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Shift equivalence and the Jordan form away from zero

MIKE BOYLE

IBM, Thomas J. Watson Research Center, P.O. Box 218, *Yorktown Heights,*
New York 10598, *USA* and *Mathematical Sciences Research Institute,*
2223 *Fulton Street, Room* 603, *Berkeley, CA* 94720, *USA*

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Abstract. Only finitely many shift equivalence classes of non-negative aperiodic integral matrices may share a given diagonal Jordan form away from zero. The diagonal assumption is necessary.

0. *Introduction*

Subshifts of finite type are fundamental in applications of symbolic dynamics to diffeomorphisms [**3**], ergodic theory [**4**], and coding theory [**2**]. These subshifts are defined by square non-negative integral matrices [**16**], and may be studied by way of invariants of matrices which define topologically conjugate subshifts. We will refer to such invariants as 'invariants of conjugacy'.

In a fundamental paper [**16**], Williams introduced two equivalence relations on square non-negative integral matrices: shift equivalence and strong shift equivalence. Strong shift equivalence is a complete but highly non-computable invariant of conjugacy. Shift equivalence is an invariant of conjugacy which is conjectured to be complete and which is more computable–it is often practical to decide if two matrices are shift equivalent, and significant partial results suggest there is a general decision procedure [**7**]. In addition, from Krieger's work we find shift equivalence intimately related to dimension groups [**9**] and the construction of factor maps [**10**].

The Jordan form away from zero is a strictly weaker invariant of conjugacy. It is still a very strong one; it determines the entropy and zeta function, classifies irreducible subshifts of finite type up to 'stable weak isomorphism' [**10**] and severely constrains equal-entropy factors [**8**]. Moreover, it is completely computable. To understand the structure of the class of subshifts of finite type, we should in particular understand how shift equivalence refines this Jordan equivalence relation.

In this paper, we find that only finitely many shift equivalence classes of aperiodic matrices may share a given diagonal Jordan form away from zero. On the other hand, examples are provided in which infinitely many aperiodic non-negative integral matrices, pairwise not shift equivalent, share the same (non-diagonal) Jordan form away from zero.

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1. *Definitions and background*

We need only consider matrices. For a thorough introduction to subshifts of finite type, see [**1**], [**4**] and [**12**]. For shift equivalence and strong shift equivalence, a good introduction is Ch. V, Sec. III of [**12**].

Let A and B be square integral matrices (not necessarily of equal size), and let $\mathcal{S}$ be one of the semi-rings $\mathbb{Z}$, $\mathbb{Q}$ or the non-negative integers $\mathbb{Z}^+$. Write $A \lhd_{\mathcal{S}} B$ if there exist matrices R, S with entries from $\mathcal{S}$ such that $SR = A$, $RS = B$. If there is a finite string $A \lhd_{\mathcal{S}} A_1 \lhd_{\mathcal{S}} \cdots \lhd_{\mathcal{S}} B$, then A and B are *strong shift equivalent* over $\mathcal{S}$ (SSE-$\mathcal{S}$). A and B are *shift equivalent* over $\mathcal{S}$ (SE-$\mathcal{S}$) if there exist matrices R, S with entries from $\mathcal{S}$ such that

(1.1) $RA = BR$, $AS = SB$, $SR = A^l$ and $RS = B^l$, for some $l > 0$.

The integer l in (1.1) is called the *lag* of the shift equivalence. Whenever the semi-ring $\mathcal{S}$ is not specified, $\mathcal{S}$ is intended to be $\mathbb{Z}^+$. Strong shift equivalence over $\mathcal{S}$ implies shift equivalence over $\mathcal{S}$; the converse is true for $\mathcal{S} = \mathbb{Z}$ or $\mathbb{Q}$ ([**16**]) and conjectured for $\mathcal{S} = \mathbb{Z}^+$.

The matrix $J^*(A)$ obtained by removing from the complex Jordan form of A all rows and columns with zero on the diagonal is the *Jordan form away from zero* of A. We (must) consider $J^*(A)$ and $J^*(B)$ to be the same if there exists a permutation matrix P such that

$$PJ^*(A)P^{-1} = J^*(B).$$

A and B are shift equivalent over $\mathbb{Q}$ if and only if $J^*(A) = J^*(B)$ ([**5**, th. 6.5]). In particular, matrices shift equivalent over $\mathbb{Z}$ or $\mathbb{Z}^+$ have the same Jordan form away from zero; the converse fails easily ([**16**, example 3]).

The characteristic polynomial of A has a unique factorization $x^n p_A(x)$ for which $p_A(0)$ is non-zero; p_A, the *characteristic polynomial of* A mod x, is an invariant of conjugacy. The periodic point counts of the subshift of finite type defined by A determine p_A, and vice versa. Obviously, $J^*(A)$ determines p_A.

