate why the bound produced in Theorem B is best possible. Let $\{e_1, e_2, \dots, e_n\}$ be a basis of the Lie algebra over a field F of characteristic $p>0$ defined via

$$[e_i, e_j] = \begin{cases} (i-j)e_{i+j}, & \text{if } i+j \leqslant n \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to check that these structural constants do in fact define a Lie algebra over F and that this Lie algebra is of maximal class whenever $2 \leqslant \dim L = n \leqslant p+1$. In addition, one may verify that this Lie algebra admits the structure of a restricted Lie algebra by defining the p-map to be trivial in the case $n < p+1$, and by setting $e_1^p = e_p$ and $e_i^p = 0$ for $i > 1$ when $n = p+1$.

One might wonder whether or not the derived length of a restricted Lie algebra of maximal class can be uniformly bounded for all $p>0$. However, the derived length of the restricted Lie algebra just constructed is approximately $\log_2 n$, and thus can be taken to be arbitrarily large (as p increases).

The more general problem of obtaining precise bounds on restricted Lie algebras of a given finite coclass greater than 1 is briefly discussed in Section 5.

Before closing this section, we would like to make it clear that the techniques employed below were heavily influenced by those of Blackburn in [**2**].

2. The degree of commutativity

Let F be a field of characteristic $p>0$. Throughout the remainder of this section, L represents a finite-dimensional (ordinary) Lie algebra over F of maximal class. We intend to adapt some group-theoretic notions originally due to Blackburn, [**2**], to our present needs. Define L_1 to be the centraliser of $\gamma_2(L)/\gamma_4(L)$ in L, and write $L_i = \gamma_i(L)$ for each $i>1$. The degree of commutativity of L, $\delta = \delta(L)$, is said to be positive if

$$[L_i, L_j] \leqslant L_{i+j+1}$$

for all $i, j \geqslant 1$. The key result of this note is as follows.

Theorem A. *Suppose that $p>2$ and the dimension of L is an odd number at most $2p+1$. Then $\delta > 0$.*

The proof of Theorem A is contained in the next section. We develop some background machinery below.

LEMMA 2.1. *Suppose that* $\dim L \geqslant 4$. *Then* $\dim L/L_1 = 1$

PROOF: We may write $L_2 = Fa + L_3$ and $L_3 = Fb + L_4$. Let $x \in L$. Then $[a,x] \in L_3$, so that

$$[a,x] \equiv \alpha b \mod L_4$$

for some $\alpha = \alpha(x) \in F$. Define a linear map $\eta : L \to F$ by $\eta(x) = \alpha$. The kernel of η is L_1, so $\dim L/L_1 \leqslant 1$. Finally, $L \neq L_1$, for otherwise $L_3 = [L_2, L] \leqslant L_4$, contrary to our assumption that $\dim L \geqslant 4$. ∎

We shall abbreviate Engel commutators by

$$[x,_m y] := [x, y, y, \ldots, y]$$

where the y appears on the right hand side exactly m times.

LEMMA 2.2. *Suppose that* $\dim L = n \geqslant 5$ *and* $\delta(L/L_{n-1}) > 0$. *Choose elements* $s \in L \setminus L_1$ *and* $s_1 \in L_1 \setminus L_2$, *and set* $s_i = [s_1,_{i-1} s]$ *for each* $i = 2, \ldots, n-2$. *Then* $L = \langle s, s_1 \rangle$, $L_i = Fs_i + L_{i+1}$ *and*

$$[s_i, s_{n-i-1}] = (-1)^{i-1}[s_1, s_{n-2}]$$

for each $i = 1, 2, \ldots, n-2$.

PROOF: The elements s and s_1 exist by the previous lemma. Since L is nilpotent, L_2 coincides with the set of non-generators of L (recall that L is an ordinary Lie algebra in this section); therefore s and s_1 generate L. Clearly $L_1 = Fs_1 + L_2$. Assume by induction that $L_{i-1} = Fs_{i-1} + L_i$. Then

$$\begin{aligned} L_i &= F[s_{i-1}, s] + F[s_{i-1}, s_1] + L_{i+1} \\ &= Fs_i + L_{i+1} \end{aligned}$$

since $\delta(L/L_{n-1}) > 0$ implies $[s_{i-1}, s_1] \in [L_{i-1}, L_1] \leqslant L_{i+1}$ for $i \leqslant n-2$.

