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BOOK REVIEW

BALL, J. M., (ed) *Systems of nonlinear partial differential equations* (D. Reidel, Dordrecht, 1983), ix+481 pp., Dfl. 150.

During the last decade quite spectacular advances have been made in the theory of non-linear partial differential equations. These advances have not led to a complete understanding of broad classes of problems but rather to a deeper understanding and in some cases a complete solution of particular problems. In the guiding spirit of analysis most of the non-linear partial differential equations studied are intimately connected with their physical or biological origins. These include the Navier–Stokes equations, the conservation laws of continuum physics and the reaction-diffusion equations of biology. In many instances progress has been achieved through the development of new techniques; e.g. weak convergence and compensated compactness or the recruitment of methods and ideas from areas somewhat removed from analysis; e.g. the topological idea of isolated invariant sets and generalised index theory.

This volume edited by John Ball presents these major developments in a series of expository papers (Part I) by authors who have made profound contributions to the theory of partial differential equations. In Part II more specialised topics are treated.

Unfortunately space does not allow for a detailed assessment of all the topics and ideas discussed in this important volume. For this reason I shall concentrate on the expository papers and indicate their links with the special sessions papers of Part II.

The article by Charles Conley and Joel Smoller studies the application of algebraic and topological invariants to reaction-diffusion equations. Prominent in this field is the concept of an isolated invariant set and its associated index. This index is a homotopy invariant and is a generalisation of the classical Morse index of an isolated rest point for a gradient flow in $\mathbb{R}^n$. Under reasonable hypotheses one can decompose an isolated invariant set into its so-called Morse sets. This implies the existence of certain exact sequences of cohomology groups defined on these Morse sets. In this way one obtains algebraic invariants associated with the given isolated invariant sets. Conley and Smoller show that in connection with reaction-diffusion equations the index carries 'dynamic' information and can be used to obtain precise statements about the dimensions of unstable manifolds, stability in general and bifurcation.

Reaction-diffusion equations are also considered by N. Alikakos, P. de Mottoni and J. K. Hale. The paper by Alikakos considers asymptotic behaviour of solutions to a system of reaction-diffusion equations with a continuum of equilibria. P. de Mottoni proves stability results for a class of degenerate parabolic problems in which the diffusivity vanishes beyond a given threshold.

J. K. Hale considers a particular parabolic-equation and gives results about the orbits which connect equilibrium points, the approach being based on the theory of dynamical systems and the maximum principle.

Despite the considerable progress in recent years, many of the fundamental problems in the analytic theory of systems of quasilinear hyperbolic equations arising from continuum mechanics remain. For example no strong a priori estimates have yet been discovered for systems of more than one equation so that the classical functional-analytic methods, so important for the study of other types of partial differential equations, are not amenable. It is only very recently that a functional-analytic treatment of these systems is emerging based on the theory of compensated compactness due to Tartar and Murat. Furthermore progress is also due to the development of the random choice method which constructs solutions to the Cauchy problem for systems in one space dimension under initial data of small total variation by monitoring the propagation and interaction of waves.

In his expository paper C. M. Dafermos presents an overview of important developments and discusses (i) nonexistence of classical solutions in the problem of breaking waves, (ii) the geometrical structure of solutions in the class of functions of bounded variation, (iii) non-uniqueness of weak solutions for Hopf's equation, and (iv) the construction of the solution to the Cauchy problem in one-space dimension by the random choice method.

Next we are treated to some interesting results concerning ill-posed problems in thermoelasticity theory by J. L. Ericksen. These include twinning of thermoelastic materials; problems for infinite elastic prisms and St.-Venant's problem for elastic prisms.

In the expository paper by L. C. Evans a study is made of the formal dynamic programming derivation of certain non-linear P.D.E. relevant to control theory. This is followed by a series of important papers by M. Giaquinta, E. Giusti and S. Hildebrandt which provide a connected account of regularity theory relating to non-linear elliptic systems and variational integrals. Many additional results concerning recent developments in the calculus of variations are given by B. Dacorogna and N. Fusco.

Bifurcation theory which is an essential tool with which to analyse non-linear boundary problems is nicely treated by D. G. Schaeffer. He begins by motivating the subject with some familiar examples, e.g. the buckling of a beam, and leading one through the ideas of Liapunov–Schmidt reduction. Schaeffer then goes on to discuss the concept of unfolding as developed by R. Thom and J. Mather. He explores these ideas in application to the problem of buckling of an arch. Recently group-theoretic methods have been applied to bifurcation theory and in his third lecture Schaeffer describes this application in relation to the effects of symmetry in bifurcation, the motivation being the Rivlin cube and planar Bénard connection.

Bifurcation is also the subject of the special session papers resulting from the session organised by J. E. Marsden. For example J. Carr studies phase transitions resulting from bifurcation from heteroclinic orbits, E. N. Dancer looks at bifurcation under continuous groups of symmetries and J. Mallet-Paret discusses bifurcation for singularly perturbed delay equations using the ideas of Morse decompositions developed by C. Conley. Also in this group J. E. Marsden considers bifurcation and stability for the traction problem while J. F. Toland studies singular elliptic eigenvalue problems.

One of the most striking and important developments in the study of non-linear partial differential equations is, as mentioned already, the idea of weak convergence and compensated compactness due to L. Tartar and F. Murat. The need for such methods is appropriately stated in the opening paragraphs of Tartar's article.

> 'One of the main difficulties in solving nonlinear partial differential equations lies in the following fact: after introducing a suitable sequence of approximations one needs enough a priori estimates to ensure the convergence of a subsequence to a solution; this argument is based on compactness results and in a nonlinear case one needs more estimates than in the linear case where weak continuity results can be used.
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> This lack of continuity in nonlinear problems has long restricted the use of weak convergence to linear problems; actually a careful analysis can render it more useful and this is the essence of the compensated compactness method. One of its most interesting applications is the recent work of R. Diperna of global existence for some 2×2 systems of conservation laws with bounded data.'

Tartar motivates these ideas through the equations of electrostatics and makes application to hyperbolic systems of conservation laws satisfying certain entropy conditions.

Space does not allow for a discussion of the remaining important papers in the special sessions. They are covered under the broad headings: problems in nonlinear elasticity; applications of bifurcation theory to mechanics; non-linear problems and phase transitions; dynamical systems and partial differential equations.

One can only give unqualified praise to this important contribution to the literature on applied analysis. The book is a must for those interested in exploring these new developments in the theory of partial differential equations.

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