A and B are *similar over* $\mathcal{S}$ if there exist matrices U, U^{-1} over $\mathcal{S}$ such that $UAU^{-1} = B$. If A and B are similar over $\mathbb{Z}$, then they are shift equivalent over $\mathbb{Z}$, with $R = UA$ and $S = U^{-1}$ in (1.1). In general, for non-negative integral A and B, similarity over $\mathbb{Z}$ does not imply shift equivalence, and shift equivalence does not imply similarity over $\mathbb{Z}$. The *similarity class* of a square integral matrix A is the set of all matrices similar to A over $\mathbb{Z}$.

We will need two basic facts about integral matrices.

(1.2) Theorem ([**11**, Ch. III, Sec. 15]). *Let A be a square integral matrix with characteristic polynomial $p_1p_2 \cdots p_n$, where each p_i is a monic polynomial with integral coefficients which is irreducible over $\mathbb{Q}$. Then A is similar over $\mathbb{Z}$ to a matrix in block lower triangular form, where the characteristic polynomial of the i'th diagonal block is p_i.*

(1.3) Theorem ([**11**, Ch. III, Sec. 16)]. *There are only finitely many similarity classes of integral matrices A such that $f(A) = 0$, where $f(x)$ is a monic polynomial with integral coefficients which is irreducible over $\mathbb{Q}$.*

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If A is a non-negative n by n matrix and every entry of some positive power of A is positive, then A is *aperiodic*; if, given integers i and j with $1 \leq i, j \leq n$, there is some positive m such that $(A^m)_{ij}$ is positive, then A is *irreducible*. The following observation, perhaps first made in [**13**], is basic to the sequel.

(1.4) Aperiodic non-negative square integral matrices are shift equivalent over $\mathbb{Z}^+$ if and only if they are shift equivalent over $\mathbb{Z}$.

For a succinct proof of (1.4), see (2.1) of [**7**]. Example (2.13) of the next section, obtained with I. Kaplansky, shows that (1.4) cannot be extended to irreducible matrices. This corrects remark 4 in § 5 of [**13**].

2. *The finiteness result*

In this section we find that only finitely many shift equivalence classes of aperiodic matrices may share a given diagonal Jordan form away from zero. For aperiodic matrices, we may neglect positivity requirements and work with shift equivalence over $\mathbb{Z}$. Over $\mathbb{Z}$, it is enough to consider non-singular matrices with a fixed diagonal Jordan form. Then we find such matrices come from only finitely many similarity classes, hence from finitely many SE-$\mathbb{Z}$ classes. We close with some related results. In particular, we find infinitely many non-negative matrices shift equivalent over $\mathbb{Z}$ may be pairwise not shift equivalent over $\mathbb{Z}^+$.

Throughout this section, a matrix is integral unless specified otherwise. We must suffer a little notation. Let $M_{nk}(\mathcal{S})$ be the set of $n \times k$ matrices with entries from $\mathcal{S}$ ($\mathcal{S}$ will be $\mathbb{Z}$, $\mathbb{Z}^+$ or $\mathbb{Q}$). In the sequel, always take $A \in M_{nn}(\mathbb{Z})$, $B \in M_{kk}(\mathbb{Z})$ and

$$[A, B]_{\mathcal{S}} = \{YA - BY\colon\ Y \in M_{kn}(\mathcal{S})\}.$$

Notice that $[A, B]_{\mathbb{Q}}$ is a rational vector space in which $[A, B]_{\mathbb{Z}}$ is a lattice of full rank. Especially,

(2.1) the quotient group $([A, B]_{\mathbb{Q}} \cap M_{kn}(\mathbb{Z}))/[A, B]_{\mathbb{Z}}$ is finite.

(2.2) LEMMA. *Any non-nilpotent square integral matrix is shift equivalent over $\mathbb{Z}$ to a non-singular matrix.*

Proof. By (1.2), any non-nilpotent square integral matrix is similar over $\mathbb{Z}$ (hence SE-$\mathbb{Z}$) to one of the form

$$B = \left[\begin{array}{c|c} A & 0 \\ \hline X & N \end{array}\right],$$

where A is non-singular and N is nilpotent. Pick l such that $N^l = 0$, so

$$B^l = \left[\begin{array}{c|c} A^l & 0 \\ \hline Y & 0 \end{array}\right]$$

for some Y. Define

$$R = \left[\begin{array}{c} A^l \\ \hline Y \end{array}\right], \qquad S = [I \mid 0].$$

Then the only non-trivial verification of the equations (1.1) involves $RA = BR$, which requires $YA = XA^l + NY$, which follows from

$$\left[\begin{array}{c|c} A & 0 \\ \hline X & N \end{array}\right]\left[\begin{array}{c|c} A^l & 0 \\ \hline Y & 0 \end{array}\right] = B^{l+1} = \left[\begin{array}{c|c} A^l & 0 \\ \hline Y & 0 \end{array}\right]\left[\begin{array}{c|c} A & 0 \\ \hline X & N \end{array}\right].$$ □