Suppose $i \leqslant n-i-1$. Then using the Jacobi identity we have

$$\begin{aligned} [s_i, s_{n-i-1}] &= \big[s_i, [s_{n-i-2}, s]\big] \\ &= [s, s_{n-i-2}, s_i] \\ &= [s_i, s_{n-i-2}, s] + [s, s_i, s_{n-i-2}] \\ &= -[s_{i+1}, s_{n-i-2}] \end{aligned}$$

because $[s_i, s_{n-i-2}, s] \in [L_{n-1}, L] = 0$. A simple induction argument now proves the lemma. ∎

Lemma 2.3. *Suppose that* $\dim L = n \geqslant 4$ *and that*

$$[L_1, L_i] \leqslant L_{i+2}$$

for $i = 1, 2, \ldots, n-2$. *Then* $\delta > 0$.

Proof: This is trivial if $n = 4$. Suppose then that $n > 4$. We shall proceed by induction on n. Thus we may assume that $\delta(L/L_{n-1}) > 0$. It remains to show

$$[L_i, L_{n-i-1}] \leqslant L_n = 0$$

for $i = 2, 3, \ldots, n-3$ since $[L_1, L_{n-2}] = 0$ by hypothesis. But by Lemma 2.2 we have

$$\begin{aligned}[L_i, L_{n-i-1}] &= [Fs_i + L_{i+1}, Fs_{n-i-1} + L_{n-i}] \\ &= F[s_i, s_{n-i-1}] \\ &= F[s_1, s_{n-2}] \\ &= 0\end{aligned}$$

for $2 \leqslant i \leqslant n-3$ since then $2 \leqslant n-i-1 \leqslant n-3$, as well. ∎

Lemma 2.4. *Suppose that* $\dim L = n \geqslant 5$ *and* $\delta(L/L_{n-1}) > 0$. *Then the following statements hold.*

1. *If* n *is odd, then* $\delta > 0$.
2. *If* n *is even, then* $\delta > 0$ *precisely when* $L_{(n/2)-1}$ *is Abelian.*

Proof: By the previous result it follows that $\delta > 0$ if and only if $[L_1, L_{n-2}] = 0$. By Lemma 2.2, this is equivalent to $[s_1, s_{n-2}] = 0$. But if n is odd, then

$$[s_1, s_{n-2}] = (-1)^{(n-1)/2}[s_{(n-1)/2}, s_{(n-1)/2}] = 0.$$

On the other hand, if n is even, then

$$[s_1, s_{n-2}] = (-1)^{(n/2)-1}[s_{(n/2)-1}, s_{n/2}].$$

Therefore $[s_1, s_{n-2}] = 0$ if and only if $L_{(n/2)-1} = Fs_{(n/2)-1} + L_{n/2}$ is Abelian. ∎

3. Proof of Theorem A

Assume that $p > 2$ and $\dim L = n$ is odd with $n \leqslant 2p + 1$. Since automatically $\delta(L/L_4) > 0$, the result follows for the case $n \leqslant 5$ from Lemma 2.4. For $n > 5$, we use induction on n. Assume then that $\delta(L/L_{n-2}) > 0$. From Lemma 2.4, we may assume, to the contrary, that $\delta(L/L_{n-1}) = 0$. Consequently, Lemma 2.3 implies that $[L_1, L_{n-3}]$ is not contained in L_{n-1}; in other words, s_1 does not centralise L_{n-3} modulo L_{n-1}. Because $L_{n-3} = Fs_{n-3} + L_{n-2}$, it follows that $t_{n-2} := [s_{n-3}, s_1]$ generates L_{n-2} modulo L_{n-1}.

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Since $\delta(L/L_{n-2}) > 0$, we may apply Lemma 2.2 to L/L_{n-1} to get $[s_i, s_j] \in L_{i+j+1}$ if $i + j \leqslant n - 3$ and $[s_i, s_{n-i-2}] = (-1)^{i-1}[s_1, s_{n-3}]$ for $i = 1, 2, \ldots, n-3$. Thus $t_{n-2} = [s_{n-3}, s_1] = [s_2, s_{n-4}]$. Using the Jacobi identity we obtain

$$\begin{aligned}[t_{n-2}, s_1] &= [s_2, s_{n-4}, s_1] \\ &= [s_1, s_{n-4}, s_2] + [s_2, s_1, s_{n-4}] \\ &= 0\end{aligned}$$

since $[s_1, s_{n-4}] \in L_{n-2}$ and $[s_2, s_1] \in L_4$. It follows that $[L_1, L_{n-2}] \leqslant L_n = 0$, and hence that $t_{n-1} := [t_{n-2}, s]$ generates L_{n-1}.

