ON NUMBERS GENERATED BY $e^{s(e^x - 1)}$

Chinthayamma and J.M. Gandhi

The numbers generated by $e^{s(e^x-1)}$ where s is an integer are discussed by many for $s = 1$. The general case also is discussed by Touchard, Riordan and others. It is proved that

$$(1) \qquad e^{s(e^x-1)} = \sum_{n=o}^{\infty} \frac{x^n}{n!} \left(\sum_{k=o}^{n} S(n,k) s^k \right)$$

where $S(n, k)$ is the stirling number of the second kind defined by

$$(t)_n = \sum_{k=o}^{n} S(n,k)\,(t)_k \ , \quad n > 0$$

$$S(o,o) = 1, \quad S(n,o) = o \ \text{ for all } \ n \geq 1$$

$$\text{and } \ (t)_k = t(t-1)(t-2)\ \ldots\ (t-k+1)$$

Setting $a_n(s) = \sum_{k=o}^{n} S(n,k) s^k$, the first few values of $a_n(s)$ have been evaluated and also some of the algebraic and number theoretic properties of these numbers have been discussed.

Here we prove the following theorems which are believed to be new.

THEOREM 1. $a_{kp^t+n}(s) \equiv a_n(s) \ (a(s) + s^p t)^k \bmod p$

where k and t are integers and p is any prime.

https://doi.org/10.4153/CMB-1967-081-3 Published online by Cambridge University Press

THEOREM 2. $$a_{\sum_r k_r p^r}(s) \equiv \prod_r (a(s) + r s^p)^{k_r}, \quad (\text{mod } p)$$

THEOREM 3. $$a_{n+(p^p-1)/p-1}(s) \equiv s^p a_n(s) \quad (\text{mod } p)$$

In order to prove the theorems, we require the known result [1]

$$(2) \qquad a_{p+k}(s) \equiv a_{k+1}(s) + s^p a_k(s) \qquad (\text{mod } p)$$

Proof of Theorem 1. First we show if the assertion is true for fixed t, all n and $k = 1$, then it is true for the same t, all n and all k. Because let us assume that the theorem is true for some t and k and all n. Then

$$(3) \qquad a_{(k+1)p^t+n}(s) = a_{kp^t+(p^t+n)}(s)$$

$$\equiv a_{p^t+n}(s) \; (a(s) + ts^p)^k, \quad (\text{mod } p)$$

by induction hypothesis.

Since it is true for $k = 1$, the left hand member of (3)

$$\equiv a_n(s) \, (a(s) + ts^p) \, (a(s) + ts^p)^k, \quad (\text{mod } p)$$

$$\equiv a_n(s) \, (a(s) + ts^p)^{k+1}, \quad (\text{mod } p)$$

Now assume the truth of the theorem for all k and n and some t.

Then for $k = p$

$$a_{p^{t+1}+n}(s) \equiv a_n(s) \, (a(s) + ts^p)^p$$

$$\equiv (a_{n+p}(s) + a_n(s) t^p s^{p^2}), \quad (\text{mod } p)$$

$$\equiv a_{n+1}(s) + s^p a_n(s) + a_n(s) t^p s^{p^2}, \quad (\text{mod } p)$$

by (2)

$$\equiv a_{n+1}(s) + s^p a_n(s)(1+t^p), \quad (\text{mod } p)$$

$$\equiv a_{n+1}(s) + s^p a_n(s)(1+t), \quad (\text{mod } p)$$

$$\equiv a_n(s)(a(s) + s^p(1+t)), \quad (\text{mod } p)$$

which implies the validity of the theorem for all k, all n and $t+1$. But the theorem is true for $t = o$. Hence it is true for all t by induction.

It can be easily seen that the theorem 2 is followed by the repeated application of theorem 1.

Now we prove the theorem 3.

By theorem 2 we have

$$a_{n+(p^p-1)/p-1} = a_{p^{p-1}+p^{p-2}+\ldots+p+(1+n)}$$

$$\equiv a_{n+1}(s)(a(s) + s^p)(a(s) + 2s^p) \ldots\ldots\ldots$$

$$\ldots.(a(s) + (p-1)s^p) \quad (\text{mod } p)$$

$$\equiv a_{n+p}(s) - a_{n+1}(s), \quad (\text{mod } p)$$

$$\equiv s^p a_n(s) \quad (\text{mod } p) \text{ by } (2)$$

COROLLARY. For $p = 2$

$$a_{n+3}(s) \equiv s^2 a_n(s) \quad (\text{mod } 2).$$

https://doi.org/10.4153/CMB-1967-081-3 Published online by Cambridge University Press

REFERENCES

1. John Riordan, An introduction to combinatorial analysis, New York, Wiley [1958].

2. J. Touchard, Propertes arithmetiques de certains nombres recurrents, Ann. Soc. Sci., Bruxells, Vol. A53 (1933) pp. 21-31.

3. G.T. Williams, Numbers generated by the function $e^{e^x - 1}$, American Mathematical Monthly, Vol. 52 (1945) pp. 323-327.

University of Alberta
Edmonton

https://doi.org/10.4153/CMB-1967-081-3 Published online by Cambridge University Press