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(2.3) LEMMA. *Suppose A and B are square matrices with no common eigenvalues, and C is a matrix such that $AC = CB$. Then $C = 0$.*

Proof. Let C, A, B act as linear transformations on complex row vectors. Let $p(x)$ be the minimal polynomial of A. Then $0 = p(A)C = Cp(B)$. So, if $C \neq 0$, then the minimal polynomial of $B|_{\operatorname{Im} C}$ divides $p(x)$; therefore, A and B share an eigenvalue, a contradiction. □

(2.4) LEMMA. *Suppose $X - Y$ is in $[A, B]_{\mathbb{Z}}$. Then*

$$\left[\begin{array}{c|c} A & 0 \\ \hline X & B \end{array}\right] \quad \textit{and} \quad \left[\begin{array}{c|c} A & 0 \\ \hline Y & B \end{array}\right]$$

are similar over $\mathbb{Z}$.

Proof. For some Z in $M_{kn}(\mathbb{Z})$, $X - Y = ZA - BZ$, so

$$\left[\begin{array}{c|c} A & 0 \\ \hline X & B \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline Z & I \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline Z & I \end{array}\right]\left[\begin{array}{c|c} A & 0 \\ \hline Y & B \end{array}\right]. \qquad □$$

(2.5) LEMMA. *Suppose $C_1, \ldots, C_n$ are square integral matrices which pairwise have no common eigenvalues. Then only finitely many similarity classes may contain matrices of the block triangular form*

$$\begin{bmatrix} C_1 & & & 0 \\ \text{---} & C_2 & & \\ \text{---} & \text{---} & \ddots & \\ \text{---} & \text{---} & \text{---} & C_n \end{bmatrix}.$$

Proof. This is trivial for $n = 1$; suppose true for $n - 1$. Then there are finitely many matrices A_i such that for any integral matrix of the form

$$C = \begin{bmatrix} C_1 & & 0 \\ \text{---} & \ddots & \\ \text{---} & \text{---} & C_{n-1} \end{bmatrix},$$

there is an integral unimodular matrix U such that $UCU^{-1} = A_i$ for some i. Therefore, given some matrix

$$\begin{bmatrix} C_1 & & 0 \\ \text{---} & \ddots & \\ \text{---} & \text{---} & C_n \end{bmatrix} = \left[\begin{array}{c|c} C & 0 \\ \hline X & C_n \end{array}\right],$$

for some U we find a similar matrix

$$\left[\begin{array}{c|c} U & 0 \\ \hline 0 & I \end{array}\right]\left[\begin{array}{c|c} C & 0 \\ \hline X & C_n \end{array}\right]\left[\begin{array}{c|c} U^{-1} & 0 \\ \hline 0 & I \end{array}\right]$$

with one of finitely many forms

$$\left[\begin{array}{c|c} A_i & 0 \\ \hline - & C_n \end{array}\right].$$

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Now it suffices to show, for fixed A and B with no common eigenvalues, that only finitely many similarity classes may contain matrices of the form

$$\left[\begin{array}{c|c} A & 0 \\ \hline X & B \end{array}\right].$$

By (2.4), it is enough to show there are only finitely many elements in the quotient group $M_{kn}(\mathbb{Z})/[A, B]_{\mathbb{Z}}$. This is true if and only if the lattice $[A, B]_{\mathbb{Z}}$ has full rank in $M_{kn}(\mathbb{Z})$. This is true if and only if the linear transformation $Z \mapsto ZA - BZ$ from $M_{kn}(\mathbb{Q})$ to itself has kernel zero. This follows from (2.3). □

The core of (2.5) was proved and used by Handelman in his study of stenotic extensions of certain dimension groups (see the remarks preceding III.7 of [**6**]).

(2.6) Lemma. *Suppose $p(x)$ in $\mathbb{Z}[x]$ is monic irreducible and n is a positive integer. Then the set of similarity classes of integral matrices with characteristic polynomial $p(x)^n$ and diagonal Jordan form is finite.*

Proof. We may suppose $p(x) \neq x$. The case $n = 1$ is (1.3). Suppose the claim is true for n, and C is diagonalizable with characteristic polynomial $\chi_C = p(x)^{n+1}$. By (1.2), C is similar over $\mathbb{Z}$ to a matrix of the form

$$\left[\begin{array}{c|c} A & 0 \\ \hline X & B \end{array}\right],$$

where $\chi_A = p(x)^n$ and $\chi_B = p(x)$. Since C is diagonalizable, so is A.