We now use induction to prove

$$[s_i, s_{n-i-1}] = (-1)^{i-1}(i-1)t_{n-1}$$

for $i = 2, 3, \ldots, n-3$. Indeed, let $i = 2$. Then because $[L_{n-2}, L_1] = 0$,

$$\begin{aligned}[s_2, s_{n-3}] &= [s_1, s, s_{n-3}] \\ &= [s_{n-3}, s, s_1] + [s_1, s_{n-3}, s] \\ &= [-t_{n-2}, s] \\ &= -t_{n-1}.\end{aligned}$$

Suppose now that

$$[s_{i-1}, s_{n-i}] = (-1)^{i-2}(i-2)t_{n-1}.$$

Then

$$\begin{aligned}[s_i, s_{n-i-1}] &= [s_{i-1}, s, s_{n-i-1}] \\ &= [s_{n-i-1}, s, s_{i-1}] + [s_{i-1}, s_{n-i-1}, s] \\ &= [s_{n-i}, s_{i-1}] + [s_{i-1}, s_{n-i-1}, s] \\ &= -[s_{i-1}, s_{n-i}] + [(-1)^{i-2}[s_1, s_{n-3}], s] \\ &= (-1)^{i-1}(i-2)t_{n-1} + (-1)^{i-1}t_{n-1} \\ &= (-1)^{i-1}(i-1)t_{n-1}\end{aligned}$$

as required.

Setting $i = (n-1)/2$ in the identity just proved yields

$$0 = [s_{(n-1)/2}, s_{(n-1)/2}] = (-1)^{(n-3)/2}\Big[\frac{1}{2}(n-3)\Big]t_{n-1},$$

contrary to our choice of p or n. ∎

Let us note the following corollary.

Corollary 3.1. *If* $\dim L = n$ *is even and* $6 \leqslant n \leqslant 2p+2$, *then* $\delta > 0$ *precisely when* $L_{(n/2)-1}$ *is Abelian.*

Proof: Applying Theorem A to L/L_{n-1} we find that $\delta(L/L_{n-1}) > 0$. The result now follows from Lemma 2.4. ∎

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4. Proof of Theorem B

In this section, we also assume that L admits the structure of a restricted Lie algebra. We shall require one last lemma.

Lemma 4.1. *Let x be an element of L. Then x^p lies in L_p.*

Proof: First notice that $x^p \in L_1$. Indeed, let a be an element of L_2. Then

$$[a, x^p] = [a,_p x] \in L_{p+2} \leqslant L_4.$$

Now let i be maximal such that $x^p \in L_i$. Then $L_i = Fx^p + L_{i+1}$, so that

$$L_{i+1} = [L, x^p] + L_{i+2} = [L,_p x] + L_{i+2} \leqslant L_{p+1} + L_{i+2}.$$

Therefore $i \geqslant p$. ∎

Assume that the conclusion of Theorem B is false. Then there exists a restricted Lie algebra L of maximal class and dimension precisely $p+2$. If $p = 2$ then $L_4 = 0$, and so $\delta > 0$ automatically. Otherwise, Theorem A guarantees that the degree of commutativity of L is positive (since $p+2$ is odd). From Section 2, we know

$$s_{p+1} = [s_1,_p s] = [s_1, s^p]$$

generates L_{p+1}. However, s^p lies in L_p by Lemma 4.1, so that $\delta > 0$ forces

$$s_{p+1} = [s_1, s^p] \in [L_1, L_p] \leqslant L_{p+2} = 0,$$

the desired contradiction. ∎

5. Another coclass conjecture

Recall from [**4**] that the coclass of a finite dimensional nilpotent Lie algebra is the difference of its dimension and nilpotency class, so that a Lie algebra is of coclass 1 precisely when it is of maximal class. The definition extends naturally to infinite dimensional Lie algebras as in the coclass 1 case. It was shown in [**4**] that restricted Lie algebras with finite coclass r have dimension at most $2p^r + r + 1$ if p is odd, and $6 \cdot 2^r + r + 1$ if $p = 2$. This result is an analogue of the so-called 'coclass conjectures' for p-groups (see [**6**] for a complete overview).

In light of Theorem B, we pose the following problem:

Problem. Suppose that a restricted Lie algebra L over a field of characteristic $p > 0$ has finite coclass r. Is it true that this implies that

$$\dim L \leqslant p^r + r?$$

If so, then this bound would be tight: see [**5**, 2.1] for examples of restricted Lie algebras of coclass r and dimension $p^r + r$, for each $r \geqslant 1$.

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