Let $\mathscr{A}$ ($\mathscr{B}$) be a finite, complete set of representatives of similarity classes of integral matrices with characteristic polynomial $p(x)^n$ ($p(x)$) and diagonal Jordan form. Possibly after replacing C with

$$\left(\begin{array}{c|c} U & 0 \\ \hline 0 & V \end{array}\right)\left(\begin{array}{c|c} A & 0 \\ \hline X & B \end{array}\right)\left(\begin{array}{c|c} U^{-1} & 0 \\ \hline 0 & V^{-1} \end{array}\right),$$

where $UAU^{-1} \in \mathscr{A}$, $VBV^{-1} \in \mathscr{B}$ and $|\det U| = |\det V| = 1$, we may assume $A \in \mathscr{A}$, $B \in \mathscr{B}$. Now it suffices to prove the finiteness claim for fixed A and B. Since

$$\left(\begin{array}{c|c} A & 0 \\ \hline X & B \end{array}\right) \text{ and } \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)$$

are diagonalizable over $\mathbb{C}$, there exists an integral matrix

$$\left(\begin{array}{c|c} U & 0 \\ \hline Y & Z \end{array}\right)$$

invertible over $\mathbb{Q}$ such that

$$\left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right)\left(\begin{array}{c|c} U & 0 \\ \hline Y & Z \end{array}\right) = \left(\begin{array}{c|c} U & 0 \\ \hline Y & Z \end{array}\right)\left(\begin{array}{c|c} A & 0 \\ \hline X & B \end{array}\right).$$

(To justify the specified zero block in

$$\left(\begin{array}{c|c} U & 0 \\ \hline Y & Z \end{array}\right),$$

consider the matrices as linear transformations on column vectors. Corresponding to each B is a natural invariant subspace of dimension k, the size of B. We may specify the zero block because we may require the conjugating transformation to map these B-subspaces to one another.)

Now Z must invert over $\mathbb{Q}$, and $BZ=ZB$; so $Z^{-1}B=BZ^{-1}$. Also $BY=YA+ZX$, so $ZX=-YA+BY$. Then

$$X=Z^{-1}ZX=Z^{-1}(-YA+BY)=(-Z^{-1}Y)A-B(-Z^{-1}Y).$$

Therefore, $X\in M_{kn}(\mathbb{Z})\cap[A, B]_{\mathbb{Q}}$. But recall (2.1), the quotient group

$$(M_{kn}(\mathbb{Z})\cap[A, B]_{\mathbb{Q}})/[A, B)_{\mathbb{Z}}$$

is finite. Now apply (2.4). □

(2.7) Lemma. *Suppose $p(x)$ is a monic polynomial with integer coefficients. Then the set of similarity classes of integral matrices with characteristic polynomial $p(x)$ and diagonal Jordan form is finite.*

Proof. Let $p(x)=\prod_{i=1}^{k}(p_i(x))^{k_i}$, where the $p_i(x)$ are distinct and irreducible. By (1.2), an integral matrix with characteristic polynomial $p(x)$ is similar over $\mathbb{Z}$ to one with the block triangular form

$$C=\begin{bmatrix} C_1 & & & 0 \\ \text{---} & C_2 & & \\ \text{------} & & \ddots & \\ \text{--------} & & & C_n \end{bmatrix},$$

where C_i has characteristic polynomial $p_i(x)^{k_i}$, $1\le i\le n$.

Let $q(x)$ be the minimal polynomial of C; since C is diagonalizable, q has no repeated roots. Since the ith diagonal block of $q(C)$ is $q(C_i)$, the minimal polynomial of C_i has no repeated roots, so C_i is diagonalizable, $1\le i\le n$. By (2.6) we may specify (by passing to a similar matrix as in the proof of (2.6)) that the n-tuple $(C_1,\dots,C_n)$ come from a finite set. Now apply (2.5). □

(2.8) Theorem. *There are only finitely many* SE-$\mathbb{Z}$ *classes of integral matrices with a given diagonal Jordan form away from zero.*

Proof. By (2.2), it suffices to consider matrices with a fixed diagonal non-singular Jordan form. By (2.7), only finitely many similarity classes contain such matrices. Since similarity over $\mathbb{Z}$ implies shift equivalence over $\mathbb{Z}$, the SE-$\mathbb{Z}$ classes containing such matrices are formed by some clumping of these similarity classes. □

(2.9) Corollary. *There are only finitely many* SE-$\mathbb{Z}^+$ *classes containing aperiodic non-negative matrices with a given diagonal Jordan form away from zero.*

Proof. Apply (1.4). □

Corollary (2.10) below is an immediate consequence of (2.9) and Kitchens' Jordan form theorem – see [8] for proof and definitions. Proposition (2.11) below, an elaboration of (3.1), gives the converse of (2.8).

I do not know if (2.9) is true without the hypothesis of aperiodicity. In general, the refinement of SE-$\mathbb{Z}$ classes into SE-$\mathbb{Z}^+$ classes is complicated. Sometimes the refinement is finite, sometimes not; (2.12) below exhibits an infinite refinement. For

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a penetrating analysis of when a shift equivalence over $\mathbb{Z}$ induces a shift equivalence over $\mathbb{Z}^+$, see [**7**].

(2.10) Corollary. *If S is a subshift of finite type defined by an aperiodic matrix with diagonal Jordan form away from zero, then only finitely many shift equivalence classes contain matrices which define equal entropy finite type factors of S.*

(2.11) Proposition. *Suppose $p(x)$ is a monic polynomial with integral coefficients with a repeated non-zero root. Then there are infinitely many matrices, pairwise not shift equivalent over $\mathbb{Z}$, with characteristic polynomial $p(x)$.*

Proof. Let $p(x)=[q(x)]^n r(x)$, where $q(x)$ is irreducible monic, n is greater than 1, and $q(x)$ does not divide $r(x)$. Let A be the companion matrix of $q(x)$, B the companion matrix of $r(x)$. Given $k\in\mathbb{Z}$, let C_k be the matrix

$$\left[\begin{array}{ccccc|c} A & & & & & \\ kI & A & & & & \\ & kI & \cdot & & & 0 \\ & & & \cdot & & \\ & & & kI & A & \\ \hline & & 0 & & & B \end{array}\right]$$

with characteristic polynomial $p(x)$.

It is an exercise to show that shift equivalence over $\mathbb{Z}$ of C_k and C_j forces a shift equivalence of

$$\begin{pmatrix} A & 0 \\ kI & A \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A & 0 \\ jI & A \end{pmatrix}$$

by matrices R, S in block lower triangular form, e.g.

$$R=\begin{pmatrix} U & 0 \\ Z & V \end{pmatrix}.$$

In particular,

$$\left(\begin{array}{c|c} A & 0 \\ \hline kI & A \end{array}\right)\left(\begin{array}{c|c} U & 0 \\ \hline Z & V \end{array}\right)=\left(\begin{array}{c|c} U & 0 \\ \hline Z & V \end{array}\right)\left(\begin{array}{c|c} A & 0 \\ \hline jI & A \end{array}\right).$$

Thus, U and V commute with A, so they are in the field $\mathbb{Q}[A]$ (see [**15**]). Therefore, $AZ-ZA=jV-kU$ is in $\mathbb{Q}[A]$. But if $0\neq AZ-ZA=C\in\mathbb{Q}[A]$, then $I=C^{-1}C=C^{-1}AZ-C^{-1}ZA=A(C^{-1}Z)-(C^{-1}Z)A$; this is a contradiction, since the trace of $A(C^{-1}Z)-(C^{-1}Z)A$ must be zero. Therefore, $jV=kU$. Then $SR=A^l$ forces det U and det V to divide some power of det A. So, if C_j and C_k are shift equivalent over $\mathbb{Z}$, and j, k are positive integers relatively prime to det A, then $j=k$. □

(2.12) *Example.* Infinitely many SE-$\mathbb{Z}^+$ classes may be contained in one SE-$\mathbb{Z}$ class.

Proof. If c is a positive integer, let

$$S_c=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1+c & c & 1 \end{bmatrix}, \qquad R_c=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -c & 1 \end{bmatrix}.$$

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Then

$$S_cR_c = A_c = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2c+2 & 0 & 1 \end{bmatrix}, \qquad R_cS_c = B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

So all the A_c are SE-$\mathbb{Z}$. It suffices to show they are pairwise not SE-$\mathbb{Z}^+$. If S, R gives a shift equivalence over $\mathbb{Z}^+$ of A_c and A_d, then R and S must take the form

$$\left[\begin{array}{c|c} 1 & 0 \\ \hline x & \\ y & P \end{array}\right],$$

where

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In R (for example), the prescribed zero block follows from evaluation of both sides of $RA_c = A_dR$ at column eigenvectors of A_c. Then 1 and P are forced on the diagonal by $SR = (A_c)^l$. Now it is easy to check that $RA_c = A_dR$ forces $c = d$. □

(2.13) *Example* (with I. Kaplansky). For irreducible non-negative integral matrices, shift equivalence over $\mathbb{Z}$ does not imply shift equivalence over $\mathbb{Z}^+$.

Proof. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 3 & 5 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 3 & 5 & 1 & 2 \\ 1 & 2 & 0 & 1 \\ 7 & 12 & 3 & 5 \end{bmatrix}.$$

Then $AC = CB$ and $\det C = 1$, so A and B are SE-$\mathbb{Z}$. We claim they are not SE-$\mathbb{Z}^+$. Suppose they are. Then

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 6 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$$

must also be SE-$\mathbb{Z}^+$. This can be seen dynamically, or directly as follows. Let non-negative S and R satisfy the equations (1.1) of shift equivalence with lag l. Let

$$S' = A^4SB^4, \qquad R' = B^4RA^{l+4}.$$

Then A^2 and B^2 are SE-$\mathbb{Z}^+$ by S', R' with lag $l+8$. In each 2×2 corner block of S' and R', the entries are all positive or all zero. The matrices R' and S' must share one of two block sign patterns,

$$\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}.$$

Now positive blocks from R' and S' can be used for a shift equivalence of D and E. Since D and E are unimodular, their shift equivalence implies similarity over $\mathbb{Z}$, say

$$\begin{bmatrix} 0 & 1 \\ 1 & 6 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix},$$

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with $ps-rq=\pm 1$. Here $r=p+3q$ and $s=2p+5q$, so

$$p(2p+5q)-(p+3q)q=2p^2+4pq-3q^2=\pm 1,$$

hence $2(p+q)^2-5q^2=\pm 1$.

The last equation is impossible mod 5. In $\mathbb{Z}/5\mathbb{Z}$, 1 and -1 are squares, but 2 is not a square. This gives the contradiction.

Alternatively, the non-similarity of D and E over $\mathbb{Z}$ can be seen from the well known connection between integral matrices and ideals in algebraic number fields: the ideal corresponding to D is principal, while the ideal corresponding to E is not. □

3. *Examples*

We will produce a sequence $\{A_n\}$ of aperiodic non-negative integral matrices, pairwise not shift equivalent, with the same (non-diagonal) Jordan form away from zero. Notice that the size of A_n must go to infinity with n, as only finitely many non-negative irreducible integral matrices of a given size may satisfy a finite upper bound on the spectral radius.

(3.1) LEMMA. *Suppose a, j and k are non-zero integers, j and k are each relatively prime to a, and $|j|\neq|k|$. Then*

$$\begin{pmatrix} a & 0 \\ j & a \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} a & 0 \\ k & a \end{pmatrix}$$

are not shift equivalent over $\mathbb{Z}$.

Proof. Suppose

$$\begin{pmatrix} a & 0 \\ j & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 0 \\ k & a \end{pmatrix}$$

are shift equivalent over $\mathbb{Z}$ by matrices S and R; say

$$R=\begin{pmatrix} b & c \\ d & e \end{pmatrix}$$

and

$$\begin{pmatrix} a & 0 \\ j & a \end{pmatrix}\begin{pmatrix} b & c \\ d & e \end{pmatrix}=\begin{pmatrix} b & c \\ d & e \end{pmatrix}\begin{pmatrix} a & 0 \\ k & a \end{pmatrix}.$$

Then $ab=ba+ck$, so $c=0$. Since det SR is a power of a, the numbers b and e must be units or products of primes dividing a. But $jb+ad=da+ek$, so $jb=ek$; therefore $|j|=|k|$, a contradiction. □

We will let χ_A denote the characteristic polynomial of a matrix A.

(3.2) LEMMA. *Suppose integral matrices*

$$A=\left(\begin{array}{c|c} B & 0 \\ \hline C & D \end{array}\right) \quad \textit{and} \quad \bar{A}=\left(\begin{array}{c|c} \bar{B} & 0 \\ \hline \bar{C} & \bar{D} \end{array}\right)$$

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are shift equivalent over $\mathbb{Z}$, $\chi_B = \chi_{\bar{B}}$ and no root of χ_B is a root of χ_D or $\chi_{\bar{D}}$. Then B and $\bar{B}$ are shift equivalent over $\mathbb{Z}$.

Proof. Suppose S, R give a shift equivalence over $\mathbb{Z}$ of A and $\bar{A}$, with $SR = A^l$ etc. Let

$$S = \left(\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right) \quad \text{and} \quad R = \left(\begin{array}{c|c} \bar{E} & \bar{F} \\ \hline \bar{G} & \bar{H} \end{array}\right),$$

where the block pattern is induced by A and $\bar{A}$. Then $AS = S\bar{A}$ forces $BF = F\bar{D}$, and by (2.3) this forces $F = 0$. Likewise, $\bar{F} = 0$. Now the pair E, $\bar{E}$ gives a shift equivalence over $\mathbb{Z}$ between B and $\bar{B}$. □

(3.3) Lemma. *Suppose V is a proper vector subspace of $\mathbb{R}^n$, $\mathcal{B} = \{v_1, \ldots, v_k\}$ is an integral basis of $\mathbb{Z}^n \cap V$ and w is a vector in $\mathbb{Z}^n \sim V$. Let W be the linear span of V and $\{w\}$. Then there is a vector v_{k+1} such that $\{v_1, \ldots, v_k, v_{k+1}\}$ is an integral basis for $\mathbb{Z}^n \cap W$.*

Proof. Define a bijective linear transformation φ from W to $\mathbb{R}^{k+1}$ by sending v_i to the i'th canonical basis vector e_i, $1 \le i \le k$, and w to e_{k+1}. Now the image under φ of $\mathbb{Z}^n \cap W$ is a lattice $\mathcal{L}$ of rank $k+1$. It suffices to find a vector v in $\mathcal{L}$ such that $\{e_1, \ldots, e_k, v\}$ is an integral basis for $\mathcal{L}$.

Because $\mathcal{L}$ is a lattice, there is a positive number γ (the volume of a fundamental domain) such that, if $x \in \mathcal{L}$, then the volume of

$$C(x) = \left\{ \alpha x + \sum_1^k \alpha_i e_i : 0 \le \alpha, \alpha_i \le 1 \right\}$$

is an integral multiple of γ (see, e.g., ch. 6 of [**14**]). Therefore, we may pick v in $\mathcal{L}$ giving $C(v)$ with minimal positive volume. We claim $\mathcal{B}' = \{e_1, \ldots, e_k, v\}$ is an integral basis for $\mathcal{L}$.

Suppose not. Then there is some u in $\mathcal{L}$ but not in the integral span of $\mathcal{B}'$, with $u = \sum_1^n \alpha_i e_i + \alpha v$ for some α and α_i from $\mathbb{R}$. Now if α were an integer, then $\sum_1^n \alpha_i e_i$ would be in $\mathcal{L}$, and since $\mathcal{B}$ is an integral basis for V the α_i would be integers, contradicting $u \in \mathcal{L}$. So, α is not an integer. By replacing u with $u - [\alpha]v$, we may assume $0 < \alpha < 1$. Let ψ be a linear bijection of $\mathbb{R}^{k+1}$ which fixes $e_1, \ldots, e_k$ and maps v to u. Then ψ takes $C(v)$ onto $C(u)$. With respect to the basis $\{e_1, \ldots, e_k, v\}$, ψ is given by the matrix

$$A = \begin{bmatrix} 1 & 0 & & \alpha_1 \\ & \ddots & & \vdots \\ & & 1 & \alpha_k \\ 0 & & & \alpha \end{bmatrix}.$$

Since $|\det A| = \alpha < 1$, vol $C(v) >$ vol $C(u)$, a contradiction. □

(3.4) *An example.* Pick positive integers M and a, with $M \ge a + 3$. Given positive

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n, let B_n be the $(n+3)\times(n+3)$ matrix

$$\begin{bmatrix} a &&&&&&&& \\ 1 & 0 &&&&&&& \\ 1 & 1 & 0 &&&& \mathbf{0} && \\ & 1 & 1 & 0 &&&&& \\ && \cdot & \cdot & \cdot &&&& \\ &&& \cdot & \cdot & \cdot &&& \\ &&&& \cdot & \cdot & \cdot && \\ & \mathbf{0} &&& 1 & 1 & 0 && \\ &&&&& 1 & 1 & a & \\ &&&&&& 1 & 1 & M \end{bmatrix}$$

Rows 3 through $n+1$ of B_n have as their only non-zero entries two 1's to the left of the diagonal. The characteristic polynomial of B_n is $x^n(x-M)(x-a)^2$.

Let U_n be the $(n+3)\times(n+3)$ unimodular matrix whose entries on the diagonal and in the last column are all 1, and whose other entries are 0. Let $A_n = U_nB_n(U_n)^{-1}$. The matrix A_n is obtained by first adding the last row of B_n to each other row, then subtracting the first $n+2$ columns of the resulting matrix from the last column. A_n is aperiodic and non-negative. Since A_n is similar over $\mathbb{Z}$ to B_n, it is shift equivalent over $\mathbb{Z}$ to B_n. So it is enough to show that the B_n are pairwise not shift equivalent over $\mathbb{Z}$.

Let matrices act on row vectors, and let V_n be the two-dimensional kernel of $(B_n - aI)^2$. Let $v^{(n)}$ be the row vector whose first entry is 1 and whose remaining $n+2$ entries are zero. Clearly, $v^{(n)} \in \operatorname{Ker}(B_n - aI) \subset V_n$. Define integral row vectors $w^{(n)}$ inductively as follows.

Let $w^{(1)} = (0, 1, a, 0)$. For $n > 1$, let:

(*) $\quad w_1^{(n)} = 0,$

(**) $\quad w_2^{(n)} = w_2^{(n-1)} + w_3^{(n-1)},$

(***) $\quad w_i^{(n)} = aw_{i-1}^{(n-1)} \quad$ if $3 \le i \le n+3$.

For example,

$w^{(2)} = (0, a+1, a, a^2, 0),$

$w^{(3)} = (0, 2a+1, a^2+a, a^2, a^3, 0),$

$w^{(4)} = (0, a^2+3a+1, 2a^2+a, a^3+a^2, a^3, a^4, 0).$

Then the following hold for each $n \ge 1$.

(i) $w_{n+2}^{(n)} = a^n$;

(ii) $w_2^{(n)}$ and a are relatively prime;

(iii) $w_3^{(n)}$ is a positive multiple of a;

(iv) $w_2^{(n+1)} > w_2^{(n)} > 0$;

(v) $w^{(n)}(B_n - aI) = k_n v^{(n)}$, with $k_n = w_2^{(n+1)}$; in particular, $w^{(n)} \in V_n$.

It is easy to check (i)-(iv) by induction. For the induction step on (v), notice that the $(n+2)\times(n+2)$ submatrices in the lower right corners of $(B_n - aI)$ and $(B_{n+1} - aI)$ are equal, and above these submatrices all entries are zero. Therefore, since

$$w^{(n+1)} = (0, w_2^{(n+1)}, aw_2^{(n)}, aw_3^{(n)}, \ldots, aw_{n+3}^{(n)}),$$

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the last $(n+2)$ entries of $w^{(n+1)}(B_{n+1}-aI)$ are just the last $(n+2)$ entries of $w^{(n)}(B_n-aI)$ multiplied by a; that is, they are zero. The second entry of $w^{(n+1)}(B_{n+1}-aI)$ is

$$-aw_2^{(n+1)}+w_3^{(n+1)}+w_4^{(n+1)}=-a(w_2^{(n)}+w_3^{(n)})+aw_2^{(n)}+aw_3^{(n)}=0.$$

The first entry is $w_2^{(n+1)}+w_3^{(n+1)}=w_2^{(n+2)}$. This shows (v).

By (i) and (ii), the g.c.d. of the entries of $w^{(n)}$ is 1, and $w_1^{(n)}=0$. Then by iteration of (3.3), $w^{(n)}$ is contained in an integral basis $\{b^{(1)},\dots,b^{(n+2)}\}=\mathcal{B}$ for the set of integral row vectors of length $n+3$ with first coordinate zero. Let A be the $(n+3)\times(n+3)$ matrix

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & B \end{array}\right)$$

whose last $n+2$ rows are the vectors from $\mathcal{B}$ and whose first row is $v^{(n)}$. Det B must be ±1, so det A is ±1, so $\mathcal{B}'=\mathcal{B}\cup\{v^{(n)}\}$ is an integral basis for $\mathbb{Z}^{n+3}$. Let $v^{(n)}$ and $w^{(n)}$ be the first and second vectors listed in $\mathcal{B}'$.

Now the linear span of $v^{(n)}$ and $w^{(n)}$ is V_n, a B_n-invariant subspace. Therefore, with respect to the basis $\mathcal{B}'$, the linear transformation defined by B_n is given by a matrix C_n of the form

$$\left[\begin{array}{cc|c} a & 0 & 0 \\ k_n & a & \\ \hline \multicolumn{2}{c|}{X} & Y \end{array}\right].$$

Since $\mathcal{B}'$ is an integral basis, C_n must be similar over $\mathbb{Z}$ (hence, SE-$\mathbb{Z}$) to B_n. But by (i)-(v), the k_n are positive, strictly increasing with n and relatively prime to a. Now (3.1) and (3.2) imply that the C_n are pairwise not shift equivalent over $\mathbb{Z}$, and we are done. □

(3.5) *Remark.* One can give a less elementary but more geometric demonstration of (3.4) which bypasses (3.2) and (3.3). Here one applies the direct limit viewpoint of Krieger [9] and considers the group automorphism $\hat{B}_n$ obtained by restriction of B_n to

$$\{x\in V_n\colon x(B_n)^k\in\mathbb{Z}^{n+3}\cap V_n,\quad \text{for some } k>0\}.$$

One sees that the shift equivalence of B_m and B_n forces the conjugacy of $\hat{B}_m$ and $\hat{B}_n$, which in turn implies that the matrices

$$\begin{pmatrix} a & 0 \\ k_m & a \end{pmatrix} \text{ and } \begin{pmatrix} a & 0 \\ k_n & a \end{pmatrix}$$

are shift equivalent over $\mathbb{Z}$.

One can vary (3.3) to produce more complicated examples. I expect the following.

(3.6) *Conjecture.* If A is a non-negative aperiodic integral matrix and its characteristic polynomial χ_A has a repeated non-zero root, then there exist infinitely many non-negative aperiodic integral matrices B, pairwise not shift equivalent, such that $\chi_A=\chi_B$ modulo powers of x.

A better understanding of the geometry behind (3.4) may show how to produce such B from A. By (2.11), there is no algebraic obstacle to (3.6).

Why try to resolve (3.6), other than to sharpen (2.8)? It is likely that the classification of aperiodic non-negative integral matrices up to shift equivalence will involve two parts: a classification of non-singular integral matrices up to shift equivalence over $\mathbb{Z}$, and a realization theory indicating when an integral matrix is shift equivalent over $\mathbb{Z}$ to a non-negative aperiodic matrix. Resolution of (3.6) would involve progress on the difficult realization problem.

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